

EXPOSITORY PAPER: A PRIMER ON HOMOGENIZATION OF ELLIPTIC PDES WITH STATIONARY AND ERGODIC RANDOM COEFFICIENT FUNCTIONS

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ABSTRACT. We study the problem of characterizing the effective (homogenized) properties of materials whose diffusive properties are modeled with random fields. Focusing on elliptic PDEs with stationary and ergodic random coefficient functions, we provide a gentle introduction to the mathematical theory of homogenization of random media. We also present numerical examples to elucidate the theoretical concepts and results.

1. Introduction. Homogenization is a branch of the theory of partial differential equations (PDEs) which provides the mathematical basis for describing effective physical properties of materials with inhomogeneous microstructures. In this article, we study homogenization of random media, i.e., materials whose physical properties are modeled with random functions. Major theoretical results on homogenization of random media were developed first by Papanicolaou and Varadhan in [38], and Kozlov in [32]. The theory of homogenization of random media (stochastic homogenization), in addition to the usual analysis and PDE theory tools, relies on results from probability and ergodic theory. This intermixing of analysis and PDE theory concepts with those of probability often makes this otherwise elegant theory difficult to penetrate for those with a more PDE oriented background and who are less familiar with the probabilistic concepts encountered in stochastic homogenization.

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This article aims to provide a gentle introduction to stochastic homogenization by focusing on a few key results and proving them in detail. We consider linear elliptic PDEs with stationary and ergodic coefficient functions, and provide proofs of homogenization results in one space dimension and in several space dimensions. A summary of the requisite background materials is provided with an expanded discussion of concepts from ergodic theory. The first homogenization result we study concerns one-dimensional elliptic equations with random coefficients. The proof of the one-dimensional result, which is considerably simpler than the general n -dimensional case, provides a first exposure to combining probabilistic and functional analytic tools to derive homogenization results. Our discussion of the homogenization theorem in the general n -dimensional case follows in similar lines as the arguments given in [31] with many details added to keep the concepts and arguments accessible. Moreover, to make the presentation beginner-friendly, throughout the article we provide a number of motivating numerical examples to illustrate the theoretical concepts and results that follow.

The target audience of this article includes graduate students who are entering this field of research as well as mathematicians who are new to stochastic homogenization. The background assumed in the following is a working knowledge of basic concepts in PDE theory, a course in linear functional analysis, and basic concepts from measure-theoretic probability. Reading this article should aid those new to the field in transitioning to advanced texts such as [15, 31] that provide a complete coverage of stochastic homogenization. One should also keep in mind that the general theory of homogenization is not limited to the cases of periodic or stationary and ergodic media, and can be applied to physical processes other than diffusion. We refer the reader to the book [42] by Tartar, where the author provides an in-depth presentation of the mathematical theory of homogenization as well as historical background on the development of homogenization theory.

Let us begin our discussion of homogenization with an example. In Figure 1, we depict what a realization of a medium with random microstructure might look like. Numerical modeling of physical processes such as diffusion through such media is generally a challenging task because the corresponding differential equations have random coefficients whose realizations are rapidly oscillating functions. Given a diffusive

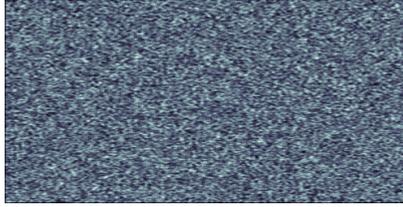


FIGURE 1. Depiction of a medium with random microstructure.

medium with inhomogeneous (random) microstructure, the goal of homogenization is to construct an effective (homogenized) medium whose conductive/diffusive properties, in macroscale, are close to the original medium. The basic motivation for this is the fact that the homogenized medium is much easier to work with.

To state the problem mathematically, we first consider a deterministic case. Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ be a matrix-valued coefficient function that is uniformly bounded and positive definite. We focus on elliptic differential operators of the form

$$(1.1) \quad \mathcal{L}^\varepsilon u = -\operatorname{div}(A^\varepsilon \nabla u), \quad \text{where } A^\varepsilon(\mathbf{x}) = A(\varepsilon^{-1}\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ indicates a microstructural length-scale. The coefficient functions A^ε characterize media with inhomogeneous microstructure. Homogenization theory studies the problem in the limit as $\varepsilon \rightarrow 0$.

In the case of materials with random microstructure, the coefficient function A in (1.1) is a random field, i.e., $A = A(\mathbf{x}, \omega)$ where ω is an element of a sample space Ω . To motivate the basic questions that arise in homogenization, we consider some specific numerical examples in Section 2 below, in the context of a problem in one space dimension. This discussion is then used to guide the reader through the subsequent sections of this article.

2. Motivation and overview. Although our discussion concerns mainly that of random structures, to develop some intuition we consider the case of a one-dimensional *periodic* structure first. Consider the problem of modeling steady-state heat diffusion in a rod whose conductivity profile is given by the function $a^\varepsilon(x) = a(\varepsilon^{-1}x)$ where a is

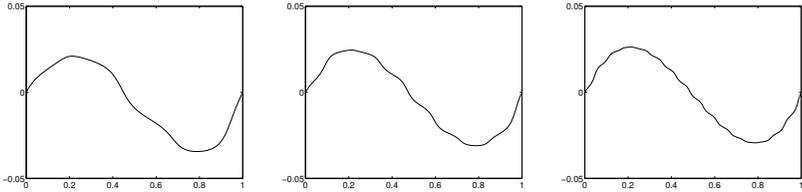


FIGURE 2. The solutions u^ε corresponding to coefficient a^ε with $\varepsilon = 1/4, 1/8,$ and $1/16,$ respectively.

a bounded periodic function defined on the physical domain \mathcal{D} ; in our example, we let $\mathcal{D} = (0, 1)$. Moreover, we assume that the temperature is fixed at zero at the end points of the interval. In this case, the following equation describes the steady-state temperature profile in the conductor,

$$(2.1) \quad \begin{aligned} -\frac{d}{dx} \left(a^\varepsilon \frac{du^\varepsilon}{dx} \right) &= f \quad \text{in } \mathcal{D} = (0, 1), \\ u^\varepsilon &= 0 \quad \text{on } \partial\mathcal{D} = \{0, 1\}. \end{aligned}$$

The right-hand side function f describes a source term. Since a is a periodic function, considering a^ε with successively smaller values of ε implies working with rapidly oscillating conductivity functions. Speaking in terms of material properties, considering successively smaller values of ε entails the consideration of conductors with successively *finer microstructure*. The basic question of homogenization is that of what happens as $\varepsilon \rightarrow 0$, and whether there is a limiting *homogenized* material.

For the purposes of illustration, let us consider a specific example. We let the function $a(x)$ and the right-hand side function $f(x)$ be given by

$$(2.2) \quad a(x) = 2 + \sin(2\pi x), \quad f(x) = -3(2x - 1).$$

It is clear that, as $\varepsilon \rightarrow 0$, the function a^ε becomes more and more oscillatory. In Figure 2, we plot the solution of the problem (2.1) for the coefficient functions a^ε with successively smaller values of ε . The results plotted in Figure 2 suggest that, as ε gets smaller, the solutions u^ε seem to converge to a limit. The following are some relevant questions: (i) Do u^ε actually converge to a limit? (ii) If so, in what topology does the

convergence take place? (iii) Can we describe/characterize the limit? The answers to these questions are all well known. In this case, the functions u^ε converge in $L^2(\mathcal{D})$ -norm to u^0 that is the solution of the following problem:

$$(2.3) \quad \begin{aligned} -\frac{d}{dx} \left(a^0 \frac{du^0}{dx} \right) &= f \quad \text{in } \mathcal{D} = (0, 1), \\ u^0 &= 0 \quad \text{on } \partial\mathcal{D} = \{0, 1\}, \end{aligned}$$

where a^0 is the harmonic mean of a over the interval $(0, 1)$,

$$a^0 = \left(\int_0^1 \frac{1}{a(x)} dx \right)^{-1}.$$

The coefficient a^0 is called the *homogenized coefficient* or the *effective conductivity*. Virtually every homogenization textbook or lecture note has some form of proof for this homogenization result. Hence, we just illustrate this result numerically here. Notice that, with our choice of a above, we have

$$\left(\int_0^1 \frac{1}{a(x)} dx \right)^{-1} = \left(\int_0^1 \frac{1}{2 + \sin(2\pi x)} dx \right)^{-1} = \sqrt{3},$$

as the homogenized coefficient. With this value of a^0 , the analytic solution of the homogenized equation (2.3) is given by

$$u^0(x) = \frac{1}{\sqrt{3}} x(x - 1/2)(x - 1).$$

In Figure 3, we plot the function u^0 (left plot) and demonstrate the convergence of u^ε to u^0 by looking at $\|u^\varepsilon - u^0\|_{L^2(\mathcal{D})}$ as $\varepsilon \rightarrow 0$ (right plot).

Now let us transition to the case of random media. In this case, the function a , which defines the conductivity profile of the material, is a random function. The stochastic version of (2.1) is given by

$$(2.4) \quad \begin{aligned} -\frac{d}{dx} \left(a^\varepsilon(\cdot, \omega) \frac{du^\varepsilon}{dx}(\cdot, \omega) \right) &= f \quad \text{in } \mathcal{D} = (0, 1), \\ u^\varepsilon(\cdot, \omega) &= 0 \quad \text{on } \partial\mathcal{D} = \{0, 1\}, \end{aligned}$$

with $a^\varepsilon(x, \omega) = a(\varepsilon^{-1}x, \omega)$, and $a(x, \omega)$ a random function (random field). The variable ω is an element of a sample space Ω , and for a fixed ω , $a(\cdot, \omega)$ is a realization of the random function a . As an example, we

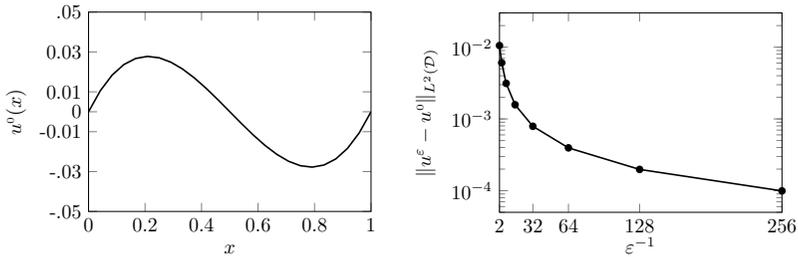


FIGURE 3. Left: Plot of the solution u^0 of the homogenized equation. Right: The convergence of the solutions u^ε to u^0 in $L^2(\mathcal{D})$; the black dots correspond to $\|u^{\varepsilon_k} - u^0\|_{L^2(\mathcal{D})}$ with $\varepsilon_k = 1/2^k$, $k = 1, \dots, 8$.

consider a material made up of tiles, each of which has conductivity of either κ_1 or κ_2 , chosen randomly with probabilities p and $1 - p$, respectively, with $p \in (0, 1)$. A realization of the conductivity function for such a structure is depicted in Figure 4, with the choices of $\kappa_1 = 1$ and $\kappa_2 = 3$ and with $p = 1/2$. In this example, the microstructural length-scale ε determines the size of the tiles in the random structure.

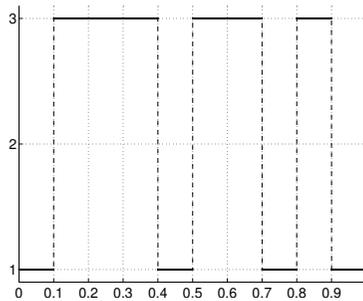


FIGURE 4. A realization of the conductivity profile for a one-dimensional random checkerboard structure.

We consider the problem (2.4) with a fixed realization (a fixed ω) of this coefficient function, and for successively smaller values of ε . (We continue to use the same right-hand side function f defined in (2.2).) The solutions $u^\varepsilon(\cdot, \omega)$ of the respective problems have been plotted in Figure 5. These plots suggest that u^ε seems to converge to a limiting

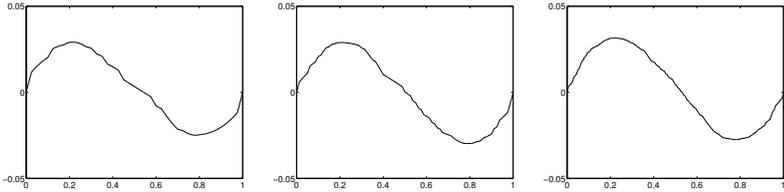


FIGURE 5. The solutions u^ε corresponding to coefficient a^ε with $\varepsilon = 1/4, 1/8,$ and $1/16$ respectively.

function. In what follows, we shall discuss the mathematical theory for such stochastic homogenization problems. Some relevant questions in this context include the following: (i) is there a homogenized problem in this stochastic setting? (ii) Is it possible to have a *constant* homogenized coefficient that is independent of ω ? (iii) Does the problem admit homogenization for all ω ? (iv) In the deterministic example above periodicity of the coefficient was the property that led to a constant homogenized coefficient; what is the stochastic counterpart of periodicity? (v) What conditions on $a(x, \omega)$ ensure existence of a deterministic homogenized coefficient? A rigorous and clear discussion of such questions, which is the main point of this article, requires a systematic synthesis of concepts from functional analysis, PDE theory, probability theory and ergodic theory.

The discussion in the rest of this article is structured as follows. In Section 3, we collect the background concepts required in our coverage of stochastic homogenization. We continue our discussion by describing the setting of the homogenization problem for random media in Section 4. Next, in Section 5, we state and prove a homogenization theorem in one space dimension. An interesting aspect of the analysis for one-dimensional random structures is the derivation of a closed-form expression for the homogenized coefficient that is analogous to the form of the homogenized coefficient for one-dimensional periodic structures. Finally, in Section 6, we study homogenization of elliptic PDEs with random coefficients in several space dimensions, where no closed-form expressions for the homogenized coefficients are available in general. In Section 7, we conclude our discussion by giving some pointers for further reading. We mention that an earlier version of the exposition of the theoretical results in Sections 5 and 6 appeared first in an introductory chapter of the PhD dissertation [1].

3. Preliminaries.

3.1. Background from functional analysis and Sobolev spaces.

Here we briefly discuss some background concepts from the theory of PDEs and functional analysis that are needed in the discussion of the homogenization results in the present work.

Poincaré inequality. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a bounded open set with piecewise smooth boundary. In what follows, we denote by $L^2(\mathcal{D})$ the space of real-valued square-integrable functions on \mathcal{D} and denote by $C_c^\infty(\mathcal{D})$ the space of smooth functions with compact support in \mathcal{D} . The Sobolev space $H^1(\mathcal{D})$ consists of functions in $L^2(\mathcal{D})$ with square integrable first-order weak derivatives and is equipped with the norm,

$$\|u\|_{H^1(\mathcal{D})}^2 = \int_{\mathcal{D}} u^2 dx + \int_{\mathcal{D}} |\nabla u|^2 dx.$$

The space $H_0^1(\mathcal{D})$ is a subspace of $H^1(\mathcal{D})$ obtained as the closure of $C_c^\infty(\mathcal{D})$ in $H^1(\mathcal{D})$. More intuitively, we may interpret $H_0^1(\mathcal{D})$ as the subspace of $H^1(\mathcal{D})$ consisting of functions in $H^1(\mathcal{D})$ that vanish on the boundary of \mathcal{D} . The well-known Poincaré inequality states that for a bounded open set $\mathcal{D} \subseteq \mathbb{R}^n$, there is a positive constant C_p (depending on \mathcal{D} only) such that for every $u \in H_0^1(\mathcal{D})$,

$$\int_{\mathcal{D}} u^2 dx \leq C_p \int_{\mathcal{D}} |\nabla u|^2 dx.$$

Weak convergence. Recall that a sequence $\{u^k\}_1^\infty$ in a Banach space X converges weakly to $u^* \in X$ if $\ell(u^k) \rightarrow \ell(u^*)$ as $k \rightarrow \infty$, for every bounded linear functional ℓ on X , in which case we write $u^k \rightharpoonup u^*$. We recall that, as a consequence of the Banach-Steinhaus theorem, weakly convergent sequences in a Banach space are bounded in norm. Moreover, it is a standard result in functional analysis that, in a reflexive Banach space, every bounded sequence has a weakly convergent subsequence. Another standard result, which will be used in our discussion below, is that compact operators on Banach spaces map weakly convergent sequences to strongly (norm) convergent sequences. In particular, this implies the following: consider a Hilbert space H and a Hilbert subspace $U \subset H$ that is compactly embedded in H ; then any bounded sequence in U will have a subsequence that converges strongly in H . We also recall that, in a Hilbert space H with inner-product $\langle \cdot, \cdot \rangle$, a

sequence $\{u^k\}$ converges weakly to u^* if $\langle u^k, \phi \rangle \rightarrow \langle u^*, \phi \rangle$ for every $\phi \in H$.

Compensated compactness. Let \mathcal{D} be a bounded domain in \mathbb{R}^n , and suppose \mathbf{u}^ε converges strongly in $\mathbf{L}^2(\mathcal{D}) = (L^2(\mathcal{D}))^n$ to \mathbf{u}^0 and $\mathbf{v}^\varepsilon \xrightarrow{w} \mathbf{v}^0$ in $\mathbf{L}^2(\mathcal{D})$. In this case, it is straightforward to show that $\mathbf{u}^\varepsilon \cdot \mathbf{v}^\varepsilon \xrightarrow{w} \mathbf{u}^0 \cdot \mathbf{v}^0$ in $L^1(\mathcal{D})$. Consider now sequences \mathbf{u}^ε and \mathbf{v}^ε in $\mathbf{L}^2(\mathcal{D})$, both of which converge weakly. In this case, additional conditions are needed to ensure the convergence of $\mathbf{u}^\varepsilon \cdot \mathbf{v}^\varepsilon$, in an appropriate sense, to the inner product of the respective weak limits. Such problems, which arise naturally in homogenization theory, led to the development of the concept of compensated compactness by Murat and Tartar [33, 41]. Here we recall an important compensated compactness lemma, which specifies conditions that enable passing to the limit in the scalar product of weakly convergent sequences and concluding the weak- \star convergence of the scalar product of the sequences to the scalar product of their weak limits. Weak- \star convergence, which is a weaker mode of convergence than weak convergence discussed above, takes the following form for a sequence of integrable functions: let $\{z^\varepsilon\}$ be a sequence in $L^1(\mathcal{D})$; then z^ε converges weak- \star to z^0 if $\{z^\varepsilon\}$ is bounded in $L^1(\mathcal{D})$, and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}} z^\varepsilon \phi \, dx = \int_{\mathcal{D}} z^0 \phi \, dx, \quad \text{for all } \phi \in C_c^\infty(\mathcal{D}).$$

We use the notation $z^\varepsilon \xrightarrow{w^*} z^0$ for weak- \star convergence. The fact that weak- \star limits are unique will be important in what follows.

The following Div-Curl lemma is a well-known compensated compactness result, and is a key in proving homogenization results; see [31] for a proof of this lemma, and [42, Chapter 7] for a more complete discussion as well as interesting historical remarks on the development of the Div-Curl lemma.

Lemma 3.1. *Let \mathcal{D} be a bounded domain in \mathbb{R}^n , and let \mathbf{p}^ε and \mathbf{v}^ε be vector-fields in $\mathbf{L}^2(\mathcal{D})$ such that*

$$\mathbf{p}^\varepsilon \xrightarrow{w} \mathbf{p}^0, \quad \mathbf{v}^\varepsilon \xrightarrow{w} \mathbf{v}^0.$$

Moreover, assume that $\text{curl } \mathbf{v}^\varepsilon = 0$ for all ε and $\text{div } \mathbf{p}^\varepsilon \rightarrow f^0$ in $H^{-1}(\mathcal{D})$. Then we have

$$\mathbf{p}^\varepsilon \cdot \mathbf{v}^\varepsilon \xrightarrow{w^*} \mathbf{p}^0 \cdot \mathbf{v}^0.$$

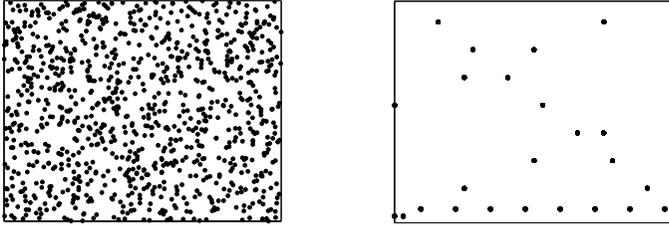


FIGURE 6. For T given in (3.1), we look at the orbits $\{T^n(\mathbf{x}_0)\}_{n=1}^N$ with $\mathbf{x}_0 = (1/32, \pi/32)$ (left plot) and $\{T^n(\mathbf{y}_0)\}_{n=1}^N$ with $\mathbf{y}_0 = (1/32, 1/32)$ (right plot), for $N = 1000$ iterations.

3.2. Background concepts from ergodic theory. Here we provide a brief coverage of the concepts from ergodic theory that are central to the discussion that follows in the rest of this article. We begin by illustrating the concept of ergodicity through a numerical example. Let \mathbb{T}^2 be the two-dimensional unit torus, given by the rectangle $[0, 1) \times [0, 1)$ with the opposite sides identified, and consider the transformation $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined by

$$(3.1) \quad T(\mathbf{x}) = \begin{bmatrix} (2x_1 + x_2) \pmod{1} \\ (x_1 + x_2) \pmod{1} \end{bmatrix}.$$

This transformation is an instance of a hyperbolic toral automorphism [12] and is commonly referred to as Arnold's Cat Map, named after V.I. Arnold who illustrated the behavior of the mapping by considering its repeated applications to an image of a cat [7].

For a given $\mathbf{x}_0 \in \mathbb{T}^2$, we call the sequence of the points $\{T^n(\mathbf{x}_0)\}_{n=1}^\infty$ the orbit of \mathbf{x}_0 , where T^n means n successive applications of T . In Figure 6, we depict a portion of the orbit of two different points given by $\mathbf{x}_0 = (1/32, \pi/32)$ and $\mathbf{y}_0 = (1/32, 1/32)$ in the left and right images, respectively. The left plot in Figure 6 suggests that the successive iterates $T^n(\mathbf{x}_0)$ do a good job of visiting the entire state space \mathbb{T}^2 . On the other hand, the plot on the right sends the opposite message. Note, however, that the coordinates of \mathbf{y}_0 in the latter case are both rational. It is known [12] that, for this specific example, the set of points with rational coordinates are precisely the set of periodic points of the transformation T ; thus, since the Lebesgue measure of this set is zero, we have that *for almost all* $\mathbf{x}_0 \in \mathbb{T}^2$, the behavior in the left

plot of Figure 6 holds. This almost sure “space filling” property of the system defined by T is a consequence of *ergodicity*.

Next, consider an integrable function $f : \mathbb{T}^2 \rightarrow \mathbb{R}$. Due to the “space filling” property of T , we may intuitively say that, for almost all \mathbf{x}_0 and for N sufficiently large, the set of points $\{f(T^n(\mathbf{x}_0))\}_{n=1}^N$ provide a sufficiently rich sampling of the function f and that

$$\frac{1}{N} \sum_{n=1}^N f(T^n(\mathbf{x}_0)) \approx \frac{1}{|\mathbb{T}^2|} \int_{\mathbb{T}^2} f(\mathbf{x}) \, d\mathbf{x}.$$

(Here $|\mathbb{T}^2|$ is the Lebesgue measure of \mathbb{T}^2 , which is equal to one, but is included in the expression for clarity.) The above observation leads to the usual intuitive understanding of ergodicity: *for an ergodic system, time averages equal space averages*. In the present example, time is specified by n , that is, we have a system with discrete time.

The remainder of this section contains a brief discussion of the concepts from probability and ergodic theory that we need in our coverage of stochastic homogenization. For more details on ergodic theory, we refer the reader to [12, 19, 44]. See also [16] for an accessible introduction to ergodic theory, where the author incorporates many illustrative computer examples in the presentation of the theoretical concepts.

Random variables and measure preserving transformations.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. The set Ω is a sample space, \mathcal{F} is an appropriate sigma-algebra on Ω , and μ is a probability measure. A random variable is an $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable function from Ω to \mathbb{R} , where $\mathfrak{B}(\mathbb{R})$ denotes the Borel sigma-algebra on \mathbb{R} . Given a random variable $f : (\Omega, \mathcal{F}, \mu) \rightarrow (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, we denote its expected value by

$$\mathbb{E} \{f\} := \int_{\Omega} f(\omega) \mu(d\omega).$$

Definition 3.2. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be measure spaces. A transformation $T : \Omega_1 \rightarrow \Omega_2$ is called measure preserving if it is measurable, i.e., for all $E \in \mathcal{F}_2$ $T^{-1}(E) \in \mathcal{F}_1$, and satisfies

$$(3.2) \quad \mu_1(T^{-1}(E)) = \mu_2(E), \quad \text{for all } E \in \mathcal{F}_2.$$

An example of a measure preserving transformation is the one defined in (3.1), which preserves the Lebesgue measure on \mathbb{T}^2 .

Dynamical systems and ergodicity. Let T be a measure preserving transformation on $(\Omega, \mathcal{F}, \mu)$. Interpreting the elements of Ω as possible states of a system, we may consider T as the law of the time evolution of the system. That is, if we denote by s_n , $n \geq 0$, the state of the system at $t = n$, and let $s_0 = \omega_0$ for some $\omega_0 \in \Omega$, then $s_1 = T(\omega_0)$, $s_2 = T(s_1) = T(T(\omega_0)) = T^2(\omega_0)$, and, in general, $s_n = T^n(\omega_0)$, for $n \geq 1$. This way, T defines a measurable dynamic on Ω . The dynamical system so constructed is called a *discrete time* measure-preserving dynamical system.

Suppose there is a set $E \in \mathcal{F}$ such that $\omega \in E$ if and only if $T(\omega) \in E$. In such a case, the study of the dynamics of T on Ω can be *reduced* to its dynamics on E and $\Omega \setminus E$. The set E so described is called a *T-invariant* set. We say that T is ergodic if, for every T -invariant set E , we have either $\mu(E) = 0$ or $\mu(E) = 1$.

n -dimensional dynamical systems. In addition to discrete time dynamical systems described above, we can also consider *continuous time* dynamical systems that are given by a family of measurable transformations $T = \{T_t\}_{t \in \mathcal{S}}$ where $\mathcal{S} \subseteq \mathbb{R}^n$ with $n = 1$. In the case $\mathcal{S} = [0, \infty)$, we call T a *semiflow* and in the case $\mathcal{S} = (-\infty, \infty)$, we call T a *flow*. In the present work, we are interested in a more general type of dynamical system where $\mathcal{S} = \mathbb{R}^n$ with $n \geq 1$.

Definition 3.3. An n -dimensional measure-preserving dynamical system T on Ω is a family of measurable mappings $T_{\mathbf{x}} : \Omega \rightarrow \Omega$, parametrized by $\mathbf{x} \in \mathbb{R}^n$, satisfying:

- (i) $T_{\mathbf{x}+\mathbf{y}} = T_{\mathbf{x}} \circ T_{\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- (ii) $T_{\mathbf{0}} = I$, where I is the identity map on Ω .
- (iii) The dynamical system is *measure preserving* in the sense that, for every $\mathbf{x} \in \mathbb{R}^n$ and $F \in \mathcal{F}$, we have $\mu(T_{\mathbf{x}}^{-1}(F)) = \mu(F)$.
- (iv) For every measurable function $g : (\Omega, \mathcal{F}, \mu) \rightarrow (X, \Sigma)$ where (X, Σ) is some measurable space, the composition $g(T_{\mathbf{x}}(\omega))$ defined on $\mathbb{R}^n \times \Omega$ is a $(\mathfrak{B}(\mathbb{R}^n) \otimes \mathcal{F})/\Sigma$ measurable function.

The notions of T -invariant functions and sets (where T is an n -dimensional dynamical system) are made precise in the following definition [19].

Definition 3.4. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^n}$ an n -dimensional measure-preserving dynamical system. A measurable function g on Ω is T -invariant if for all $\mathbf{x} \in \mathbb{R}^n$,

$$(3.3) \quad g(T_{\mathbf{x}}(\omega)) = g(\omega), \quad \text{for all } \omega \in \Omega.$$

A measurable set $E \in \mathcal{F}$ is T -invariant if its characteristic function $\mathbb{1}_E$ is T -invariant.

It is straightforward to show that a T -invariant set E defined according to the above definition can be defined equivalently as follows: a set E is T -invariant if

$$T_{\mathbf{x}}^{-1}(E) = E, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

As is often the case in measure theory, we can replace “for all $\omega \in \Omega$ ” by “for almost all $\omega \in \Omega$ ” in Definition 3.4. A function that satisfies (3.3) for all \mathbf{x} and almost all $\omega \in \Omega$ is called T -invariant mod 0. Also, given two measurable sets A and B , we write $A = B \text{ mod } 0$, if their symmetric difference, $A \Delta B = (A \setminus B) \cup (B \setminus A)$ has measure zero; note that this means A and B agree modulo a set of measure zero. We call a measurable set T -invariant mod 0 if its characteristic function is T -invariant mod 0.

One can show (cf., [19]) that for any measurable function g on Ω that is T -invariant mod 0, there exists a T -invariant function \tilde{g} such that $g = \tilde{g}$ almost everywhere. Similarly, for any T -invariant mod 0 set E , there exists a T -invariant set \tilde{E} such that $\mu(\tilde{E} \Delta E) = 0$. Hence, in what follows, we will not distinguish between T -invariance mod 0 and T -invariance.

With these background ideas in place, we define the notion of an n -dimensional ergodic dynamical system.

Definition 3.5. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $T = \{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^n}$ an n -dimensional measure-preserving dynamical system. We say T is ergodic if all T -invariant sets have measure of either zero or one.

Let us also recall the following useful characterization of an ergodic dynamical system [19, 31], in terms of invariant functions: a *dynamical*

system is ergodic if every T -invariant function is constant almost everywhere; that is,

$$\left[g(T_{\mathbf{x}}(\omega)) = g(\omega) \text{ for all } \mathbf{x} \text{ and almost all } \omega \right] \implies g \equiv \text{const } \mu\text{-a.e.}$$

Let $\{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^n}$ be a dynamical system. Corresponding to a function $g : \Omega \rightarrow X$ (where X is any set) we define the function $g_T : \mathbb{R}^n \times \Omega \rightarrow X$ by

$$(3.4) \quad g_T(\mathbf{x}, \omega) = g(T_{\mathbf{x}}(\omega)), \quad \mathbf{x} \in \mathbb{R}^n, \omega \in \Omega.$$

For each $\omega \in \Omega$, the function $g_T(\cdot, \omega) : \mathbb{R}^n \rightarrow X$ is called a *realization* of g .

The Birkhoff ergodic theorem. Ergodicity of a dynamical system has many profound implications. Of particular importance to our discussion is the Birkhoff Ergodic theorem. Before stating Birkhoff's theorem, we define the following notion of mean-value for functions.

Definition 3.6. Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$. A number M_g is called the mean-value of g if, for every Lebesgue measurable bounded set $K \subset \mathbb{R}^n$,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|K|} \int_K g(\varepsilon^{-1} \mathbf{x}) d\mathbf{x} = M_g.$$

Here $|K|$ denotes the Lebesgue measure of K .

The following result, due to Birkhoff, is a major result in ergodic theory [19], which, as we will see shortly, plays a central role in proving homogenization results for random elliptic operators. The statement of Birkhoff's theorem given below follows the presentation in [31].

Theorem 3.7. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and suppose $T = \{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^n}$ is a measure-preserving dynamical system on Ω . Let $g \in L^p(\Omega)$ with $p \geq 1$. Then, for almost all $\omega \in \Omega$, the realization $g_T(\mathbf{x}, \omega)$, as defined in (3.4), has a mean value $M_g(\omega)$ in the following sense: defining $g_T^\varepsilon(\mathbf{x}, \omega) = g_T(\varepsilon^{-1} \mathbf{x}, \omega)$ for $\varepsilon > 0$, one has

$$g_T^\varepsilon(\cdot, \omega) \xrightarrow{w} M_g(\omega) \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^n), \quad \text{as } \varepsilon \rightarrow 0,$$

for almost all $\omega \in \Omega$. Moreover, M_g is a T -invariant function, that is,

$$(3.5) \quad M_g(T_{\mathbf{x}}(\omega)) = M_g(\omega) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n, \mu\text{-a.e.}$$

Also,

$$(3.6) \quad \int_{\Omega} g(\omega) \mu(d\omega) = \int_{\Omega} M_g(\omega) \mu(d\omega).$$

Notice that, if the dynamical system T in Birkhoff’s theorem is ergodic, then, the mean value M_g is constant almost everywhere and is given by $M_g = \mathbb{E} \{g\}$. We record this observation in the following Corollary of Theorem 3.7:

Corollary 3.8. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and suppose $T = \{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^n}$ is a measure-preserving and ergodic dynamical system on Ω . Let $g \in L^p(\Omega)$ with $p \geq 1$. Define $g_T^\varepsilon(\mathbf{x}, \omega) = g_T(\varepsilon^{-1}\mathbf{x}, \omega)$ for $\varepsilon > 0$. Then, for almost all $\omega \in \Omega$,*

$$g_T^\varepsilon(\cdot, \omega) \xrightarrow{w} \int_{\Omega} g(\omega) \mu(d\omega) \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^n), \text{ as } \varepsilon \rightarrow 0.$$

Stationary random fields. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $G : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$ be a random field. We say G is stationary if, for any finite collection of points $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, \dots, k$ and any $\mathbf{h} \in \mathbb{R}^n$, the joint distribution of the random k -vector $(G(\mathbf{x}_1 + \mathbf{h}, \omega), \dots, G(\mathbf{x}_k + \mathbf{h}, \omega))^T$ is the same as that of $(G(\mathbf{x}_1, \omega), \dots, G(\mathbf{x}_k, \omega))^T$. It is straightforward to show that, if G can be written in the form

$$(3.7) \quad G(\mathbf{x}, \omega) = g(T_{\mathbf{x}}(\omega)),$$

where $g : \Omega \rightarrow \mathbb{R}$ is a measurable function and T is a measure preserving dynamical system, then G is stationary. For G to be stationary and ergodic, we need the dynamical system T in (3.7) to be ergodic.

Note that, when working with stationary and ergodic random functions, the Birkhoff ergodic theorem enables the type of averaging that is relevant in the context of homogenization. It is also interesting to recall the following Riemann-Lebesgue lemma that plays a similar role as Birkhoff’s theorem, in the problems of averaging of elliptic differential operators with *periodic* coefficient functions (see [20, page 21] for a more general statement of the Riemann-Lebesgue lemma and its proof).

Lemma 3.9. *Let $Y = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ be a rectangle in \mathbb{R}^n , and let $g \in L^2(Y)$. Extend g by periodicity from Y to \mathbb{R}^n . For $\varepsilon > 0$, let $g^\varepsilon(\mathbf{x}) = g(\varepsilon^{-1}\mathbf{x})$. Then, as $\varepsilon \rightarrow 0$, $g^\varepsilon \xrightarrow{w} \bar{g}$ in $L^2(Y)$, where $\bar{g} := \frac{1}{|Y|} \int_Y g(\mathbf{x}) d\mathbf{x}$.*

Solenoidal and potential vector fields and Weyl’s decomposition theorem. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Here we briefly recall an important decomposition of the space $\mathbf{L}^2(\Omega) = L^2(\Omega; \mathbb{R}^n)$ of square integrable vector-fields on Ω —the Weyl decomposition theorem. This result will be important in homogenization results for random elliptic operators in the general n -dimensional case. Recall that a locally square integrable vector-field \mathbf{v} on \mathbb{R}^n is called *potential* if $\mathbf{v} = \nabla\phi$ for some $\phi \in H^1_{\text{loc}}(\mathbb{R}^n)$ and called *solenoidal* if it is divergence free. Letting T be an n -dimensional measure-preserving dynamical system on Ω , we consider the following spaces:

$$\begin{aligned}
 \mathbf{L}^2_{\text{pot}}(\Omega, T) &= \{\mathbf{f} \in \mathbf{L}^2(\Omega) : \\
 &\quad \mathbf{f}_T(\cdot, \omega) \text{ is potential on } \mathbb{R}^n \text{ for almost all } \omega \in \Omega\}, \\
 \mathbf{L}^2_{\text{sol}}(\Omega, T) &= \{\mathbf{f} \in \mathbf{L}^2(\Omega) : \\
 (3.8) \quad &\quad \mathbf{f}_T(\cdot, \omega) \text{ is solenoidal on } \mathbb{R}^n \text{ for almost all } \omega \in \Omega\}, \\
 \mathcal{V}^2_{\text{pot}}(\Omega, T) &= \{\mathbf{f} \in \mathbf{L}^2_{\text{pot}}(\Omega, T) : \mathbb{E}\{\mathbf{f}\} = \mathbf{0}\}, \\
 \mathcal{V}^2_{\text{sol}}(\Omega, T) &= \{\mathbf{f} \in \mathbf{L}^2_{\text{sol}}(\Omega, T) : \mathbb{E}\{\mathbf{f}\} = \mathbf{0}\}.
 \end{aligned}$$

The Weyl decomposition theorem (see, e.g., [31, page 228]) states that the subspaces $\mathcal{V}^2_{\text{pot}}(\Omega, T)$ and $\mathbf{L}^2_{\text{sol}}(\Omega, T)$ of $\mathbf{L}^2(\Omega)$ are mutually orthogonal and complementary, given that T is ergodic.

Theorem 3.10 (Weyl decomposition). *If the dynamical system T is ergodic, then $\mathbf{L}^2(\Omega)$ admits the following orthogonal decompositions:*

$$(3.9) \quad \mathbf{L}^2(\Omega) = \mathcal{V}^2_{\text{pot}}(\Omega, T) \oplus \mathbf{L}^2_{\text{sol}}(\Omega, T) = \mathcal{V}^2_{\text{sol}}(\Omega, T) \oplus \mathbf{L}^2_{\text{pot}}(\Omega, T).$$

4. Mathematical definition of homogenization. As before, we let $(\Omega, \mathcal{F}, \mu)$ be a probability space. The conductivity function of a medium with random microstructure is specified by a random function $A(\mathbf{x}, \omega)$ where, for each $\omega \in \Omega$, $A(\cdot, \omega)$ is a matrix-valued function $A(\cdot, \omega) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}_{\text{sym}}$. Here $\mathbb{R}^{n \times n}_{\text{sym}}$ denotes the space of symmetric

$n \times n$ matrices with real entries. Let the physical domain be given by a bounded open set $\mathcal{D} \subset \mathbb{R}^n$ (with $n = 1, 2$, or 3). Assume for simplicity that the temperature u is fixed at zero on the boundary of \mathcal{D} . The PDE governing heat conduction in the medium with microstructure is given by

$$(4.1) \quad \begin{cases} -\operatorname{div}_{\mathbf{x}}(A(\varepsilon^{-1}\mathbf{x}, \omega)\nabla u^\varepsilon(\mathbf{x}, \omega)) = f(\mathbf{x}) & \text{in } \mathcal{D}, \\ u^\varepsilon(\mathbf{x}, \omega) = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where $f \in H^{-1}(\mathcal{D})$ specifies a (deterministic) source term. The goal of homogenization theory is to specify a problem of the form

$$(4.2) \quad \begin{cases} -\operatorname{div}_{\mathbf{x}}(A^0\nabla u^0) = f & \text{in } \mathcal{D}, \\ u^0 = 0 & \text{on } \partial\mathcal{D} \end{cases}$$

where A^0 in (4.2) is a constant matrix such that the solution u^0 of (4.2) provides a reasonable approximation (for almost all ω) to the solution of (4.1) in the limit as $\varepsilon \rightarrow 0$. The following definition makes the notion of homogenization precise for a single deterministic conductivity function.

Definition 4.1. Consider a matrix valued function, $A : \mathbb{R}^n \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$, and suppose there exist real numbers $0 < \nu_1 < \nu_2$ such that, for each $\mathbf{x} \in \mathbb{R}^n$,

$$\nu_1|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi} \cdot A(\mathbf{x})\boldsymbol{\xi} \leq \nu_2|\boldsymbol{\xi}|^2, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^n.$$

That is, A is uniformly bounded and positive definite. For $\varepsilon > 0$, denote $A^\varepsilon(\mathbf{x}) = A(\varepsilon^{-1}\mathbf{x})$. Then, we say that A admits homogenization if there exists a constant symmetric positive definite matrix A^0 such that for any bounded domain $\mathcal{D} \subset \mathbb{R}^n$ and any $f \in H^{-1}(\mathcal{D})$, the solutions u^ε of the problems

$$(4.3) \quad \begin{cases} -\operatorname{div}(A^\varepsilon\nabla u^\varepsilon) = f & \text{in } \mathcal{D}, \\ u^\varepsilon = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

satisfy the following convergence properties:

$$u^\varepsilon \xrightarrow{w} u^0 \text{ in } H_0^1(\mathcal{D}),$$

and

$$A^\varepsilon \nabla u^\varepsilon \xrightarrow{w} A^0 \nabla u^0 \quad \text{in } \mathbf{L}^2(\mathcal{D}),$$

as $\varepsilon \rightarrow 0$, where u^0 satisfies the problem

$$(4.4) \quad \begin{cases} -\operatorname{div}(A^0 \nabla u^0) = f & \text{in } \mathcal{D}, \\ u^0 = 0 & \text{on } \partial \mathcal{D}. \end{cases}$$

Remark 4.2. In practice, it is sufficient to verify the convergence relations in the above definition for right-hand side functions $f \in L^2(\mathcal{D})$; see also the discussion in [31, Remark 1.5].

Remark 4.3. A family of operators $\{A^\varepsilon\}_{\varepsilon>0}$ satisfying the above definition is said to *G-converge* to A^0 . The uniqueness of the homogenized matrix A^0 is also guaranteed by the uniqueness of *G*-limits, see e.g., [31, page 150] or [30, page 229] for basic properties of *G*-convergence.

Note that Definition 4.1 concerns the homogenization of a single conductivity function $A(\mathbf{x})$. In the case where A is a periodic function, i.e., the case of periodic media, the existence of the homogenized matrix is well-known [8, 17, 34, 40]. In the random case [10, 32, 31, 37, 38, 39, 45], where we work with a random conductivity function $A = A(\mathbf{x}, \omega)$, we say A admits homogenization if, for almost all $\omega \in \Omega$, $A(\cdot, \omega)$ admits homogenization A^0 (with A^0 a constant matrix independent of ω) in the sense of Definition 4.1.

5. Stochastic homogenization: The one-dimensional case. In this section, we discuss the homogenization of an elliptic boundary value problem, in one space dimension, with a random coefficient function. As we shall see shortly, under assumptions of stationarity and ergodicity, there is a closed-form expression for the (deterministic) homogenized coefficient. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $T = \{T_x\}_{x \in \mathbb{R}}$ be a one-dimensional measure preserving and ergodic dynamical system. Let $a : \Omega \rightarrow \mathbb{R}$ be a measurable function, and suppose there exist positive constants ν_1 and ν_2 such that

$$(5.1) \quad \nu_1 \leq a(\omega) \leq \nu_2, \quad \text{for almost all } \omega \in \Omega.$$

For $\omega \in \Omega$, we consider the following problem

$$(5.2) \quad \begin{aligned} -\frac{d}{dx} \left(a_T(\cdot, \omega) \frac{du}{dx}(\cdot, \omega) \right) &= f \quad \text{in } \mathcal{D} = (s, t), \\ u(\cdot, \omega) &= 0 \quad \text{on } \partial\mathcal{D} = \{s, t\}. \end{aligned}$$

Here $\mathcal{D} = (s, t)$ is an open interval, $f \in L^2(\mathcal{D})$ is a deterministic source term and $a_T(x, \omega) = a(T_x(\omega))$ denotes realizations of a with respect to T . Note that, by construction, $a_T(x, \omega)$ is a stationary and ergodic random field.

Theorem 5.1. *For almost all $\omega \in \Omega$, $a_T(x, \omega)$ defined above admits homogenization and*

$$(5.3) \quad a^0 = \frac{1}{\mathbb{E} \{1/a\}}$$

is the corresponding homogenized coefficient.

Proof. Since the dynamical system is ergodic, by the Birkhoff ergodic theorem, we know that there is a set $E \in \mathcal{F}$, with $\mu(E) = 1$ such that, for all $\omega \in E$,

$$(5.4) \quad \frac{1}{a_T^\varepsilon(\cdot, \omega)} \xrightarrow{w} \mathbb{E} \left\{ \frac{1}{a} \right\} := \frac{1}{a^0} \quad \text{in } L^2(\mathcal{D}),$$

as $\varepsilon \rightarrow 0$. Let $\omega \in E$ be fixed but arbitrary and, for $\varepsilon > 0$, consider the problem

$$(5.5) \quad \begin{aligned} -\frac{d}{dx} \left(a_T^\varepsilon(\cdot, \omega) \frac{du^\varepsilon}{dx}(\cdot, \omega) \right) &= f \quad \text{in } \mathcal{D} = (s, t), \\ u^\varepsilon(\cdot, \omega) &= 0 \quad \text{on } \partial\mathcal{D} = \{s, t\}, \end{aligned}$$

with the weak formulation given by

$$(5.6) \quad \int_{\mathcal{D}} a_T^\varepsilon(\cdot, \omega) \frac{du^\varepsilon}{dx} \frac{d\phi}{dx} dx = \int_{\mathcal{D}} f \phi dx, \quad \text{for all } \phi \in H_0^1(\mathcal{D}).$$

We know that, for each $\varepsilon > 0$, (5.6) has a unique solution $u^\varepsilon = u^\varepsilon(\cdot, \omega)$. First, we show that $\{u^\varepsilon(\cdot, \omega)\}_{\varepsilon > 0}$ is bounded in the $H_0^1(\mathcal{D})$

norm. To see this, we begin by letting $\phi = u^\varepsilon$ in (5.6) and note that

$$\begin{aligned} \nu_1 \int_{\mathcal{D}} \left| \frac{du^\varepsilon}{dx} \right|^2 dx &\leq \int_{\mathcal{D}} a_T^\varepsilon \frac{du^\varepsilon}{dx} \frac{du^\varepsilon}{dx} dx = \int_{\mathcal{D}} f u^\varepsilon dx \\ &\leq \|f\|_{L^2(\mathcal{D})} \|u^\varepsilon\|_{L^2(\mathcal{D})} \leq C_p \|f\|_{L^2(\mathcal{D})} \left\| \frac{du^\varepsilon}{dx} \right\|_{L^2(\mathcal{D})}, \end{aligned}$$

where the last two inequalities use Cauchy-Schwarz and Poincaré inequalities respectively. Thus,

$$(5.7) \quad \left\| \frac{du^\varepsilon}{dx} \right\|_{L^2(\mathcal{D})} \leq \frac{C_p}{\nu_1} \|f\|_{L^2(\mathcal{D})}.$$

Moreover, applying the Poincaré inequality again, we have

$$\|u^\varepsilon\|_{L^2(\mathcal{D})} \leq C_p \left\| \frac{du^\varepsilon}{dx} \right\|_{L^2(\mathcal{D})}$$

and, therefore, the sequence $\{u^\varepsilon\}$ is bounded in $L^2(\mathcal{D})$ as well. Thus, we conclude that $\{u^\varepsilon(\cdot, \omega)\}_{\varepsilon>0}$ is bounded in $H_0^1(\mathcal{D})$. Consequently, we have as $\varepsilon \rightarrow 0$, along a subsequence (not relabeled),

$$(5.8) \quad u^\varepsilon(\cdot, \omega) \xrightarrow{w} u^0 \quad \text{in } H_0^1(\mathcal{D}).$$

Moreover, by compact embedding of $H_0^1(\mathcal{D})$ into $L^2(\mathcal{D})$, we have that $u^\varepsilon(\cdot, \omega) \rightarrow u^0$ strongly in $L^2(\mathcal{D})$. Note that, at this point, it is not clear whether u^0 is independent of ω . From (5.8), we immediately get that

$$(5.9) \quad \frac{du^\varepsilon}{dx}(\cdot, \omega) \xrightarrow{w} \frac{du^0}{dx} \quad \text{in } L^2(\mathcal{D}).$$

Next, we let

$$(5.10) \quad \sigma^\varepsilon(x, \omega) = a_T^\varepsilon(x, \omega) \frac{du^\varepsilon}{dx}(x, \omega).$$

Using the fact that $\{a_T^\varepsilon(\cdot, \omega)\}_{\varepsilon>0}$ is bounded in $L^\infty(\mathcal{D})$ and (5.7), we have $\{\sigma^\varepsilon(\cdot, \omega)\}_{\varepsilon>0}$ is bounded in $L^2(\mathcal{D})$. Moreover, we note that

$$\frac{d\sigma^\varepsilon}{dx} = -f$$

and, therefore,

$$\left\{ \frac{d\sigma^\varepsilon}{dx}(\cdot, \omega) \right\}_{\varepsilon>0}$$

is bounded in $L^2(\mathcal{D})$ as well. Therefore, we conclude that $\{\sigma^\varepsilon(\cdot, \omega)\}_{\varepsilon>0}$ is bounded in $H^1(\mathcal{D})$. Thus, $\sigma^\varepsilon(\cdot, \omega) \rightharpoonup \sigma^0(\cdot, \omega)$ in $H^1(\mathcal{D})$ (along a subsequence), and therefore, by compact embedding of $H^1(\mathcal{D})$ into $L^2(\mathcal{D})$ we have, as $\varepsilon \rightarrow 0$,

$$(5.11) \quad \sigma^\varepsilon(\cdot, \omega) \longrightarrow \sigma^0(\cdot, \omega) \quad \text{in } L^2(\mathcal{D}).$$

Next, consider the following obvious equality

$$(5.12) \quad \frac{du^\varepsilon}{dx}(\cdot, \omega) = \frac{a_T^\varepsilon(\cdot, \omega)}{a_T^\varepsilon(\cdot, \omega)} \frac{du^\varepsilon}{dx}(\cdot, \omega) = \sigma^\varepsilon(\cdot, \omega) \frac{1}{a_T^\varepsilon(\cdot, \omega)}.$$

In view of (5.9) and using (5.4) and (5.11), we have, as $\varepsilon \rightarrow 0$.

$$\sigma^\varepsilon(\cdot, \omega) \frac{1}{a_T^\varepsilon(\cdot, \omega)} \rightharpoonup \sigma^0(\cdot, \omega) \frac{1}{a^0} \quad \text{in } L^2(\mathcal{D}),$$

and

$$\frac{du^0}{dx} = \sigma^0(\cdot, \omega) \frac{1}{a^0}.$$

Thus, we have

$$\sigma^0(\cdot, \omega) = a^0 \frac{du^0}{dx},$$

and, recalling the definition of σ^ε in (5.10), we can rewrite (5.11) as follows:

$$(5.13) \quad a_T^\varepsilon(x, \omega) \frac{du^\varepsilon}{dx}(x, \omega) \longrightarrow a^0 \frac{du^0}{dx}, \quad \text{in } L^2(\mathcal{D}).$$

Hence, passing to the limit as $\varepsilon \rightarrow 0$ in (5.6) gives

$$\int_{\mathcal{D}} a^0 \frac{du^0}{dx} \frac{d\phi}{dx} dx = \int_{\mathcal{D}} f \phi dx, \quad \text{for all } \phi \in H_0^1(\mathcal{D}),$$

which says that u^0 is the weak solution to

$$(5.14) \quad \begin{aligned} -\frac{d}{dx} \left(a^0 \frac{du^0}{dx} \right) &= f \quad \text{in } \mathcal{D} = (s, t), \\ u^0 &= 0 \quad \text{on } \partial\mathcal{D} = \{s, t\}. \end{aligned}$$

Note also that, by (5.1), we have that $\nu_1 \leq a^0 \leq \nu_2$. The problem (5.14) has a unique solution u^0 that is independent of ω , because a^0 is a constant independent of ω and the right-hand side function f is deterministic. Also, since the solution u^0 is unique, any subsequence of $u^\varepsilon(\cdot, \omega)$ converges to the same limit u^0 (weakly in $H_0^1(\mathcal{D})$ and thus

strongly in $L^2(\mathcal{D})$), and thus the entire sequence $\{u^\varepsilon(\cdot, \omega)\}_{\varepsilon>0}$ converges to u^0 , not just such a subsequence. Finally, since the domain \mathcal{D} was any arbitrary open interval and the right-hand side function $f \in L^2(\mathcal{D})$ was arbitrary, (5.8), (5.13) and (5.14) lead to the conclusion that $a_T^\varepsilon(\cdot, \omega)$ admits homogenization with homogenized coefficient given by $a^0 = \mathbb{E}\{1/a\}^{-1}$. Note also that this conclusion holds for almost all $\omega \in \Omega$. \square

Remark 5.2. Note that Theorem 5.1 says the effective coefficient a^0 is a constant function on \mathcal{D} with $a^0(x) = \mathbb{E}\{1/a\}^{-1}$ for all $x \in \mathcal{D}$. Also, observe that a^0 is the one-dimensional counterpart of the homogenized coefficient A^0 in (4.4).

6. Stochastic homogenization: The n -dimensional case. Before delving into the theory, we consider a numerical illustration of homogenization in a two-dimensional example. We consider:

$$(6.1) \quad \begin{cases} -\operatorname{div}(A(\varepsilon^{-1}\mathbf{x}, \omega)\nabla u^\varepsilon(\mathbf{x}, \omega)) = f(\mathbf{x}) & \text{in } \mathcal{D} = (0, 1) \times (0, 1), \\ u^\varepsilon(\mathbf{x}, \omega) = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where the source term is given by

$$f(\mathbf{x}) = \frac{C}{2\pi L} \exp\left\{-\frac{1}{2L}[(x_1 - 1/2)^2 + (x_2 - 1/2)^2]\right\},$$

with $C = 5$ and $L = 0.05$.

We describe the diffusive properties of the medium, modeled by the conductivity function $A(\mathbf{x}, \omega)$, by a random tile-based structure similar to the one-dimensional example presented at the beginning of the article. Consider a checkerboard like structure where the conductivity of each tile is a random variable that can take four possible values $\kappa_1, \dots, \kappa_4$, with probabilities $p_i \in (0, 1)$, $\sum_{i=1}^4 p_i = 1$. For the present example, we let $\kappa_1 = 1$, $\kappa_2 = 10$, $\kappa_3 = 50$ and $\kappa_4 = 100$, which can occur with probabilities $p_1 = 0.4$ and $p_2 = p_3 = p_4 = 0.2$, respectively. We depict a realization of the resulting (scalar-valued) random conductivity function $A(\mathbf{x}, \omega)$ in Figure 7 (left) and the solution $u(\mathbf{x}, \omega)$ of the corresponding diffusion problem (6.1) in the right image of the same figure. Note that, in the plot of the random checkerboard, lighter colors correspond to tiles with larger conductivities.

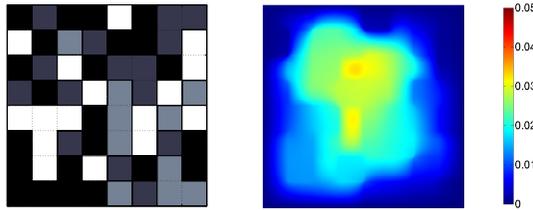


FIGURE 7. Left: a realization of the random checkerboard conductivity function described above; right: the solution $u(\mathbf{x}, \omega)$ corresponding to the realization of $A(\mathbf{x}, \omega)$.

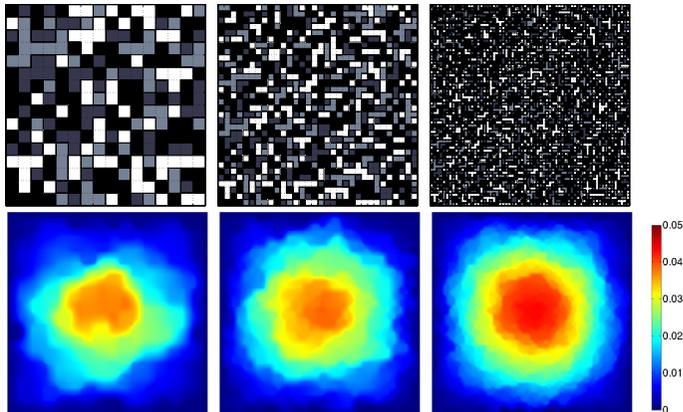


FIGURE 8. Top row: $A(\varepsilon^{-1}\mathbf{x}, \omega)$, for a fixed ω , with $\varepsilon = 1/2, 1/4$, and $1/8$; bottom row, the respective solutions $u^\varepsilon(\mathbf{x}, \omega)$.

For a numerical illustration of homogenization, we compute the solutions of problem (6.1) with successively smaller values of ε . Specifically, using the same realization of the medium shown in Figure 7 (left), we solve the problem (6.1) with $\varepsilon = 1/2, 1/4$, and $1/8$. Results are reported in Figure 8, where we plot the coefficient fields $A(\varepsilon^{-1}\mathbf{x}, \omega)$ (top row) and the corresponding solutions $u^\varepsilon(\mathbf{x}, \omega)$ (bottom row). Note that, as ε gets smaller, the solutions u^ε seem to approach that of a diffusion problem with a constant diffusion coefficient. This is the expected outcome when working with structures that admit homogenization. We mention that these problems were solved numerically using a continuous Galerkin finite-element discretization with a 200×200 mesh of

quadratic quadrilateral elements. COMSOL Multiphysics was used for the finite-element discretization, and computations were performed in Matlab.

Below, we study a homogenization result in \mathbb{R}^n , which shows that, under assumptions of stationarity and ergodicity, a homogenized medium exists. As we shall see shortly, in this general n -dimensional case, unlike the one-dimensional problem, there is no closed-form analytic formula for the homogenized coefficients. (Analytic formulas for the homogenized coefficients are available only in some special cases in two dimensions [31].) Note that, even in the case of periodic structures in several space dimensions, analytic formulas for the homogenized coefficient are not available; however, in the periodic case, the characterization of the effective coefficients suggests a straightforward computational method for computing the homogenized conductivity matrix. This is no longer the case in the stochastic case, where the numerical approximation of homogenized coefficients is generally a difficult problem; see also Remark 6.2 below.

6.1. The homogenization theorem in \mathbb{R}^n . In this section, we present the stochastic homogenization theorem for linear elliptic operators in \mathbb{R}^n . The discussion in this section follows along similar lines as that presented in [31]. Consider the problem

$$(6.2) \quad \begin{cases} -\operatorname{div}(A(\varepsilon^{-1}\mathbf{x}, \omega)\nabla u^\varepsilon(\mathbf{x}, \omega)) = f(\mathbf{x}) & \text{in } \mathcal{D}, \\ u^\varepsilon(\mathbf{x}, \omega) = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

Here \mathcal{D} is a bounded domain in \mathbb{R}^n , $f \in L^2(\mathcal{D})$ is a deterministic source term, and A is a stationary and ergodic random field. That is, we assume that

$$(6.3) \quad A(\mathbf{x}, \omega) = \mathcal{A}(T_{\mathbf{x}}(\omega)), \quad \text{for all } \mathbf{x} \in \mathbb{R}^n, \omega \in \Omega,$$

where $T = \{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^n}$ is an n -dimensional measure preserving and ergodic dynamical system, and \mathcal{A} is a measurable function from Ω to $\mathbb{R}_{\text{sym}}^{n \times n}$ that is uniformly bounded and positive definite. We define the set of all such \mathcal{A} as follows. For positive constants $0 < \nu_1 \leq \nu_2$, let

$$\begin{aligned} \mathcal{E}(\nu_1, \nu_2, \Omega) = \{ & \mathcal{A} : \Omega \rightarrow \mathbb{R}_{\text{sym}}^{n \times n} : \\ & \mathcal{A} \text{ is measurable and } \nu_1|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi} \cdot \mathcal{A}(\omega)\boldsymbol{\xi} \leq \nu_2|\boldsymbol{\xi}|^2 \\ & \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n, \text{ for almost all } \omega \in \Omega\}. \end{aligned}$$

Note that here $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^n , i.e., $|\xi|^2 = \sum_{i=1}^n \xi_i^2$. The following homogenization result (cf. [31, Theorem 7.4]) provides a characterization of the homogenized matrix for stationary and ergodic diffusive media.

Theorem 6.1. *Let $\mathcal{A} : \Omega \rightarrow \mathbb{R}^{n \times n}$ be in $\mathcal{E}(\nu_1, \nu_2, \Omega)$ for some $0 < \nu_1 \leq \nu_2$. Moreover, assume that $T = \{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^n}$ is a measure preserving and ergodic dynamical system. Then, for almost all $\omega \in \Omega$, the realization $\mathcal{A}_T(\cdot, \omega)$ admits homogenization, and the homogenized matrix \mathcal{A}^0 is characterized by*

$$(6.4) \quad \mathcal{A}^0 \xi = \int_{\Omega} \mathcal{A}(\omega)(\xi + \mathbf{v}_{\xi}(\omega)) \mu(d\omega), \quad \text{for all } \xi \in \mathbb{R}^n,$$

where \mathbf{v}_{ξ} is the solution to the following auxiliary problem: Find $\mathbf{v} \in \mathcal{V}_{\text{pot}}^2(\Omega, T)$ (recall the definition of $\mathcal{V}_{\text{pot}}^2(\Omega, T)$ in (3.8)) such that

$$(6.5) \quad \int_{\Omega} \mathcal{A}(\omega)(\xi + \mathbf{v}(\omega)) \cdot \varphi(\omega) \mu(d\omega) = 0, \quad \text{for all } \varphi \in \mathcal{V}_{\text{pot}}^2(\Omega, T).$$

Before presenting the proof of this result, we collect some observations.

Remark 6.2. Note that Theorem 6.1 provides an abstract characterization for \mathcal{A}^0 , which does not lend itself directly to a numerical recipe for computing \mathcal{A}^0 . While the discussion in the present note does not include numerical methods, we point out that numerical approaches for computing \mathcal{A}^0 are available. See, e.g., [11, 35] that describe the method of periodization, which can be used to compute approximations to the homogenized matrix \mathcal{A}^0 .

Remark 6.3. The above homogenization result applies to random diffusive media whose conductivity functions are described by stationary and ergodic random fields. From a practical point of view, such ergodicity assumptions are mathematical niceties that cannot be verified in real-world problems. One possible idea is to construct mathematical definitions of certain “idealized” random structures for which one can prove ergodicity and use such structures as potential modeling tools in real applications. An example of such an effort is done in [2] where,

starting from their physical descriptions, a class of stationary and ergodic tile-based random structures has been constructed. See also the book [43], which provides a comprehensive treatment of means for statistical characterization of random heterogeneous materials.

Remark 6.4. The form of the homogenized coefficient in one space dimension given by Theorem 5.1 can be derived by specializing Theorem 6.1 to the case of $n = 1$. To see this, we note that, in the one-dimensional case, the homogenized coefficient is characterized as follows: For $\xi \in \mathbb{R}$,

$$(6.6) \quad a_0 \xi = \int_{\Omega} a(\omega)(\xi + v_{\xi}(\omega)) \mu(d\omega),$$

where $v_{\xi} \in \mathcal{V}_{\text{pot}}^2(\Omega, T)$ is a solution to auxiliary problem (6.5). Hence, using Weyl's theorem, we may write

$$(6.7) \quad a(\xi + v_{\xi}) \in \mathbf{L}_{\text{sol}}^2(\Omega, T).$$

To find a_0 , we need only to consider $\xi = 1$ in (6.6). Denote

$$(6.8) \quad q(\omega) = a(\omega)(1 + v_1(\omega)),$$

and note that, by (6.7), and recalling the definition of $\mathbf{L}_{\text{sol}}^2(\Omega, T)$, we have that, for almost all ω , $q(T_x(\omega))$ is a constant (depending on ω). That is, for almost all $\omega \in \Omega$, $q(T_x(\omega)) = q(\omega)$, for all $x \in \mathbb{R}$. Therefore, by ergodicity of the dynamical system T , we have $q(\omega) \equiv \text{const} =: \bar{q}$ almost everywhere. Thus, using (6.8), we have $v_1(\omega) = \bar{q}/a(\omega) - 1$, and since $\mathbb{E}\{v_1\} = 0$, we have $\bar{q} = \mathbb{E}\{1/a\}^{-1}$. Then, (6.6) gives

$$a_0 = \int_{\Omega} a(\omega)(1 + v_1(\omega)) \mu(d\omega) = \int_{\Omega} \bar{q} \mu(d\omega) = \bar{q} = \mathbb{E}\{1/a\}^{-1}.$$

Next, we turn to the proof of Theorem 6.1.

Proof. First we note that the characterization of \mathcal{A}^0 in the statement of the theorem along with the properties of \mathcal{A} allows us to, through a standard argument, conclude that \mathcal{A}^0 is a symmetric positive definite matrix (see subsection 6.2 for a proof of this fact). Consider the family of Dirichlet problems

$$\begin{cases} -\operatorname{div}(\mathcal{A}_T^{\varepsilon}(\mathbf{x}, \omega) \nabla u^{\varepsilon}(\mathbf{x}, \omega)) = f(\mathbf{x}) & \text{in } \mathcal{D}, \\ u^{\varepsilon}(\mathbf{x}, \omega) = 0 & \text{on } \partial \mathcal{D}, \end{cases}$$

whose weak formulation is given by

$$(6.9) \quad \int_{\mathcal{D}} \mathcal{A}_T^\varepsilon(\cdot, \omega) \nabla u^\varepsilon(\cdot, \omega) \cdot \nabla \phi \, d\mathbf{x} = \int_{\mathcal{D}} f \phi \, d\mathbf{x}, \quad \text{for all } \phi \in H_0^1(\mathcal{D}).$$

For a fixed ω , we can use arguments similar to those in the one-dimensional case, to show that the family of functions $u^\varepsilon(\cdot, \omega)$ is bounded in $H_0^1(\mathcal{D})$, and the family of functions $\sigma^\varepsilon(\cdot, \omega) = \mathcal{A}_T^\varepsilon(\cdot, \omega) \nabla u^\varepsilon(\cdot, \omega)$ is bounded in $L^2(\mathcal{D})$. Therefore, (along a subsequence) as $\varepsilon \rightarrow 0$,

$$(6.10) \quad u^\varepsilon(\cdot, \omega) \xrightarrow{w} u^0 \quad \text{in } H_0^1(\mathcal{D}),$$

$$(6.11) \quad \sigma^\varepsilon(\cdot, \omega) = \mathcal{A}_T^\varepsilon(\cdot, \omega) \nabla u^\varepsilon(\cdot, \omega) \xrightarrow{w} \sigma^0 \quad \text{in } L^2(\mathcal{D}).$$

Note that (6.10) also implies that $\nabla u^\varepsilon(\cdot, \omega) \xrightarrow{w} \nabla u^0$ in $L^2(\mathcal{D})$. Our goal is to show that $\sigma^0 = \mathcal{A}^0 \nabla u^0$ and that the limit u^0 is the (weak) solution of the problem

$$(6.12) \quad \begin{cases} -\operatorname{div}(\mathcal{A}^0 \nabla u^0) = f & \text{in } \mathcal{D}, \\ u^0 = 0 & \text{on } \partial \mathcal{D}. \end{cases}$$

Let $\xi \in \mathbb{R}^n$ be fixed but arbitrary, and let $\mathbf{p} = \mathbf{p}_\xi$ be given by

$$(6.13) \quad \mathbf{p} = \xi + \mathbf{v}_\xi,$$

where $\mathbf{v}_\xi \in \mathcal{V}_{\text{pot}}^2(\Omega, T)$ solves (6.5). Note that $\mathbf{p} \in \mathbf{L}_{\text{pot}}^2(\Omega, T)$ with $\mathbb{E}\{\mathbf{p}\} = \xi$. Moreover, let $\mathbf{q}(\omega) = \mathcal{A}(\omega)\mathbf{p}(\omega)$, and note that

$$\mathbb{E}\{\mathbf{q}\} = \int_{\Omega} \mathcal{A}(\omega)\mathbf{p}(\omega) \, \mu(d\omega) = \int_{\Omega} \mathcal{A}(\omega)(\xi + \mathbf{v}_\xi(\omega)) \, \mu(d\omega) = \mathcal{A}^0 \xi,$$

where the last equality follows from (6.4). Moreover, let us note that, since \mathbf{v}_ξ satisfies (6.5), invoking Weyl’s decomposition theorem, we have that $\mathbf{q}(\omega) = \mathcal{A}(\omega)(\xi + \mathbf{v}_\xi(\omega))$ belongs to the space $\mathbf{L}_{\text{sol}}^2(\Omega, T)$.

By ergodicity of the dynamical system T , we can invoke the Birkhoff ergodic theorem to conclude that, for almost all $\omega \in \Omega$,

$$\mathbf{p}_T^\varepsilon(\cdot, \omega) \xrightarrow{w} \xi, \quad \text{in } L^2(\mathcal{D}), \quad \text{and} \quad \mathbf{q}_T^\varepsilon(\cdot, \omega) \xrightarrow{w} \mathcal{A}^0 \xi, \quad \text{in } L^2(\mathcal{D}).$$

Next, since $\mathcal{A}(\omega) \in \mathbb{R}_{\text{sym}}^{n \times n}$, we can write,

$$(6.14) \quad \begin{aligned} \sigma^\varepsilon(\mathbf{x}, \omega) \cdot \mathbf{p}_T^\varepsilon(\mathbf{x}, \omega) &= \mathcal{A}_T^\varepsilon(\mathbf{x}, \omega) \nabla u^\varepsilon(\omega, \omega) \cdot \mathbf{p}_T^\varepsilon(\mathbf{x}, \omega) \\ &= \nabla u^\varepsilon(\mathbf{x}, \omega) \cdot \mathbf{q}_T^\varepsilon(\mathbf{x}, \omega). \end{aligned}$$

Let us consider both sides of (6.14). Note that $-\operatorname{div} \sigma^\varepsilon(\cdot, \omega) = f$ and $\operatorname{curl} \mathbf{p}^\varepsilon(\cdot, \omega) = 0$ for every ε ; this along with the weak convergence of $\sigma^\varepsilon(\cdot, \omega)$ and $\mathbf{p}_T^\varepsilon(\cdot, \omega)$ allows us to use Lemma 3.1 to get

$$(6.15) \quad \sigma^\varepsilon(\cdot, \omega) \cdot \mathbf{p}_T^\varepsilon(\cdot, \omega) \xrightarrow{w^*} \sigma^0 \cdot \xi.$$

On the other hand, considering the right-hand side of (6.14), we note that for every ε , we have $\operatorname{curl} \nabla u^\varepsilon = 0$ and (for almost all $\omega \in \Omega$) $\operatorname{div} \mathbf{q}^\varepsilon(\cdot, \omega) = 0$. Therefore, again we use Lemma 3.1 to get

$$(6.16) \quad \nabla u^\varepsilon(\cdot, \omega) \cdot \mathbf{q}_T^\varepsilon(\cdot, \omega) \xrightarrow{w^*} \nabla u^0 \cdot \mathcal{A}^0 \xi.$$

Finally, using (6.14) along with (6.15) and (6.16), we have $\nabla u^0 \cdot \mathcal{A}^0 \xi = \sigma^0 \cdot \xi$. Therefore, by symmetry of \mathcal{A}^0

$$\sigma^0 \cdot \xi = \mathcal{A}^0 \nabla u^0 \cdot \xi,$$

and since ξ was arbitrary we have $\sigma^0 = \mathcal{A}^0 \nabla u^0$. Therefore, recalling the definition of σ^ε and (6.11), we have that

$$\mathcal{A}_T^\varepsilon(\cdot, \omega) \nabla u^\varepsilon(\cdot, \omega) \xrightarrow{w} \mathcal{A}^0 \nabla u^0, \quad \text{in } L^2(\mathcal{D}).$$

Hence, we can pass to limit $\varepsilon \rightarrow 0$ in (6.9) to get

$$\int_{\mathcal{D}} \mathcal{A}^0 \nabla u^0 \cdot \nabla \phi \, dx = \int_{\mathcal{D}} f \phi \, dx, \quad \text{for all } \phi \in H_0^1(\mathcal{D}),$$

which says that u^0 is a weak solution to the problem (6.12). Note also that since \mathcal{A}^0 and f are deterministic, u^0 does not depend on ω . \square

6.2. Variational characterization of the homogenized matrix.

Let the probability space $(\Omega, \mathcal{F}, \mu)$ be as in the previous subsection, and let $\mathcal{A} \in \mathcal{E}(\nu_1, \nu_2, \Omega)$ be as in Theorem 6.1. For an arbitrary $\xi \in \mathbb{R}^n$, we let $\mathcal{J}_\xi : \mathcal{V}_{\text{pot}}^2(\Omega, T) \rightarrow \mathbb{R}$ be the quadratic functional below:

$$(6.17) \quad \mathcal{J}_\xi(\mathbf{v}) = \int_{\Omega} (\xi + \mathbf{v}(\omega)) \cdot \mathcal{A}(\omega) (\xi + \mathbf{v}(\omega)) \, \mu(d\omega),$$

$$\mathbf{v} \in \mathcal{V}_{\text{pot}}^2(\Omega, T).$$

Note that the dynamical system T in definition of $\mathcal{V}_{\text{pot}}^2(\Omega, T)$ here is as in Theorem 6.1. The functional \mathcal{J}_ξ is strictly convex, coercive and bounded from below, and therefore, it has a unique minimizer in $\mathcal{V}_{\text{pot}}^2(\Omega, T)$. The Fréchet derivative of \mathcal{J}_ξ at the minimizer \mathbf{v}_ξ in any

direction φ is zero, that is:

$$(6.18) \quad \int_{\Omega} \mathcal{A}(\omega)(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \cdot \boldsymbol{\varphi}(\omega) \mu(d\omega) = 0,$$

for all $\boldsymbol{\varphi} \in \mathcal{V}_{\text{pot}}^2(\Omega, T)$.

Therefore, in view of Weyl’s decomposition, we have

$$\mathcal{A}(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}) \in \mathbf{L}_{\text{sol}}^2(\Omega, T).$$

It is clear from (6.18) that $\mathbf{v}_{\boldsymbol{\xi}}$ is linear in $\boldsymbol{\xi}$. Hence, the expected value $\mathbb{E} \{ \mathcal{A}(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}) \}$, viewed as a function of $\boldsymbol{\xi}$, is a linear mapping from \mathbb{R}^n to \mathbb{R}^n . Consequently, we define the matrix \mathcal{A}^0 by

$$(6.19) \quad \mathcal{A}^0 \boldsymbol{\xi} = \int_{\Omega} \mathcal{A}(\omega)(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \mu(d\omega), \quad \boldsymbol{\xi} \in \mathbb{R}^n.$$

Notice that \mathcal{A}^0 defined above is the same as the homogenized matrix in Theorem 6.1.

Proposition 6.5. *The homogenized matrix \mathcal{A}^0 satisfies the following:*

- (i) *For every $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{\xi} \cdot \mathcal{A}^0 \boldsymbol{\xi} = \inf_{\mathbf{v} \in \mathcal{V}_{\text{pot}}^2(\Omega, T)} \mathcal{J}_{\boldsymbol{\xi}}(\mathbf{v})$.*
- (ii) *The matrix \mathcal{A}^0 is symmetric and positive definite.*

Proof. Let us note that

$$\begin{aligned} \inf_{\mathbf{v} \in \mathcal{V}_{\text{pot}}^2(\Omega, T)} \mathcal{J}_{\boldsymbol{\xi}}(\mathbf{v}) &= \mathcal{J}_{\boldsymbol{\xi}}(\mathbf{v}_{\boldsymbol{\xi}}) \\ &= \int_{\Omega} (\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \cdot \mathcal{A}(\omega)(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \mu(d\omega) \\ &= \boldsymbol{\xi} \cdot \int_{\Omega} \mathcal{A}(\omega)(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \mu(d\omega) \\ &\quad + \int_{\Omega} \mathbf{v}_{\boldsymbol{\xi}}(\omega) \cdot \mathcal{A}(\omega)(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \mu(d\omega). \end{aligned}$$

Now, the first integral on the right-hand side reduces to $\boldsymbol{\xi} \cdot \mathcal{A}^0 \boldsymbol{\xi}$ due to (6.19), and the second integral vanishes because $\mathbf{v}_{\boldsymbol{\xi}}$ and $\mathcal{A}(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}})$ are orthogonal in $\mathbf{L}^2(\Omega)$.

To show \mathcal{A}^0 is symmetric, we proceed as follows. Let \mathbf{e}_i and \mathbf{e}_j be i th and j th standard basis vectors in \mathbb{R}^n , and let \mathbf{v}_i and \mathbf{v}_j be minimizers in $\mathcal{V}_{\text{pot}}^2(\Omega, T)$ of $\mathcal{J}_{\mathbf{e}_i}$ and $\mathcal{J}_{\mathbf{e}_j}$, respectively. It is straightforward to see

$\mathbf{e}_i \cdot \mathcal{A}^0 \mathbf{e}_j = \int_{\Omega} (\mathbf{e}_i + \mathbf{v}_i) \cdot \mathcal{A}(\mathbf{e}_j + \mathbf{v}_j) d\mu$. Thus, symmetry of \mathcal{A}^0 follows from symmetry of \mathcal{A} . As for positive definiteness, we note

$$\begin{aligned} \boldsymbol{\xi} \cdot \mathcal{A}^0 \boldsymbol{\xi} &= \int_{\Omega} (\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \cdot \mathcal{A}(\omega)(\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \mu(d\omega) \\ &\geq \nu_1 \int_{\Omega} |\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)|^2 \mu(d\omega) \\ &\geq \nu_1 \left| \int_{\Omega} (\boldsymbol{\xi} + \mathbf{v}_{\boldsymbol{\xi}}(\omega)) \mu(d\omega) \right|^2 = \nu_1 |\boldsymbol{\xi}|^2. \quad \square \end{aligned}$$

7. Epilogue. In this article, we took a brief tour of stochastic homogenization by studying homogenization of linear elliptic PDEs of divergence form with stationary and ergodic coefficient functions. The goal of our discussion was to provide an accessible entry into a very rich theory that is elaborated in detail in books such as [15, 31], which we refer to for in-depth coverage of various aspects of stochastic homogenization. Also, we mention again the book [42] by Tartar, on the general theory of homogenization, that is an excellent resource for mathematicians working in the area as well as those who are entering the field. We end our discussion by giving some pointers for further reading.

Our discussion focused on homogenization of linear elliptic PDEs with random coefficients. The homogenization of *nonlinear* PDEs involves many additional difficulties both in theory as well as in numerical computations. We refer to the book [36] as well as the articles [13, 14, 21, 22] for stochastic homogenization theory for nonlinear problems. See also [23, 24], which concern numerical methods for stochastic homogenization of nonlinear PDEs.

Stochastic homogenization continues to be an active area of research. Recent developments in the area include the works [9, 18, 25, 27, 28, 29]. We also point to the survey article [26], which provides a review of the state-of-the-art of numerical methods for homogenization of linear elliptic equations with random coefficients. Recent work in homogenization of random nonlinear PDEs includes the articles [4, 5]. See also [3, 6], which concern stochastic homogenization of Hamilton-Jacobi equations.

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REFERENCES

1. A. Alexanderian, *Random composite media: Homogenization, modeling, simulation, and material symmetry*, Ph.D. thesis, University of Maryland, Baltimore County, 2010.
2. A. Alexanderian, M. Rathinam and R. Rostamian, *Homogenization, symmetry, and periodization in diffusive random media*, Acta Math. Sci. **32** (2012), 129–154.
3. S.N. Armstrong, P. Cardaliaguet and P.E. Souganidis, *Error estimates and convergence rates for the stochastic homogenization of Hamilton-Jacobi equations*, J. Amer. Math. Soc. **27** (2014), 479–540.
4. S.N. Armstrong and C.K. Smart, *Stochastic homogenization of fully nonlinear uniformly elliptic equations revisited*, Calc. Var. Part. Diff. Equat. (2012), 1–14.
5. ———, *Regularity and stochastic homogenization of fully nonlinear equations without uniform ellipticity*, Ann. Prob. **42** (2014), 2558–2594.
6. S.N. Armstrong and P.E. Souganidis, *Stochastic homogenization of level-set convex Hamilton-Jacobi equations*, Int. Math. Res. Not. IMRN **15** (2013), 3420–3449.
7. V.I. Arnold and A. Avez, *Ergodic problems of classical mechanics*, W.A. Benjamin, Inc., New York, 1968.
8. A. Bensoussan, J.-L. Lions and G.C. Papanicolaou, *Asymptotic analysis for periodic structures*, Stud. Math. Appl. **5**, North-Holland Publishing Co., Amsterdam, 1978.
9. X. Blanc, R. Costaouec, C. Le Bris and F. Legoll, *Variance reduction in stochastic homogenization using antithetic variables*, Markov Proc. Rel. Fields **18** (2012), 31–66.
10. X. Blanc, C. Le Bris and P.-L. Lions, *Stochastic homogenization and random lattices*, J. Math. Pure Appl. **88** (2007), 34–63.
11. A. Bourgeat and A. Piatnitski, *Approximations of effective coefficients in stochastic homogenization*, Ann. Inst. H. Poin. Prob. Stat. **40** (2004), 153–165.
12. M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002.
13. L.A. Caffarelli and P.E. Souganidis, *Rates of convergence for the homogenization of fully nonlinear uniformly elliptic pde in random media*, Invent. Math. **180** (2010), 301–360.
14. L.A. Caffarelli, P.E. Souganidis and L. Wang, *Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media*, Comm. Pure Appl. Math. **58** (2005), 319–361.

15. G.A. Chechkin, A.L. Piatnitski and A.S. Shamaev, *Homogenization: Methods and applications*, Trans. Math. Mono. **2345**, American Mathematical Society, Providence, RI, 2007.
16. G.H. Choe, *Computational ergodic theory*, Alg. Comp. Math. **13**, Springer-Verlag, Berlin, 2005.
17. D. Cioranescu and P. Donato, *An introduction to homogenization*, Oxford Lect. Ser. Math. Appl. **17**, The Clarendon Press Oxford University Press, New York, 1999.
18. J. Conlon and T. Spencer, *Strong convergence to the homogenized limit of elliptic equations with random coefficients*, Trans. Amer. Math. Soc. **366** (2014), 1257–1288.
19. I.P. Cornfeld, S.V. Fomin and Y.G. Sinai, *Ergodic theory*, Grundle Math. Wissen. **245**, Springer-Verlag, New York, 1982.
20. B. Dacorogna, *Direct methods in the calculus of variations*, Appl. Math. Sci. **78**, Springer, New York, 1989.
21. G. Dal Maso and L. Modica, *Nonlinear stochastic homogenization*, Ann. Mat. Pura Appl. **144** (1986), 347–389.
22. ———, *Nonlinear stochastic homogenization and ergodic theory*, J. reine angew. Math. **368** (1986), 28–42.
23. Y. Efendiev and A. Pankov, *Numerical homogenization and correctors for nonlinear elliptic equations*, SIAM J. Appl. Math. **65** (2004), 43–68.
24. ———, *Numerical homogenization of nonlinear random parabolic operators*, Multiscale Model. Sim. **2** (2004), 237–268.
25. A. Gloria, *Numerical approximation of effective coefficients in stochastic homogenization of discrete elliptic equations*, ESAIM: Math. Model. Numer. Anal. **46** (2012), 1–38.
26. ———, *Numerical homogenization: survey, new results, and perspectives*, in ESAIM: Proceedings **37** (2012), 50–116.
27. A. Gloria, S. Neukamm and F. Otto, *Quantification of ergodicity in stochastic homogenization: Optimal bounds via spectral gap on glauher dynamics*, Invent. Math. (2013), 1–61.
28. ———, *A quantitative two-scale expansion in stochastic homogenization of discrete linear elliptic equations*, Model. Math. Anal. Numer., 2013.
29. A. Gloria and F. Otto, *An optimal variance estimate in stochastic homogenization of discrete elliptic equations*, Ann. Prob. **39** (2011), 779–856.
30. U. Hornung, *Homogenization and porous media*, Volume 6, Springer, 1997.
31. V.V. Jikov, S.M. Kozlov and O.A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994 (in English).
32. S.M. Kozlov, *The averaging of random operators*, Mat. Sb. (N.S.), **109** (1979), 188–202, 327.
33. François Murat, *Compacité par compensation*, Ann. Scuol. Norm. Sup. Pisa **5** (1978), 489–507.

- 34.** O.A. Oleĭnik, A.S. Shamaev and G.A. Yosifian, *Mathematical problems in elasticity and homogenization*, Stud. Math. Appl. **26**, North-Holland Publishing Co., Amsterdam, 1992.
- 35.** H. Owhadi, *Approximation of the effective conductivity of ergodic media by periodization*, Prob. Theor. Rel. Fields **125** (2003), 225–258.
- 36.** A. Pankov, *G-convergence and homogenization of nonlinear partial differential operators*, Math. Appl. **422**, Kluwer Academic Publishers, Dordrecht, 1997.
- 37.** G.C. Papanicolaou, *Diffusion in random media*, in Surveys Appl. Math. **1**, (1995), 205–253.
- 38.** G.C. Papanicolaou and S.R.S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, in *Random fields*, Volumes I, II, Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam, 1981.
- 39.** K. Sab, *On the homogenization and the simulation of random materials*, Europ. J. Mech. Solids **11** (1992), 585–607.
- 40.** E. Sánchez-Palencia, *Nonhomogeneous media and vibration theory*, Lect. Notes Phys. **127**, Springer-Verlag, Berlin, 1980.
- 41.** L. Tartar, *Compensated compactness and applications to partial differential equations*, in *Nonlinear analysis and mechanics* **4** (1979), 136–211.
- 42.** ———, *The general theory of homogenization. A personalized introduction*, Springer, Berlin, 2009.
- 43.** S. Torquato, *Random heterogeneous materials*, Interdiscipl. Appl. Math. **16**, Springer-Verlag, New York, 2002.
- 44.** P. Walters, *An introduction to ergodic theory*, Grad. Texts Math. **79**, Springer-Verlag, New York, 1982.
- 45.** V. Yurinskii, *Averaging of symmetric diffusion in random medium*, Siber. Math. J. **27** (1986), 603–613.

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