

THE COMPLEX OF NON-SEPARATING EMBEDDED SPHERES

SUHAS PANDIT

ABSTRACT. For $n \geq 3$, we shall show that the group $\text{Aut}(NS(M))$ of simplicial automorphisms of the complex $NS(M)$ of non-separating embedded spheres in the manifold $M = \sharp_n S^2 \times S^1$ is isomorphic to the group $\text{Out}(\mathbf{F}_n)$ of outer automorphisms of the free group \mathbf{F}_n , where \mathbf{F}_n is identified with the fundamental group of the manifold M up to conjugacy of the base point in M .

1. Introduction. Let \mathbf{F}_n be a free group of rank n . Let $\text{Aut}(\mathbf{F}_n)$ and $\text{Out}(\mathbf{F}_n)$ be the groups of automorphisms and outer automorphisms of \mathbf{F}_n . The group $\text{Out}(\mathbf{F}_n)$ is the quotient of $\text{Aut}(\mathbf{F}_n)$ by the subgroup of inner automorphisms of \mathbf{F}_n . These groups are important objects in geometric group theory and combinatorial group theory. They have been studied extensively in analogy with the fundamental groups of surfaces and the mapping class groups of surfaces, for instance, see [2, 3, 5, 6], where these groups have been studied via a one-dimensional model which arises from regarding \mathbf{F}_n as the fundamental group of a graph. Recall that the mapping class group of a surface is the group of isotopy classes of homeomorphisms of the surface. In order to study these groups, people are using topological and geometric objects on which these groups act.

Culler and Vogtmann [11] introduced a space CV_n called the *outer space* of a free group of rank n on which the group $\text{Out}(\mathbf{F}_n)$ acts with finite point stabilizers. The outer space with the action of $\text{Out}(\mathbf{F}_n)$ can be thought of as a free group analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface. Culler and Morgan have constructed a compactification of outer space much like Thurston's compactification of the Teichmüller space of a surface, see [10].

2010 AMS *Mathematics subject classification*. Primary 57M07, Secondary 57M05, 20E05.

Received by the editors on May 12, 2012.

DOI:10.1216/RMJ-2014-44-6-2029

Copyright ©2014 Rocky Mountain Mathematics Consortium

In the case of surfaces, to study mapping class groups of the surfaces and their Teichmüller spaces, we have objects like the curve complex associated to a surface, the complex of non-separating curves, the complex of pant decompositions. Recall that for a closed hyperbolic surface F , the curve complex $\mathbf{C}(F)$ is the simplicial complex with vertices free homotopy classes of essential simple closed curves on F . A finite collection of isotopy classes of simple closed curves in F is deemed to span a simplex if it can be realized disjointly (up to isotopy) in F . There is a natural simplicial action of the mapping class group $\mathcal{M}\text{ap}(F)$ on the curve complex. This yields a homomorphism

$$\mathcal{M}\text{ap}(F) \longrightarrow \text{Aut}(\mathbf{C}(F)),$$

where $\text{Aut}(\mathbf{C}(F))$ is the group of simplicial automorphisms of the curve complex $\mathbf{C}(F)$. Ivanov proved that the group $\text{Aut}(\mathbf{C}(F))$ is isomorphic to the mapping class group $\mathcal{M}\text{ap}(F)$ for connected orientable surfaces of genus at least 2 in [21]. Ivanov's results were extended to lower genus cases by Korkmaz in [23]. Luo gave a new proof that the automorphisms of the curve complex are induced by homeomorphisms of the surface if the dimension of the curve complex is at least one and the surface is not a torus with two holes in [27]. After Ivanov's work, the mapping class group was viewed as the automorphism group of various geometric objects on surfaces, for instance, the complex of pants decompositions in [29], the complex of non-separating curves in [19], the complex of separating curves in [7] and the arc and curve complex in [24].

In the free group context, people have considered complexes like the splitting complex, the complex of free factors of \mathbf{F}_n ([17]) and the complex of separating embedded spheres ([35]). In [14], Handel and Mosher have proved that the splitting complex of a free group of finite rank with its geodesic simplicial metric is Gromov hyperbolic. In [4], Bestvina and Feighn have proved that the complex of free factors of a free group of finite rank is hyperbolic. The 3-manifold $M = \natural_n S^2 \times S^1$ has the fundamental group a free group of rank n . The manifold M is used as a model for studying the free group and its automorphisms, in particular, the group of outer automorphisms of free groups. It was first used by Whitehead which involves looking at embedded spheres in M ([34]). An embedded sphere in M corresponds to a free splitting of the fundamental group $G = \pi_1(M)$. The *sphere*

complex $\mathbf{S}(M)$ (see Definition 2.3) associated to M is equivalent to the splitting complex associated to \mathbf{F}_n , where we identify \mathbf{F}_n with $\pi_1(M)$. There is a natural action of the mapping class group $\text{Map}(M)$ of M on the sphere complex which yields an action of $\text{Out}(\mathbf{F}_n)$ on the sphere complex $\mathbf{S}(M)$. This gives a homomorphism $\text{Out}(\mathbf{F}_n) \rightarrow \text{Aut}(\mathbf{S}(M))$, where $\text{Aut}(\mathbf{S}(M))$ is the group of simplicial automorphisms of the sphere complex $\mathbf{S}(M)$. In [1], it is proved that the group $\text{Aut}(\mathbf{S}(M))$ is isomorphic to the group $\text{Out}(\mathbf{F}_n)$ for $n \geq 3$.

In this paper, we consider the complex $NS(M)$ of non-separating embedded spheres in M (see Definition 5.1) on which the group $\text{Out}(\mathbf{F}_n)$ acts. An embedded sphere S in M is said to be *non-separating* if the complement $M - S$ of S in M is connected. We shall prove that the group $\text{Aut}(NS(M))$ of simplicial automorphisms of the complex $NS(M)$ is isomorphic to $\text{Out}(\mathbf{F}_n)$, for $n \geq 3$ (see Theorem 6.1). It is due to Hatcher that the sphere complex contains an embedded copy of the spine K_n of the reduced outer space cv_n . The complex $NS(M)$ also contains an embedded copy of the spine K_n of the reduced outer space cv_n . As in [1], we shall show that the spine K_n is invariant under the group $\text{Aut}(NS(M))$ (Lemma 6.7) by showing that, given any simplicial automorphism ϕ of $NS(M)$, it maps simplices in $NS(M)$ corresponding to reduced sphere systems in M to simplices in $NS(M)$ corresponding to reduced sphere systems in M (see Lemma 6.2). The idea is to look at the links of such simplices in $NS(M)$. We shall observe that the link of a simplex in $NS(M)$ corresponding to a reduced sphere system in M is spanned by finitely many vertices of $NS(M)$, and the link of an $(n - 1)$ -simplex which corresponds to a sphere system which is not reduced in M is spanned by infinitely many vertices of $NS(M)$. We call a simplex in $NS(M)$ which corresponds to a simple sphere system in M a *simple simplex*. Using Lemma 6.2, we see that every simplicial automorphism ϕ of $NS(M)$ maps a simple simplex in $NS(M)$ to a simple simplex in $NS(M)$ (Lemma 6.5). From this, Lemma 6.7 follows and then the rest of the proof follows as in [1].

Now, we outline the paper. In Section 2, we discuss the model three-manifold M and some definitions. In Section 3, we show that embedded spheres in M correspond to splittings of $G = \pi_1(M)$. This result is well known in the field (see [18]), but here we give another proof of this in M for the sake of completeness. In Section 4, we show that, given a graph of groups decomposition \mathcal{G} of the fundamental group G of M ,

there exist a system of 2-spheres in M which gives the same graph of groups decomposition \mathcal{G} of G . In Section 5, we discuss complex $NS(M)$ of non-separating spheres and some facts about it. In Section 6, we give a proof of Theorem 6.1.

2. Preliminaries.

2.1. The model 3-manifold. Consider the 3-manifold $M = \#_n S^2 \times S^1$, i.e., the connected sum of n copies of $S^2 \times S^1$. A description of M can be given as follows. Consider the 3-sphere S^3 , and let A_i, B_i , $1 \leq i \leq n$, be a collection of $2n$ disjointly embedded 3-balls in S^3 . Let S_i (respectively, T_i) denote the boundary of A_i (respectively, B_i). Let P be the complement of the union of the interiors of all the balls A_i and B_i . Then, M is obtained from P by gluing together S_i and T_i with an orientation reversing diffeomorphism φ_i for each i , $1 \leq i \leq n$. The image of S_i (hence T_i) in M will be denoted Σ_i^0 . The fundamental group $\pi_1(M)$ of M is a free group of rank n . We identify the free group F_n of rank n with $\pi_1(M)$ up to conjugacy of the base point.

Definition 2.1. A smooth embedded 2-sphere S in M is said to be *essential* if it does not bound a 3-ball in M .

We shall always consider essential smooth embedded 2-spheres in M throughout this paper. So by an embedded sphere, we mean an essential embedded 2-sphere in M .

Definition 2.2. Two embedded 2-spheres S and S' in M are *parallel* if they are isotopic, i.e., they bound a manifold of the form $S^2 \times (0, 1)$.

By Laudenbach's work ([25]), the embedded spheres S and S' in M are parallel if and only if they are homotopic to each other.

Definition 2.3. A *system of 2-spheres* in M is defined to be a finite collection of disjointly embedded essential 2-spheres $S_i \subset M$ such that no two spheres in this collection are parallel.

A system of spheres in M is *maximal* if it is not properly contained in another system of spheres in M .

Definition 2.4. A *reduced sphere system* is a sphere system in M with connected, simply-connected complement, i.e., it cuts the manifold M into a single simply-connected component.

Note that a reduced sphere system in $M = \sharp_n S^2 \times S^1$ contains exactly n non-separating sphere components, and it defines a basis of $H_2(M, \mathbf{Z})$. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ be a reduced sphere system in M . We choose a transverse orientation for each sphere Σ_i , so we may speak of a positive and a negative side of Σ_i . Let $p \in M$ be a base point in the complement of Σ . A basis dual to Σ is a set of homotopy classes of loops $\gamma_1, \dots, \gamma_n$ in M based at p such that the loop γ_i is disjoint from Σ_j for all $j \neq i$ and intersects Σ_i in a single point. We orient γ_i such that it approaches Σ_i from the positive side. Since the complement of Σ is simply connected, the homotopy classes of loops γ_i define a basis of $\pi_1(M, p)$. Thus, a reduced sphere system gives a basis of the fundamental group of M . Note that the sphere system Σ_i^0 described in subsection 2.1 is a reduced sphere system. By splitting M along a reduced sphere system $\Sigma = \cup_i \Sigma_i$, we get a manifold M_S which is a $2n$ -punctured S^3 , i.e., S^3 with interiors of $2n$ disjointly embedded 3-balls removed. Corresponding to each sphere Σ_i , we get two boundary spheres Σ_i^+ and Σ_i^- in M_S .

2.2. The mapping class group of M and $\text{Out}(\mathbf{F}_n)$.

Definition 2.5. The *mapping class group* $\text{Map}(M)$ of M is the group of isotopy classes of orientation-preserving diffeomorphisms of M .

It is due to Laudenbach that we can replace diffeomorphisms by homeomorphisms in the definition of the mapping class group of M (see [25]). As we have identified \mathbf{F}_n with $\pi_1(M)$, we have a natural homomorphism

$$\Phi : \mathcal{M}\text{ap}(M) \longrightarrow \text{Out}(\mathbf{F}_n),$$

where $\text{Out}(\mathbf{F}_n)$ is the group of outer automorphisms of \mathbf{F}_n . By Laudenbach's work ([26]), the above homomorphism Φ is surjective and has finite kernel generated by Dehn-twists along essential embedded 2-spheres in M .

2.3. Sphere complex. We shall recall the following definition:

Definition 2.6. The *sphere complex* $\mathbf{S}(M)$ associated to M is the simplicial complex whose vertices are isotopy classes of essential embedded spheres in M . A finite collection of isotopy classes of embedded spheres in M is deemed to span a simplex in $\mathbf{S}(M)$ if they can be realized disjointly (up to isotopy) in M .

The mapping class group $\mathcal{M}\text{ap}(M)$ of M acts simplicially on the sphere complex $\mathbf{S}(M)$. This yields a homomorphism

$$\Phi' : \mathcal{M}\text{ap}(M) \longrightarrow \text{Aut}(\mathbf{S}(M)),$$

where $\text{Aut}(\mathbf{S}(M))$ is the group of simplicial automorphisms of the sphere complex $\mathbf{S}(M)$. It follows from the work of Laudenbach that kernels of Φ and Φ' are equal. In particular, the action of $\mathcal{M}\text{ap}(M)$ on the sphere complex $\mathbf{S}(M)$ induces a simplicial action of $\text{Out}(\mathbf{F}_n)$ on the sphere complex. This yields a homomorphism

$$\Phi'' : \text{Out}(\mathbf{F}_n) \longrightarrow \text{Aut}(\mathbf{S}(M)).$$

In [1], Aramayona and Souto have shown that this homomorphism is an isomorphism.

3. Embedded spheres in M . Group theoretically, embedded spheres in M correspond to splittings of the fundamental group of M . We recall the following definitions.

Let G_1 and G_2 be two groups with presentations $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Let $f_1 : H \rightarrow G_1$ and $f_2 : H \rightarrow G_2$ be two homomorphisms from group H .

Definition 3.1. The *free product* $G_1 * G_2$ of G_1 and G_2 is the group with a presentation $\langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \rangle$, where \sqcup denotes disjoint union.

Definition 3.2. The *amalgamated free product* $G_1 *_H G_2$ is defined as $(G_1 * G_2)/N$, where N is the normal subgroup of $G_1 * G_2$ generated by elements of the form $f_1(h)(f_2(h))^{-1}$ for $h \in H$.

Definition 3.3. Let G be a group with presentation $G = \langle S \mid R \rangle$, and let α be an isomorphism between two subgroups H and K of G . Let t be a new symbol not in S , and define

$$G^*_\alpha = \langle S, t \mid R, tht^{-1} = \alpha(h), \text{ for all } h \in H \rangle.$$

The group G^*_α is called the HNN-extension of G relative to α . The original group G is called the base group for the construction, and the subgroups H and K are the associated subgroups. The new generator t is called the stable letter. Sometimes, we also write G^*_H for G^*_α .

Definition 3.4. A *splitting of a group* G consists either of proper subgroups A and B of G and a subgroup H of $A \cap B$ with $A \neq H \neq B$ such that the natural map $A *_H B \rightarrow G$ is an isomorphism or it consists of a subgroup A of G and subgroups H_0 and H_1 of A such that there is an element t of G which conjugates H_0 to H_1 and the natural map $A *_H \rightarrow G$ is an isomorphism.

Note that the condition that $A \neq H \neq B$ is needed as otherwise any group G would split over any subgroup H . For, one can always write $G = G *_H H$.

We shall see how an embedded sphere in M corresponds to a splitting of $G = \pi_1(M) = \mathbf{F}_n$.

We shall recall the description of M given in the beginning of Section 2. Fix a base point x_0 away from Σ^0_i . For each $1 \leq i \leq n$, consider the element $\alpha_i \in \pi_1(M)$ represented by a closed path γ_i in M starting from x_0 , going to A_i , piercing Σ^0_i and returning to the base point from B_i . We choose this closed path γ_i such that it does not intersect any Σ^0_j , $j \neq i$. Then, as the complement of Σ^0_i in M is simply-connected, the collection $\{\alpha_1, \dots, \alpha_n\}$ forms a free basis of $G = \pi_1(M)$. So, we have $G = \langle \alpha_1, \dots, \alpha_n \rangle$. Any directed closed path in M hitting the Σ^0_i transversely represents a word in $\{\alpha_1, \dots, \alpha_n\}$ by the way it pierces each Σ^0_i and the order in which it does so. Without a chosen base point, such a closed path represents a conjugacy class or equivalently the cyclic word in \mathbf{F}_n . We call the basis $\{\alpha_1, \dots, \alpha_n\}$ a standard basis and the spheres $\Sigma^0_1, \dots, \Sigma^0_n$ standard basic spheres in M .

Proposition 3.5. *Let S be an essential embedded sphere in M . Then, S gives a splitting of the fundamental group of M .*

Proof. If S separates M , then using Van Kampen's theorem, we get a splitting of the fundamental group G of M as a free product of its subgroups.

Now, suppose S is non-separating. Choose a regular neighborhood $V = S^2 \times [-1, 1]$ of S and an embedded path γ in $M - V$ from a point of $S^2 \times -1$ to a point $S^2 \times 1$. The sphere S' which is the connected sum of $S^2 \times -1$ with $S^2 \times 1$ along with the boundary of a regular neighborhood U of γ , clearly bounds $U \cup V$ in M . We have $M = U \cup V \cup (M - (U \cup V))$. The set $U \cup V$ is a one punctured $(S^2 \times S^1)$ with boundary S' and $(M - (U \cup V))$ is a one punctured 3-manifold M' with boundary S' . Thus, M is a connected sum of $S^2 \times S^1$ with the 3-manifold M' . Then, we get a splitting of $G = G' * \langle t \rangle$, where $\pi_1(M') = G'$ and $\pi_1(S^2 \times S^1) = \langle t \rangle$. Thus, G can be viewed as an HNN-extension of G' over the trivial subgroup $\{1\}$ of G' . \square

Note that, when we apply Van Kampen's theorem, we choose a base point y_0 on the sphere S . If we choose a path γ from x_0 to y_0 in M , then we get a splitting, say L , of G which consists of subgroups of G . Choosing another path γ' from x_0 to y_0 , we get a splitting of G conjugate to L . So, the sphere S gives a splitting of G up to conjugacy.

Now, we shall see the converse.

Proposition 3.6. *Given a splitting of G , there exists an embedded sphere S in M which gives that splitting of G .*

Proof. Suppose $G = F_1 * F_2$. As subgroups of free groups are free, both F_1 and F_2 are free. Choose free bases $\{a_1, \dots, a_m\}$ and $\{b_{m+1}, \dots, b_{m+k}\}$ of F_1 and F_2 , respectively. The set $\{a_1, \dots, a_m, b_{m+1}, \dots, b_{m+k}\}$ forms a free basis for G . Therefore, $m + k = n$. Any two bases of a free group are equivalent in the sense that there exists an automorphism of that free group sending one basis to another. So, we have an automorphism ϕ of G sending the basis $\{a_1, \dots, a_m, b_{m+1}, \dots, b_n\}$ to the standard basis with $\phi(a_i) = \alpha_i$, for $1 \leq i \leq m$ and $\phi(b_{m+j}) = \alpha_{m+j}$, for $1 \leq j \leq k$.

Every automorphism of a free group is finite composition of Nielsen automorphisms, and every Nielsen automorphism of G is induced by a homeomorphism of M which fixes the base point x_0 (see [28], [32], [34]). Thus, every automorphism of G is induced by a homeomorphism of M fixing the base point. Let h be a homeomorphism of M which fixes the base point and induces the automorphism ϕ on G . The element $\phi(a_i) = h_*(a_i) = \alpha_i$, for $1 \leq i \leq m$, corresponds to the basic standard sphere Σ_i^0 and $\phi(b_{m+j}) = h_*(b_{m+j}) = \alpha_{m+j}$, for $1 \leq j \leq k$ corresponds to the basic standard sphere Σ_{m+j}^0 . We can choose an embedded sphere S , disjoint from all Σ_i^0 such that it partitions the collection of basic standard spheres into two sets, namely, $\{\Sigma_i^0, \dots, \Sigma_m^0\}$ and $\{\Sigma_{m+1}^0, \dots, \Sigma_{m+k}^0\}$ and $x_0 \in S$. Then, S gives a free splitting of $G = A * B$, where $A = \langle \alpha_1, \dots, \alpha_m \rangle$ and $B = \langle \alpha_{m+1}, \dots, \alpha_{m+k} \rangle$. Now, the sphere $h^{-1}(S) = S'$ gives partition of the collection of spheres $\{h^{-1}(\Sigma_1^0), \dots, h^{-1}(\Sigma_{m+k}^0)\}$ into two sets $\{h^{-1}(\Sigma_i^0), \dots, h^{-1}(\Sigma_m^0)\}$ and $\{h^{-1}(\Sigma_{m+1}^0), \dots, h^{-1}(\Sigma_{m+k}^0)\}$. The sphere structure $\{h^{-1}(\Sigma_1^0), \dots, h^{-1}(\Sigma_{m+k}^0)\}$ corresponds to the basis $\{a_1, \dots, a_m, b_{m+1}, \dots, b_{m+k}\}$. Then, by applying Van-Kampen's theorem, we can see that S' gives the splitting $G = F_1 * F_2$ as $h(x_0) = x_0 \in S'$.

If we choose a sphere S such that the base point x_0 does not lie in S , then we choose a path γ from x_0 to a point y_0 in S such that γ does not intersect any standard basic sphere. We choose curves α'_i starting from y_0 , going to A_i , piercing Σ_i^0 , and returning to the base point from B_i without hitting any Σ_j^0 , $j \neq i$. Note the homotopy class of closed curve the $\gamma * \alpha'_i * \bar{\gamma}$ gives the generator α_i . The homotopy class of closed curve $h^{-1}(\gamma * \alpha'_i * \bar{\gamma})$ is a_i . From this, we can see that the sphere S' gives the splitting $G = F_1 * F_2$.

Note that if G is an HNN-extension of a subgroup G' of G relative to the subgroups H, K of G' and an isomorphism $\theta : H \rightarrow K$, then $H = K = \{1\}$ and G' is a subgroup of rank $n - 1$ as G is a free group of rank n . Thus, $G = G' * \langle t \rangle$, where $t \in G$. We choose a basis $\{c_1, \dots, c_{n-1}\}$ of G' . The set $\{c_1, \dots, c_{n-1}, t\}$ forms a basis of G . Then, we have an isomorphism ϕ' of G sending the basis $\{c_1, \dots, c_{n-1}, t\}$ to the standard basis with $\phi'(c_i) = \alpha_i$, for $1 \leq i \leq n - 1$ and $\phi'(t) = \alpha_n$. Let h' be a homeomorphism of M which fixes the base point and induces the automorphism ϕ' on G . The element $\phi'(c_i) = h'_*(c_i) = \alpha_i$, for $1 \leq i \leq n - 1$, corresponds to the basic standard sphere Σ_i^0

and $\phi'(t) = h'_*(t) = \alpha_n$ corresponds to the basic sphere Σ_n^0 . The sphere structure $\{h'^{-1}(\Sigma_1^0), \dots, h'^{-1}(\Sigma_{n-1}^0), h'^{-1}(\Sigma_n^0)\}$ corresponds to the basis $\{c_1, \dots, c_{n-1}, t\}$ of G . Now, one can easily see that the sphere $h'^{-1}(\Sigma_n^0)$ gives a splitting of G as an HNN-extension of G' over the trivial subgroup $\{1\}$. \square

3.1. Isotopy classes of embedded spheres and conjugacy classes of corresponding splittings of G . Now, we shall see that the isotopy classes of embedded spheres in M correspond to the conjugacy classes of splittings of G .

We need the following: For a set $E \subset G$, we denote the complement of E by E^* and by $E^{(*)}$, we mean one of the sets E and E^* .

Definition 3.7. Two subsets E and E' of the group G are said to be *almost equal* if their symmetric difference is finite.

A set E is said to be *non-trivial* if both E and E^* are infinite.

Definition 3.8. A set $E \subset G$ is said to be *almost invariant* if E is almost equal to Eg for all $g \in G$.

An equivalent condition in terms of the Cayley graph $\Gamma(G)$ is that the set δE of edges of $\Gamma(G)$ with one vertex in E and the other in E^* is finite. This is due to Cohen, [9].

Definition 3.9. Two almost invariant sets E and E' are said to be *small* if $E \cap E'$ is finite.

A description of \mathcal{M} related to the description of M given in the beginning of Section 2 is as follows. Let $\Gamma(G)$ be the Cayley graph of G with respect to the standard basis $\{\alpha_1, \dots, \alpha_n\}$. Then, $\Gamma(G)$ is a tree with vertices G . The space \widetilde{M} is obtained by taking a copy gP of P for each vertex g of $\Gamma(G)$ and identifying gS_i with $g\alpha_i T_i$ using φ_i , for each $g \in G$ and for each generator $\alpha_i, 1 \leq i \leq n$. We observe that, by the Hurewicz theorem ([16]), $\pi_2(M) = \pi_2(\widetilde{M}) = H_2(\widetilde{M})$.

There is a natural embedding of $\Gamma(G)$ in \mathcal{M} and in particular, G can be identified with a subset of \mathcal{M} . Note that the set of ends of $\Gamma(G)$

can be identified with the set of ends of \mathcal{M} . Suppose that $\tilde{S} \subset \mathcal{M}$ is an embedded sphere. As \mathcal{M} is simply-connected, \tilde{S} separates \mathcal{M} into two non-compact complementary components and hence gives a partition of the set $E(\mathcal{M})$ of ends of \mathcal{M} into two non-empty open subsets of $E(\mathcal{M})$. Let X_1 and X_2 be the closures of the complementary components of \tilde{S} . Let $E_i = X_i \cap G$. As only finitely many edges of the Cayley graph $\Gamma(G)$ intersect \tilde{S} , the set δE_1 of edges of $\Gamma(G)$ with one vertex in E_1 and the other in E_1^* is finite. This implies that the sets E_1 and E_2 form complementary almost invariant subsets of G . The sets $E_1^{(*)}$ are called the almost invariant sets of \tilde{S} .

Note that the sets $E(X_i)$ of ends of the sets X_i are determined by the sets E_i . Hence, for two embedded spheres \tilde{S} and \tilde{S}' , the corresponding almost invariant sets are almost equal if and only if \tilde{S} and \tilde{S}' are homologous as homology classes of embedded spheres in \mathcal{M} are determined by the partitions of the set $E(\mathcal{M})$ into two non-empty open sets, see [13].

An embedded sphere S in M lifts to a collection of embedded spheres in \mathcal{M} . The collection of almost invariant sets in G of the lifts of S to \mathcal{M} is called the collection of almost invariant sets of the splitting of G corresponding to S .

Consider the lifts \tilde{S} and \tilde{S}' to \mathcal{M} of embedded spheres S and S' in M . Let $E^{(*)}$ and $E'^{(*)}$ be almost invariant sets associated to \tilde{S} and \tilde{S}' , respectively. If \tilde{S} and \tilde{S}' are isotopic, then their almost invariant sets are almost equal. Therefore, two of the four sets $E^{(*)} \cap E'^{(*)}$ are small. Then, by [30, Lemma 2.3], the two splittings of G given by S and S' are conjugate. For more details of the splittings of groups and their almost invariant sets, refer to [30].

Conversely, suppose that the splitting of G associated to S' is conjugate of the splitting of G associated to S by an element $\underline{g} \in G$. Let $E^{(*)}$ be the almost invariant sets associated to some lift \tilde{S} of S to \mathcal{M} . Then, the sets gEg^{-1} and gE^*g^{-1} are the almost invariant sets associated to some lift \tilde{S}' of S' to \mathcal{M} . As E (respectively, E^*) is an almost invariant subset of G , Eg^{-1} (respectively, E^*g^{-1}) is almost equal to E (respectively, E^*). This implies that $gE^{(*)}g^{-1}$ is almost equal to $gE^{(*)}$ which are almost invariant sets associated to the translate $g\tilde{S}$ of S . Therefore, \tilde{S}' and $g\tilde{S}$ are homologous and hence

isotopic. This implies S and S' are homotopic and hence isotopic by Laudenbach's theorem.

Thus, in group theoretic terms, isotopy classes of embedded spheres in M correspond to conjugacy classes of splittings of the free group $G = \pi_1(M)$.

4. Graph of groups decomposition of $G = \pi_1(M)$.

4.1. Graph of groups. Recall the following:

Definition 4.1. A *graph* \mathbf{T} consists of two sets $E(\mathbf{T})$ and $V(\mathbf{T})$, called the *edges* and *vertices* of \mathbf{T} , a mapping from $E(\mathbf{T})$ to $E(\mathbf{T})$ denoted $e \mapsto \bar{e}$, for which $e \neq \bar{e}$ and $\bar{\bar{e}} = e$ and a mapping from $E(\mathbf{T})$ to $V(\mathbf{T}) \times V(\mathbf{T})$, denoted $e \mapsto (o(e), t(e))$, such that $\bar{e} \mapsto (t(e), o(e))$ for every $e \in E(\mathbf{T})$.

A graph \mathbf{T} has an obvious geometric realization $|\mathbf{T}|$ with vertices $V(\mathbf{T})$ and edges corresponding to pairs (e, \bar{e}) . When we say that \mathbf{T} is connected or has some topological property, we shall mean that the realization of \mathbf{T} has the appropriate property.

Definition 4.2. A *graph of groups* \mathcal{G} consists of a graph Γ together with a function assigning to each vertex v of Γ a group G_v and to each edge e a group G_e , with $G_{\bar{e}} = G_e$ and an injective homomorphism $f_e : G_e \rightarrow G_{\partial_0 e}$.

4.2. Fundamental group of a graph of groups. We shall see an algebraic definition of the fundamental group of graph of group.

First, choose a spanning tree \mathbf{T} in Γ .

Definition 4.3. The *fundamental group of \mathcal{G}* with respect to \mathbf{T} , denoted $\mathcal{G}_\Gamma(\mathbf{T})$, is defined as the quotient of the free product $(*_{v \in \Gamma(V)}(G_v)) * F(E)$, where $F(E)$ is a free group with free basis $E = \text{Edge}(\Gamma)$, subject to the following relations:

- (i) $\bar{e}\alpha_e(g)e = \omega_e(g)$, for every e in E and every $g \in G_e$.
- (ii) $\bar{e}e = 1$, for every $e \in E$.
- (iii) $e = 1$, for every edge e of the spanning tree \mathbf{T} .

One can see that the above definition is independent of the choice of the spanning tree \mathbf{T} of Γ .

Observe that in the case when Γ has just one pair (e, \bar{e}) of edges and two vertices v_1 and v_2 and if groups associated to v_1, v_2 and (e, \bar{e}) are A, B and C , respectively, the fundamental group $\pi_1(\mathcal{G})$ is $A *_C B$. In the case, when Γ has just one pair (e, \bar{e}) of edges and one vertex v and if the associated groups are C and A respectively, then the fundamental group $\pi_1(\mathcal{G})$ is $A *_C$.

We call a graph of groups \mathcal{G} with the fundamental group equal to $G = \pi_1(M)$ a graph of groups structure of G or graph of groups decomposition of G . A splitting of a group $G = \pi_1(M)$ is a graph of groups \mathcal{G} with a connected graph Γ containing only one edge such that the fundamental group \mathcal{G}_Γ is G . Note that for a graph of groups structure of G , edge groups are trivial. A graph of groups structure of G on a graph with one edge and two vertices corresponds to a free product of vertex groups. A graph of groups structure of G on a single vertex with a loop corresponds to an HNN-extension.

Let \mathcal{G} be a graph of groups structure of G with the underlying graph Γ containing k edges. We number the edges as $1, 2, \dots, k$. We call the splitting of G obtained by collapsing all the edges except the i th edge as i th elementary splitting of \mathcal{G} (or elementary splitting associated to i th edge of Γ).

Now, we shall show that, given any graph of groups structure \mathcal{G} of $G = \pi_1(M)$ up to conjugacy of vertex groups, there is a corresponding isotopy class of a sphere system in M . We shall recall the following.

4.3. Graph of topological spaces. We define a *graph χ of topological spaces* or of spaces with preferred base point as we defined graph of groups; here, it is not necessary for the map $X_e \rightarrow X_{\partial_0 e}$ to be injective, as we can use the mapping cylinder construction to replace the maps by inclusions and this does not alter the total space defined below. Given a graph χ of spaces, we can define total space χ_Γ as the quotient of

$$\cup\{X_v : v \in V(\Gamma)\} \cup \{\cup\{X_e \times I : e \in E(\Gamma)\}\}$$

by identifications

$$\begin{aligned} X_e \times I &\longrightarrow X_{\bar{e}} \times I && \text{by } (x, t) \longrightarrow (x, 1 - t) \\ X_e \times 0 &\longrightarrow X_{\partial_0 e} && \text{by } (x, 0) \longrightarrow f_e(x). \end{aligned}$$

If χ is a graph of (connected) base spaces, then by taking fundamental groups we obtain a graph of groups \mathcal{G} with the same underlying graph Γ . The fundamental group $\pi_1(\mathcal{G})$ of the graph of groups \mathcal{G} is defined to be the fundamental group of the total space χ_Γ .

Note that the fundamental group of the graph of groups \mathcal{G} defined in subsection 4.2 is the same (isomorphic) as defined here. So, we now denote $\pi_1(\mathcal{G})$ by \mathcal{G}_Γ . One can show that \mathcal{G}_Γ is independent of the choice of χ . Note that each map $G_v \rightarrow \mathcal{G}_\Gamma$ is injective.

4.4. Graph of groups structure of $\pi_1(M)$ associated to a sphere system. Given a sphere system $S = \cup_i S_i$ in M , we can associate a graph of groups \mathcal{G} to the sphere system S with underlying graph Γ such that the fundamental group \mathcal{G}_Γ of \mathcal{G} is $G = \pi_1(M)$ as follows: The vertices of Γ are the closures of components of $M - S$ and edges are the spheres S_i . An edge e in Γ is adjacent to v if the sphere corresponding to e is a boundary component of the component X_v corresponding to v and the maps from the edges to vertices are inclusion maps of spheres. This gives the graph χ of topological spaces with the same underlying graph Γ . We can consider base spaces by fixing a base point in each space which is a complementary component of $M - S$. Then, by taking fundamental groups of the spaces associated to vertices and edges of Γ , we get a graph of groups \mathcal{G} with the underlying graph Γ . The edge groups G_e are trivial. We can see that the total space χ_Γ is M and $\mathcal{G}_\Gamma = \pi_1(M)$. The fundamental group G_v of X_v can be viewed as subgroup (up to conjugacy) of G as the map $G_v \rightarrow G$ is injective. Note that Γ has no terminal vertex with trivial vertex group, and no two elementary splittings of \mathcal{G} are conjugates of each other as sphere components of S are essential and no two spheres components are isotopic.

Now, we prove the following theorem.

Theorem 4.4. *Let \mathcal{G} be a graph of groups structure of $G = \pi_1(M)$ with underlying graph Γ such that Γ has no terminal vertex with trivial vertex group associated to it and no two elementary splittings of \mathcal{G} are conjugates of each other. Then, there exist a system of embedded spheres in M such that the graph of groups structure of G given by this sphere system is $\mathcal{G}G$ (up to conjugacy of vertex groups).*

Proof. Note that, as G is a free group and G is the fundamental group of \mathcal{G} , each edge group must be trivial and all the groups associated to vertices are free groups (subgroups of G).

First, choose a spanning tree \mathbf{T} in Γ . The fundamental group $G = \mathcal{G}_\Gamma$ of \mathcal{G} with respect to \mathbf{T} is $*_{v \in \Gamma(V)}(G_v) * F(E')$, where $F(E')$ is a free group with free basis E' which contains edges of Γ that are not in the spanning tree \mathbf{T} . Choose a basis $A_v = \{a_1^v, \dots, a_{n_v}^v\}$ of the subgroup G_v of G associated to vertex v . Let $B = \{b_1, \dots, b_{n'}\}$ be a free basis of $F(E')$. The set $C = \cup_v A_v \cup B$ forms a free basis of $G = \pi_1(M)$. Suppose Γ has l vertices v_1, \dots, v_l . We write the set C as $\{a_1^{v_1} \cdots a_{n_{v_1}}^{v_1}, a_{n_{v_1}+1}^{v_2}, \dots, a_{n_{v_1}+n_{v_2}}^{v_2}, \dots, a_{n_{v_1}+\dots+n_{v_l}}^{v_l}, b_{n_{v_1}+\dots+n_{v_l}+1}, \dots, b_n\}$. We have an automorphism ϕ of G sending the basis C to the standard basis with $\phi(a_i^{v_1}) = \alpha_i$, for $1 \leq i \leq n_{v_1}$, $\phi(a_i^{v_2}) = \alpha_i$, for $n_{v_1} < i \leq n_{v_1} + n_{v_2}, \dots, \phi(a_i^{v_l}) = \alpha_i$, for $n_{v_1} + \dots + n_{v_{l-1}} < i \leq n_{v_1} + \dots + n_{v_l}$ and $\phi(b_i) = \alpha_i$, for $n_{v_1} + \dots + n_{v_l} < i \leq n$. Every automorphism of G is induced by a homeomorphism of M fixing the base point. Let h be a homeomorphism of M which fixes the base point and induces the automorphism ϕ on G .

For the vertex v_i , let $e_1^{v_i}, \dots, e_{k_{v_i}}^{v_i}$ be the edges adjacent to v_i such that the edge $e_j^{v_i}$ is not a loop base at v_i . Let $e_1'^{v_i}, \dots, e_{k_{v_i}}'^{v_i}$ be the edges which are loop based at v_i . As the vertex group G_{v_i} is a free group of rank n_{v_i} , we consider the space N_i as a connected sum of n_{v_i} copies of $S^2 \times S^1$ with a fixed basic (reduced) sphere system $\mathcal{B}_i = \{B_1^{v_i}, \dots, B_{n_{v_i}}^{v_i}\}$ in each space N_i . We remove interiors of $k_{v_i} + 2k_{v_i}'$ disjointly embedded 3-balls from N_i such that these balls are disjoint from the basic sphere system \mathcal{B}_i of N_i . To each edge $e_j^{v_i}$, we label exactly one boundary spheres by $S_j^{v_i}$ and for each loop $e_k'^{v_i}$, we label exactly two boundary spheres $S_k^{v_i}$. We denote this new space by N_i' . If the vertex group is trivial, then we consider N_i as the 3-sphere S^3 and then construct space N_i' as described above.

Now, we identify the boundary spheres of all N_i' 's which correspond to the same edge in Γ using diffeomorphisms with appropriate orientation preserving or reversing property so that the resultant space M' is homeomorphic to M and images in M of the spheres corresponding to edges of Γ give the graph of topological spaces structure of M with the underlying graph Γ .

Let $B_{n_{v_1}+\dots+n_{v_l}+1}, \dots, B_n$ be the spheres corresponding to the edges not in the spanning tree. Note that the set \mathbf{B} which is a union of all B_i together with the spheres in M' which correspond to the edges not in the spanning tree \mathbf{T} , forms a reduced sphere system in M' .

Let h_1 be a homeomorphism from M' to M which maps a fixed base point y_o in M' to the base point x_o . We can assume that y_o is away from all the sphere components of \mathbf{B} . Consider the h_1 images of the spheres in M' corresponding to the edges in Γ . There exist a homeomorphism $h_2 : M \rightarrow M$ such that $h_2(h_1(B_i^{v_1})) = \Sigma_i^0$, for $1 \leq i \leq n_{v_1}$, $h_2(h_1(B_i^{v_2})) = \Sigma_i^0$, for $n_{v_1} < i \leq n_{v_1} + n_{v_2}, \dots, h_2(h_1(B_i^{v_l})) = \Sigma_i^0$, for $n_{v_1} + \dots + n_{v_{l-1}} < i \leq n_{v_1} + \dots + n_{v_l}$ and $h_2(h_1(B_i)) = \Sigma_i^0$, for $n_{v_1} + \dots + n_{v_l} < i \leq n$. Consider the h_2 images of h_1 spheres. We call them h_2 -spheres. Then, we can see that h^{-1} images of h_2 -spheres give the decomposition of M such that we can associate the graph of groups structure \mathcal{G} to G (up to conjugacy of vertex groups). As Γ has no terminal vertex with trivial vertex group and no two elementary splittings of \mathcal{G} are conjugates of each other, an h^{-1} image of an h_2 -sphere is a sphere system in M . \square

5. Complex of non-separating spheres. In this section, we shall discuss the complex of non-separating embedded spheres in M .

Definition 5.1. The complex $NS(M)$ of *non-separating* spheres in M is the simplicial complex whose vertices are isotopy classes of non-separating embedded spheres in M . A finite collection of isotopy classes of non-separating embedded spheres in M is deemed to span a simplex in $NS(M)$ if they can be realized disjointly (up to isotopy) in M .

Clearly, we have a natural inclusion of $NS(M)$ into the sphere complex $\mathbf{S}(M)$, i.e., $NS(M)$ is a subcomplex of $\mathbf{S}(M)$. It is a subcomplex of the sphere complex spanned by the vertices which are isotopy classes of non-separating embedded spheres in M .

5.1. Path-connectedness of $NS(M)$. The 1-skeleton of the complex $NS(M)$ is path-connected. For, let $A = [S]$ and $B = [S']$ be two vertices in $NS(M)$. Let $A = A_1, \dots, A_n = B$ be a shortest path in the sphere graph of M , i.e., the 1-skeleton of the sphere complex $\mathbf{S}(M)$. We can choose each A_i to be an isotopy class of a non-separating

embedded sphere in M as follows. Suppose that A_i is an isotopy class of a separating embedded sphere S_i in M , and let M_1 and M_2 be its two complementary components of S_i in M . Suppose $A_{i-1} = [S_{i-1}]$ and $A_{i+1} = [S_{i+1}]$ such that both S_{i-1} and S_{i+1} are disjoint from S_i . If S_{i-1} and S_{i+1} are contained in different complementary components of S_i , then they are disjoint and hence A_{i-1} and A_{i+1} connected by an edge in the sphere graph. In this case, the vertex A_i can be removed from the edge-path. If S_{i-1} and S_{i+1} are contained in the same component, say M_1 , then one can replace A_i by a isotopy class A'_i of a non-separating embedded sphere S'_i contained in M_2 .

5.2. Dimension and the diameter of $NS(M)$. One can see that the dimension of the complex $NS(M)$ is $3n - 4$ as there is a maximal system $S = \cup_i S_i$ of 2-spheres in M such that each S_i is non-separating in M . The diameter of the 1-skeleton of the complex $NS(M)$ is infinite. For, if the diameter of the 1-skeleton of $NS(M)$ is finite, then it will imply that the diameter of sphere graph is finite as every vertex in the sphere graph corresponding to an isotopy class of a separating embedded sphere in M is at a distance one from a vertex in the sphere graph corresponding to an isotopy class of a non-separating embedded sphere in M . But this is not true as the diameter of the sphere graph is infinite (see [22]).

5.3. The complex $NS(M)$ is flag.

Definition 5.2. A simplicial complex \mathbf{K} is flag when, in dimensions two and higher, a simplex is present if its faces are, i.e., every complete subgraph on $r + 1$ vertices contained in the 1-skeleton is the 1-skeleton of an r -simplex.

Note that every automorphism of the 1-skeleton of a flag simplicial complex extends uniquely to an automorphism of the whole complex. By [12, Theorem 3.3], one can see that the complex $NS(M)$ is a flag complex. Thus, every automorphism of the 1-skeleton of the complex $NS(M)$ extends uniquely to an automorphism of $NS(M)$.

5.4. Action of $\text{Out}(\mathbf{F}_n)$ on $NS(M)$. One can see that a homeomorphism of M maps a non-separating embedded sphere in M to a

non-separating embedded sphere in M . This defines a simplicial action of the mapping class group $\mathcal{M}\text{ap}(M)$ on the complex $NS(M)$ and hence a simplicial action of the group $\text{Out}(\mathbf{F}_n)$ of outer automorphisms of F_n . In fact, $NS(M)$ is a subcomplex of the sphere complex invariant under the action of $\text{Out}(\mathbf{F}_n)$. Hence, $NS(M)$ is invariant under the action of $\text{Aut}(\mathbf{S}(M))$. The action of $\text{Out}(\mathbf{F}_n)$ on $NS(M)$ defines a homomorphism

$$\Upsilon : \text{Out}(\mathbf{F}_n) \longrightarrow \text{Aut}(NS(M)),$$

where $\text{Aut}(NS(M))$ is the group of simplicial automorphisms of $NS(M)$. We shall show that this is an isomorphism in Section 6.

We recall the following:

Definition 5.3. A *simple* sphere system $\Sigma = \cup_i \Sigma_i$ in M is a sphere system in M such each complementary component of Σ in M is simply-connected.

A maximal sphere system in M is simple. Splitting M along a maximal sphere system produces a finite collection of 3-punctured 3-spheres. Here, a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere.

We make a note of the following known facts:

- Every non-separating embedded sphere in M is contained in a maximal system Σ of 2-spheres such that each component of Σ is a non-separating sphere in M .
- If S is a non-separating component of a maximal system Σ of spheres in M , then there is a reduced sphere system $\Sigma' \subset \Sigma$ with $S \in \Sigma'$. In particular, every non-separating embedded sphere in M can be extended to a reduced sphere system in M .

6. The groups $\text{Aut}(NS(M))$ and $\text{Out}(\mathbf{F}_n)$. In this section, we shall prove the main result of this paper.

Theorem 6.1. *The group $\text{Aut}(NS(M))$ of simplicial automorphisms of the complex $NS(M)$ is isomorphic to the group $\text{Out}(\mathbf{F}_n)$.*

We call a simplex in $NS(M)$ *simple* (respectively, *reduced*) if it corresponds to a simple (respectively, reduced) sphere system in M .

Let ϕ be an automorphism of $NS(M)$. Firstly, we shall prove the following lemma.

Lemma 6.2. *The automorphism ϕ maps a reduced simplex in $NS(M)$ to a reduced simplex in $NS(M)$.*

To prove the above lemma, we recall the following definition.

Definition 6.3. The *link* of a simplex σ in a simplicial complex \mathbf{K} is a subcomplex of \mathbf{K} consisting of the simplices τ that are disjoint from σ and such that both σ and τ are faces of some higher-dimensional simplex in \mathbf{K} .

So, if σ is a simplex in $NS(M)$, σ corresponds to a system $S = \cup_i S_i$ of non-separating embedded spheres in M . The link(σ) is the subcomplex of $NS(M)$ spanned by the isotopy classes $[T]$ of non-separating embedded spheres T in M such that all T 's are disjoint from all and distinct from all S_i up to isotopy. We shall look at the links of reduced simplices and non-reduced simplices in $NS(M)$.

6.1. Link of a reduced simplex. Consider a reduced simplex σ in $NS(M)$ corresponding to a reduced sphere system $\Sigma = \cup_i \Sigma_i$ in M . Cutting M along Σ then gives a simply-connected 3-manifold M' with $2n$ boundary components Σ_i^+, Σ_i^- , for $1 \leq i \leq n$, where Σ_i^+ and Σ_i^- correspond to the sphere Σ_i for each i . Note that M' is homeomorphic to a 3-sphere with interiors of $2n$ disjointly embedded 3-balls removed. Now, every non-boundary parallel embedded sphere in M' gives a partition of the boundary spheres of M' into two sets, say X_S and Y_S . Moreover, its isotopy class in M' is determined by such a partition of the boundary spheres of M' .

Consider an embedded sphere S in M' such that the partition $\{X_S, Y_S\}$ of the boundary spheres of M' has the following property: There is a $\Sigma_i^\varepsilon \in X_S$ for some sign $\varepsilon \in \{+, -\}$ and for some i such that $\Sigma_i^{-\varepsilon} \in Y_S$. Such a sphere in M' corresponds to a non-separating embedded sphere in M which is disjoint from all and distinct from all the spheres Σ_i . Note that any embedded sphere in M disjoint and

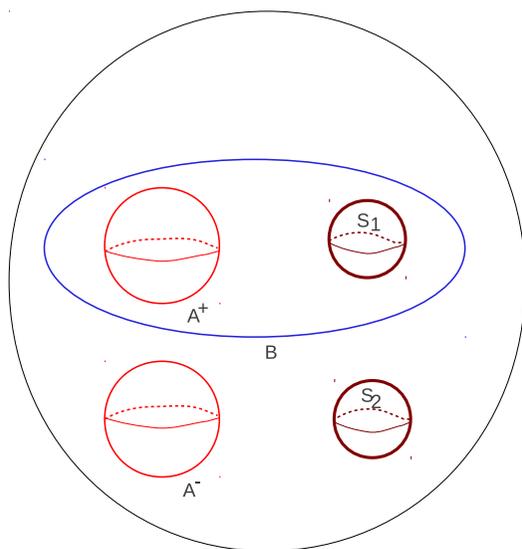
distinct from all Σ_i which corresponds to a partition of the boundary spheres of M' with the above property is non-separating in M . The isotopy classes of such non-separating embedded spheres in M span the link of the simplex σ in $NS(M)$. One can easily see that the link (σ) has a finite number of vertices.

6.2. Link of an $(n - 1)$ -simplex which is not reduced. Consider an $(n - 1)$ -simplex σ' in $NS(M)$ such that it corresponds to a system $\Sigma' = \cup_i \Sigma'_i$ of non-separating spheres in M which is not reduced.

Lemma 6.4. *The sphere system Σ' is not a simple sphere system.*

Proof. If Σ' is simple, then it must have more than one complementary component as Σ' is not a reduced sphere system. Consider the graph of groups \mathcal{G} with underlying graph Γ associated to Σ' . The edges of Γ correspond to the components of Σ' and vertices correspond to the complementary components of Σ' in M . Then, Γ has more than one vertex with all the vertex groups and edge groups trivial. In this case, $F_n = \pi_1(M) = \pi_1(\mathcal{G})$ is the fundamental group of Γ . But Γ has more than one vertex and n edges, so $\pi_1(\Gamma)$ cannot be a free group on n generators. Therefore, Σ' cannot be simple. \square

Cutting M along Σ' then gives a (not necessarily connected) 3-manifold M' with $2n$ boundary components Σ'_i^+, Σ'_i^- for $1 \leq i \leq n$ such that Σ'_i^+ and Σ'_i^- correspond to the sphere Σ'_i for each i . Now, as Σ' is not reduced and each Σ'_i is non-separating, by Lemma 6.4, there is at least one non simply-connected component of M' , say R , with at least two boundary spheres. Note that there are at least two boundary spheres of R which correspond to two distinct components of Σ' and, if M' has more than one component, then there are at least two boundary spheres, say $S_1 = \Sigma'_i^{\varepsilon}$ and $S_2 = \Sigma'_j^{\eta}$, for some i and j and for some signs $\varepsilon, \eta \in \{+, -\}$ such that $\Sigma'_i^{-\varepsilon}$ and $\Sigma'_j^{-\eta}$ are not boundary spheres of R , i.e., they are boundary components of some other component of M' . Note that R is a p -punctured $\sharp_k S^2 \times S^1$, for appropriate integers $p \geq 2$ and $k \geq 1$. Now, we shall see that, for $k = 1$ and $p = 2$, $\text{link}(\sigma')$ has infinitely many vertices by constructing infinitely many non-boundary parallel non-separating embedded spheres in R . The same construction will work for all the other possibilities of k and p obviously.

FIGURE 1. Sphere B in $P=2$ -punctured $S^2 \times S^1$

Now, suppose $k = 1$ and $p = 2$. In this case, R is a 2-punctured $S^2 \times S^1$. Let S_1 and S_2 be the two boundary components of R . These boundary spheres correspond to two distinct components of Σ' . In this case, R has infinitely many embedded spheres which separate the boundary spheres S_1 and S_2 . These spheres in R are non-separating embedded spheres in M . We construct such examples as follows. Let A be a non-boundary parallel, non-separating embedded sphere in R disjoint from S_1 and S_2 . If we cut R along A , then we get a manifold R' which is a 4-punctured S^3 with boundary components S_1, S_2, A^+ and A^- , where A^+ and A^- correspond to the embedded sphere A in R . Consider an embedded sphere B in R' which separates A^+ and S_1 from A^- and S_2 as shown in Figure 1.

The (image under identification map from R' to R of) embedded sphere B is non-separating in R . Let S'_1 be an embedded sphere parallel to S_1 in R which is disjoint from all the spheres S_1, S_2 and A . Now, we consider an embedded sphere T_m which is connected sum of B and S'_1 in R along with a tube t_m such that:

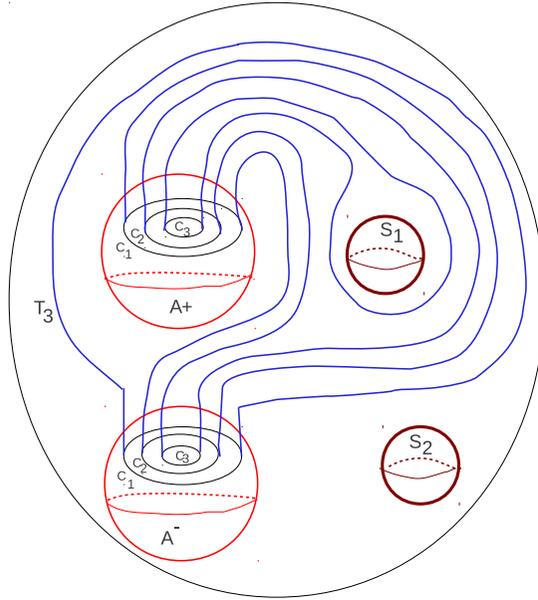


FIGURE 2. Sphere T_3 in $P=2$ -punctured $S^2 \times S^1$

- (i) the tube t_m intersects transversely m times the embedded sphere A as follows: Starting from B it hits A^- , piercing A , again hitting A^- through A^+ , it hits $(m - 1)$ times the sphere A in this fashion, it ends at S'_1 as shown in Figure 2, for $m = 3$.
- (ii) The circles of intersection of t_m with A split t_m into components called *pieces* such that each piece separates the boundary spheres S_1 and S_2 .

One can see that each sphere T_m is non-separating in M as it separates the boundary spheres of R . Each T_m is disjoint from all and distinct from all the spheres Σ'_i . Thus, $\text{link}(\sigma')$ is infinite.

6.3. Automorphisms of $NS(M)$ and simple simplices. Firstly we shall see the proof of Lemma 6.2.

Proof of Lemma 6.2. If an automorphism of $NS(M)$ maps a simplex τ of $NS(M)$ to another simplex τ' of $NS(M)$, then it maps $\text{link}(\tau)$

to $\text{link}(\tau')$ isomorphically. As $\text{link}(\sigma)$ of a reduced simplex σ has finitely many vertices and $\text{link}(\sigma')$ of a non-reduced $(n-1)$ -simplex has infinitely many vertices, the automorphism ϕ cannot map $\text{link}(\sigma)$ to $\text{link}(\sigma')$, and hence it cannot map the reduced simplex σ to the non-reduced simplex σ' . This shows that ϕ maps a reduced simplex in $NS(M)$ to a reduced simplex in $NS(M)$. \square

Lemma 6.5. *Every automorphism ϕ of $NS(M)$ maps a simple simplex in $NS(M)$ to a simple simplex in $NS(M)$.*

Proof. Note that a sphere system in M which contains a reduced sphere system as its subset is simple. Moreover, every simple sphere system $\Sigma = \cup_i \Sigma_i$ in M contains a reduced sphere system in M as its subset. For, consider the graph of groups \mathcal{G} with underlying graph Γ associated to Σ . The edges of Γ correspond to the embedded spheres Σ_i and the vertices correspond to the complementary components of Σ in M . As Σ is simple, Γ has with all the vertex groups and edge groups trivial. In this case, $F_n = \pi_1(M) = \pi_1(\mathcal{G})$ is the fundamental group of Γ . Consider a maximal tree \mathbf{T} in Γ . Then, $\Gamma \setminus \mathbf{T}$ has n edges, say e_1, \dots, e_n . Let $\Sigma_{i_1}, \dots, \Sigma_{i_n}$ be the components of Σ corresponding to the edges e_1, \dots, e_n . As the tree \mathbf{T} is connected and contains all the vertices of Γ , one can see that the complement of the embedded spheres Σ_{i_k} in M is connected. The fundamental group of the complement of these spheres Σ_{i_k} in M is the fundamental group of the graph of groups \mathcal{G}' with underlying graph \mathbf{T} . As \mathbf{T} is a tree with all vertex groups and edge groups trivial, the fundamental group of \mathcal{G}' is trivial. Hence, the complement of the spheres Σ_{i_k} in M is simply-connected. This shows that the n embedded spheres Σ_{i_k} 's corresponding to these n edges outside the maximal tree \mathbf{T} form a reduced sphere system in M .

If σ is a simple simplex in $NS(M)$, it has a face τ which is reduced. Then, $\phi(\sigma)$ is a simplex in $NS(M)$ with face $\phi(\tau)$. By Lemma 6.2, $\phi(\tau)$ is a reduced simplex which implies that $\phi(\sigma)$ is a simple simplex in $NS(M)$. Thus, ϕ maps a simple simplex in $NS(M)$ to a simple simplex in $NS(M)$. \square

6.4. Culler-Vogtmann space and the reduced outer space. We briefly recall the definition of *Culler-Vogtmann space* CV_n for free group

\mathbf{F}_n of rank n ([11, 36]). It is also known as *outer space*. A point in CV_n is an equivalence class of marked metric graphs (h, X) such that

- (i) X is a metric graph with $\pi_1(X) = \mathbf{F}_n$ having edge lengths which sum to 1, with all vertices of valence at least 3.
- (ii) The marking is given by a homotopy equivalence $h : R_n \rightarrow X$, where R_n is a graph with one vertex v and n edges. The free group of rank n is identified with $\pi_1(R_n, v)$ where the identification takes generators of the free group to the edges of R_n .
- (iii) Two such marked graphs (h, X) and (h', X') are equivalent if they are isometric via an isometry $g : X \rightarrow X'$ such that $g \circ h$ is homotopic to h' .

In [15], the outer space CV_n is denoted by \mathbf{O}_n . We shall follow the same notation as in [15]. An edge e of a graph X is called a separating edge if $X - e$ is disconnected. There is a natural equivariant deformation retraction of \mathbf{O}_n onto the subspace cv_n consisting of marked metric graphs (h, X) such that X has no separating edges. The deformation proceeds by uniformly collapsing all separating edges in all marked graphs in \mathbf{O}_n . The subspace cv_n of \mathbf{O}_n is also known as *reduced outer space*.

The group $\text{Out}(\mathbf{F}_n)$ acts on \mathbf{O}_n on the right by changing the markings: given $\phi \in \text{Out}(\mathbf{F}_n)$, choose a representative $f : R_n \rightarrow R_n$ for ϕ ; then $(g, X)\phi = (g \circ f, X)$. This action preserves the reduced outer space cv_n as a subset of \mathbf{O}_n . In particular, we have an $\text{Out}(\mathbf{F}_n)$ -action on the reduced outer space cv_n .

In [15], it was shown that the space \mathbf{O}_n is $\text{Out}(\mathbf{F}_n)$ -equivariantly homeomorphic to a subset of sphere complex $\mathbf{S}(M)$, described as follows. We interpret points in the sphere complex $\mathbf{S}(M)$ as (positively) weighted sphere systems in M . Let \mathbf{S}_∞ be the subcomplex of $\mathbf{S}(M)$ consisting of those elements $\sum_i a_i S_i \in \mathbf{S}(M)$ such that $M \setminus \cup_i S_i$ has at least one non-simply connected component. To a point $\sum_i a_i S_i \in \mathbf{S}(M) \setminus \mathbf{S}_\infty$, we associate the dual graph to $\cup_i S_i$ and assign the edge corresponding to S_i to have length a_i . This gives a map $\theta : \mathbf{O}_n \rightarrow \mathbf{S}(M) \setminus \mathbf{S}_\infty$. In [15], it is shown that θ is a homeomorphism.

Let $NS_\infty = NS(M) \cap \mathbf{S}_\infty$. Then, one can see that the reduced outer space cv_n is $\text{Out}(\mathbf{F}_n)$ -equivariantly homeomorphic to $NS(M) \setminus NS_\infty$, where the homeomorphism is given by the restriction of θ to $cv_n \subset \mathbf{O}_n$.

Culler and Vogtmann defined what is called the *spine* K_n of the reduced outer space cv_n . Considering cv_n as a subset of $NS(M)$, the spine K_n is the maximal full subcomplex of the first barycentric subdivision of $NS(M)$ which is contained in cv_n and is disjoint from NS_∞ . The interior of every simplex contained in cv_n intersects K_n . We have an action of $\text{Out}(\mathbf{F}_n)$ on $NS(M)$ which preserves cv_n , i.e., we have an action $\text{Out}(\mathbf{F}_n)$ on cv_n . This action of $\text{Out}(\mathbf{F}_n)$ on cv_n preserves the spine which yields an action of $\text{Out}(\mathbf{F}_n)$ on the spine K_n . This gives a homomorphism

$$\Omega : \text{Out}(\mathbf{F}_n) \longrightarrow \text{Aut}(K_n),$$

where $\text{Aut}(K_n)$ is the group of simplicial automorphisms of K_n .

We have the following theorem of Bridson and Vogtmann (see [8]),

Theorem 6.6. *The homomorphism Ω is an isomorphism for $n \geq 3$.*

Now, we have the following lemma.

Lemma 6.7. *Every automorphism $\phi \in \text{Aut}(NS(M))$ preserves the reduced outer space cv_n and hence the spine K_n .*

Proof. We have seen that every automorphism ϕ of $NS(M)$ maps simple simplices of $NS(M)$ to simple simplices $NS(M)$, Lemma 6.5. From this, it follows that every automorphism $\phi \in \text{Aut}(NS(M))$ preserves the reduced outer space cv_n and hence preserves the spine K_n of cv_n . \square

Thus, we have a homomorphism

$$\Lambda : \text{Aut}(NS(M)) \longrightarrow \text{Aut}(K_n) \cong \text{Out}(\mathbf{F}_n).$$

6.5. Proof of Theorem 6.1. Now, Theorem 6.1 follows if we show that the homomorphism Λ is injective. In [1], it is shown for automorphisms of the sphere complex. The same arguments hold for automorphisms of $NS(M)$. But, for the sake of completeness, we write the proof here again.

Lemma 6.8. *The identity is the only automorphism of $NS(M)$ acting trivially on the spine K_n .*

Proof. Let ϕ be an automorphism of $NS(M)$ which acts trivially on the spine K_n . Then, ϕ maps every simplex in cv_n to itself. Let $\Sigma = \cup_i \Sigma_i$ be a sphere system in M determining a top-dimensional simplex σ in cv_n . One can see that the codimension 1 face given by $\Sigma \setminus \Sigma_i$ for each i is also contained in cv_n . In order to see this it suffices to prove that it is contained in $NS(M) \setminus NS_\infty$. If this were not the case, then there would be a unique component U of $M \setminus (\Sigma \setminus \Sigma_i)$ homeomorphic to $S^2 \times S^1$ with a 3-ball removed. The boundary of U is a connected component of Σ which separates M , a contradiction to the assumption that $\sigma \in cv_n$. It follows that the automorphism ϕ maps the codimension 1 face of σ determined by Σ to itself. In particular, ϕ has to fix the opposite vertex $[\Sigma_i]$ of σ . As i is arbitrary, this shows that ϕ is the identity on σ . As σ is arbitrary, it follows that ϕ is the identity on cv_n .

Now, consider any vertex $[S]$ in $NS(M)$. We can extend S to a maximal sphere system Σ of non-separating spheres in M . The simplex determined by Σ is contained in cv_n and is hence fixed by ϕ . In particular, ϕ is identity on $[S]$. This shows that ϕ is an identity on $NS(M)$. \square

Thus, the group $\text{Aut}(NS(M))$ of simplicial automorphisms of $NS(M)$ is isomorphic to the group $\text{Out}(F_n)$. In particular, this shows that every automorphism of $NS(M)$ is geometric, i.e., it is given by a homeomorphism of the manifold M by the natural action of the mapping class group.

Now, the 1-skeleton \mathbf{K} of the complex $NS(M)$ is a subgraph of the sphere graph. Given any simplicial automorphism ϕ of \mathbf{K} , it has unique extension to a simplicial automorphism ϕ' of $NS(M)$ as the complex $NS(M)$ is flag. The automorphism ϕ' is geometric, i.e., there exists a homeomorphism h of the manifold M which induces the automorphism ϕ' on $NS(M)$. The homeomorphism h induces an automorphism ϕ'' on the sphere complex $\mathbf{S}(M)$ such that restriction of ϕ'' to $NS(M) \subset \mathbf{S}(M)$ is ϕ' . Now, restriction of ϕ'' to the sphere graph gives an extension of the automorphism ϕ of \mathbf{K} to an automorphism of the sphere graph. Thus, we have:

Corollary 6.9. *Every automorphism ϕ of \mathbf{K} has an extension to an automorphism of the sphere graph.*

REFERENCES

1. Javier Aramayona and Juan Souto, *Automorphisms of the graph of free splittings*, Michigan Math J. **60** (2011), 483–493.
2. M. Bestvina, M. Feighn and M. Handel, *Laminations, trees, and irreducible automorphisms of free groups*, Geom. Funct. Anal. **7** (1997), 215–244.
3. Mladen Bestvina and Mark Feighn, *The topology at infinity of $\text{Out}(F_n)$* , Invent. Math. **140** (2000), 651–692.
4. ———, *Hyperbolicity of the complex of free factors*, arXiv:1107.3308v2 [math.GR] 31 Oct. 2011.
5. Mladen Bestvina, Mark Feighn and Michael Handel, *The Tits alternative for $\text{Out}(F_n)$ I, Dynamics of exponentially-growing automorphisms*, Ann. Math. **151** (2000), 517–623.
6. Mladen Bestvina and Michael Handel, *Train tracks and automorphisms of free groups*, Ann. Math. **135** (1992), 1–51.
7. Tara E. Brendle and Dan Margalit, *Commensurations of the Johnson kernel*, Geom. Topol. **8** (2004), 1361–1384 (electronic).
8. Martin R. Bridson and Karen Vogtmann, *The symmetries of outer space*, Duke Math. J. **106** (2001), 391–409.
9. D.E. Cohen, *Groups of cohomological dimension one*, Lect. Notes Math. **245**, SpringerVerlag, Berlin, 1972.
10. M. Culler and J.M. Morgan, *Group actions on R -trees* Proc. Lond. Math. Soc. **55** (1987), 571–604.
11. M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free group*, Invent. Math. **87** (1986), 91–119.
12. S. Gadgil and S. Pandit, *Algebraic and geometric intersection numbers for free groups*, Topol. Appl. **156**, 1615–1619.
13. ———, *Splittings of free groups, normal forms and partitions of ends*, Proc. Indian Acad. Sci. **120** (2010), 217–241.
14. M. Handel and Lee Mosher, *The free splitting complex of a free group I: Hyperbolicity*, arXiv:1111.1994v1 [math.GR], 8 Nov. 2011.
15. Allen Hatcher, *Homological stability for automorphism groups of free groups*, Comment. Math. Helv. **70** (1995), 39–62.
16. ———, *Algebraic topology*, Cambridge University Press, Cambridge, 2001.
17. A. Hatcher and Karen Vogtmann, *The complex of free factors of a free group*, Quart. J. Math. Oxford **49** (1998), 459–468.
18. John Hempel, *3-Manifolds*, Ann. Math. Stud. **86**, Princeton University Press, Princeton, 1976.
19. E. Irmak, *Complexes of nonseparating curves and mapping class groups*, Michigan Math. J. **54** (2006), 81–110.
20. Nikolai V. Ivanov, *Complexes of curves and Teichmüller spaces* Mat. Z. **49** (1991), 54–61, 158.

21. Nikolai V. Ivanov, *Automorphism of complexes of curves and of Teichmüller spaces*, Inter. Math. Res. Not. **14** (1997), 651–666.
22. I. Kapovich and M. Lustig, *Geometric intersection number and analogues of the curve complex for free groups*, Geom. Topol. **13** (2009), 1805–1833.
23. M. Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori*, Topol. Appl. **95** (1999), 85–111.
24. M. Korkmaz and A. Papadopoulos, *On the arc and curve complex of a surface*, Math. Proc. Camb. Philos. Soc. **148** (2010), 473–483.
25. Francois Laudenbach, *Topologie de la dimension trois: Homotopie et isotopie*, Aster. **12**, Soc. Math. France, Paris, 1974.
26. ———, *Sur les 2-spheres d'une variété de dimension 3*, Ann. Math. **97** (1973), 57–81.
27. Feng Luo, *Automorphisms of the complex of curves*, Topology **39** (2000), 283–298.
28. R.C. Lyndon and P.E. Schupp, *Combinatorial group theory* Springer-Verlag, Berlin, 1977.
29. D. Margalit, *Automorphisms of the pants complex*, Duke Math. J. **121** (2004), 457–479.
30. Peter Scott and Gadde A. Swarup, *Splittings of groups and intersection numbers*, Geom. Topol. **4** (2000), 179–218.
31. Peter Scott and Terry Wall, *Topological methods in group theory*, Lond. Math. Soc. **36**, Cambridge University Press, Cambridge, 1979.
32. John R. Stallings, *Group theory and three-dimensional manifolds*, Yale Math. Mono. **4**, Yale University Press, New Haven, Connecticut, 1971.
33. ———, *A topological proof of Grushko's theorem on free products*, Math. Z. **90** (1965), 1–8.
34. ———, *Whitehead graphs on handlebodies*, deGruyter, Berlin, 1999.
35. Lucas Sabalka and Dmytro Savchuk, *On the geometry of a proposed curve complex analogue for $\text{Out}(F_n)$* , [arXiv:1007.1998v3](https://arxiv.org/abs/1007.1998v3) [math.GR].
36. Karen Vogtmann, *Automorphisms of free groups and outer space*, Geom. Ded. **94** (2002), 1–31.

MATHEMATICS SECTION, THE ABDUS SALAM INTERNATIONAL CENTER FOR THEORETICAL PHYSICS, TRIESTE, ITALY

Email address: jsuhas@gmail.com, suhas@iitm.ac.in