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$\mathfrak{m}$-FULL AND BASICALLY FULL IDEALS IN RINGS OF CHARACTERISTIC p

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ABSTRACT. We generalize the notion of $\mathfrak{m}$-full and basically full ideals in the setting of tight closure and demonstrate some $\mathfrak{m}$-full and basically full ideals in non-regular rings.

1. Introduction. For simplicity, let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p. However, everything that we discuss in characteristic p can be generalized to an equicharacteristic Noetherian local ring.

After hearing some lectures by Rees in Japan in the late 80's, Watanabe wrote a paper on $\mathfrak{m}$-full ideals. In a ring with infinite residue field, he defined an ideal I to be $\mathfrak{m}$-*full* if $(\mathfrak{m}I : x) = I$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. He showed that an ideal I which is $\mathfrak{m}$-full satisfies *the Rees property*: $\mu(I) \geq \mu(J)$ for all $J \supseteq I$. In a normal domain, he proved that any integrally closed ideal is $\mathfrak{m}$-full [**14,** Theorem 5].

In regular rings, several authors have given criteria which help to determine if an ideal is $\mathfrak{m}$-full. In particular, Watanabe has shown the following for two-dimensional regular local rings:

Theorem 1.1. [**14,** Theorem 4]. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring, I an $\mathfrak{m}$-primary ideal with $I \subseteq \mathfrak{m}^n$ and $I \nsubseteq \mathfrak{m}^{n+1}$. The following are equivalent*:

(a) *I is $\mathfrak{m}$-full.*
(b) $\mu(I) = n + 1$.
(c) *I satisfies the Rees property.*
(d) $(I : \mathfrak{m}) = (I : x)$ *for some* $x \in \mathfrak{m}$.

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In both [**6**] and [**9**], ideals, I, satisfying item (d) in the above theorem have been called *full*. Another result pertaining to $\mathfrak{m}$-full parameter ideals in regular local rings of any dimension can be summed up in the following theorem.

Theorem 1.2. [**2,** Proposition 2.3], [**6,** Theorem 4.1]. *Let* $(R, \mathfrak{m})$ *be a regular local ring, I a parameter ideal. Then the following are equivalent*:

(a) $I^n = \overline{I^n}$ *for all* $n \geq 1$ *(I is normal.)*
(b) *I is integrally closed.*
(c) *I is* $\mathfrak{m}$*-full.*
(d) *I is full.*
(e) $\lambda((I + \mathfrak{m}^2)/\mathfrak{m}^2) \geq d - 1$

There is not as much known about $\mathfrak{m}$-primary, $\mathfrak{m}$-full ideals in non-regular rings. However, the following are interesting results of Goto and Hayasaka [**3**] and Ciuperca [**1**]:

Proposition 1.3. [**3,** Proposition 2.4]. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and I an* $\mathfrak{m}$*-primary ideal with* $n = \mu(I)$. *Then the following are equivalent*:

(a) *I is* $\mathfrak{m}$*-full and R/I Gorenstein.*
(b) $\mu(\mathfrak{m}) = n$ *and there exists a minimal basis* $a_1, \ldots a_n$ *of* $\mathfrak{m}$ *such that* $I = (a_1, \ldots, a_{n-1}, a_n^s)$ *for some* $s = \min\{r \mid \mathfrak{m}^r \subseteq I\}$.

Proposition 1.4. [**1,** Proposition 4.1]. *Let* $(R, \mathfrak{m})$ *be a d-dimensional Noetherian local ring with* $\operatorname{edim} R = d + 1$. *Suppose I is an* $\mathfrak{m}$*-primary ideal generated by $d + 1$ elements. I is* $\mathfrak{m}$*-full if and only if there exist generators* $x, y, a_1, \ldots, a_{d-1}$ *of* $\mathfrak{m}$ *such that*:

(a) $I + xR = \mathfrak{m}$, *or*
(b) $\mathfrak{m}^2 = (x, a_1, \ldots, a_{d-1})\mathfrak{m}$ *and* $I + xR = (x, a_1, \ldots, a_{d-1})$.

However, neither of the above propositions give us a feeling for what $\mathfrak{m}$-full ideals look like which sit much deeper inside of the maximal ideal.

In 2002, Heinzer, Ratliff and Rush [**5**] defined the related concept: basically full ideals. An ideal is *basically full* if no minimal set of generators of I can be extended to a minimal set of generators for J, an ideal containing I. It was also shown in [**5**] that a basically full ideal is $\mathfrak{m}$-primary and satisfies $(\mathfrak{m}I : \mathfrak{m}) = I$. Recall that monomials in a regular local ring are partially ordered as follows:

$$x_1^{a_1} \cdots x_d^{a_d} \leq x_1^{b_1} \cdots x_d^{b_d}$$

if and only if $a_i \leq b_i$ for all $1 \leq i \leq d$. A set of monomials form an antichain if the generators are pairwise incomparable. In [**5,** Proposition 8.5], Heinzer, Ratliff and Rush give a nice criterion for determining if a monomial ideal in a regular local ring is basically full: a monomial ideal I is basically full if the minimal set of generators for I is a maximal antichain. Note that, for $\mathfrak{m}$-primary ideals I, $(\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : x)$ for all $x \in \mathfrak{m} \setminus \mathfrak{m}^2$; hence, all $\mathfrak{m}$-full ideals are basically full.

In this paper, we will use tight closure, in particular colon capturing, to illustrate a way to find $\mathfrak{m}$-full ideals and basically full ideals in non-regular rings. We also define tight closure notions of $\mathfrak{m}$-full and basically full ideals. In particular, we can show:

$$\begin{tikzcd}
\boxed{*\text{-}\mathfrak{m}\text{-full}} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\mathfrak{m}=\tau"] & \boxed{\text{weakly } *\text{-}\mathfrak{m}\text{-full}} \arrow[d, Rightarrow] & \boxed{\mathfrak{m}\text{-full}} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\boxed{*\text{-basically full}} \arrow[r, Rightarrow] & \boxed{\text{weakly } *\text{-basically full}} & \boxed{\text{basically full}} \arrow[l, Rightarrow]
\end{tikzcd}$$

2. Tight closure and test ideals. Tight closure, introduced by Hochster and Huneke, is a closure operation for rings containing a field. Let R be a ring of characteristic p. We say an element $x \in R$ is in the *tight closure* I^* of I if there exists a $c \in R \setminus \bigcup_{P \in \operatorname{Min}(R)} P$ with $cx^q \in I^{[q]}$ for all large $q = p^e$ where $I^{[q]}$ is the ideal generated by all the qth powers of elements in I. If $I = I^*$, we say that I is tightly closed. Tight closure has given easy proofs to some very hard problems in commutative algebra such as the Briançon-Skoda Theorem and many others. When Hochster and Huneke first defined tight closure; they noted that the tight closure I^* is contained in the integral closure. It sits much closer to I in general than the integral closure; hence, a tighter fit. Although, there are not many rings for which all ideals are integrally closed, there are many rings for which all ideals are tightly closed. We call a local ring $(R, \mathfrak{m})$ *weakly F-regular* if all ideals are tightly closed. Of course, all regular rings are weakly F-regular, but these are not the only rings which are weakly F-regular. For example, $R = k[[x, y, z]]/(x^2 - y^3 - z^5)$ is weakly F-regular when the characteristic of k is greater than 5.

Recall that, if $x_1, \ldots, x_n$ is a regular sequence, then

$$(x_1, \ldots, \widehat{x_i}, \ldots, x_n) : x_i = (x_1, \ldots, \widehat{x_i}, \ldots, x_n).$$

In a Noetherian ring of characteristic $p > 0$ which is a homomorphic image of a Cohen Macaulay ring, we say that parameters $x_1, \ldots, x_n$ satisfy colon capturing if $(x_1, \ldots, x_{n-1}) : x_n \subseteq (x_1, \ldots, x_{n-1})^*$. Since complete local domains are always the homomorphic image of a Cohen Macaulay ring, we would like to mention a stronger version of colon capturing in this instance:

Theorem 2.1. [**7,** Theorem 9.2]. *Let $(R, \mathfrak{m})$ be a d-dimensional complete local domain of characteristic p with coefficient field k. Let $x_1, \ldots, x_d$ be a system of parameters for R, and let I and J be ideals of the subring $A = k[[x_1, \ldots, x_d]]$. Then*

(a) $(IR)^* :_R JR \subseteq ((I : J)R)^*$ *and*
(b) $(IR)^* \cap (JR)^* \subseteq ((I \cap J)R)^*$.

As we saw from the definition of tight closure, it is necessary to have an element c which is not contained in any minimal prime, to find the elements in the tight closure of an ideal. How do we know if any given c will multiply a qth power of an element into $I^{[q]}$? We say an element c is a *test element* if $cI^* \subseteq I$ for all ideals $I \subseteq R$. Having test elements enables us to compute the tight closure of an ideal. The test ideal, $\tau = \bigcap_{I \subseteq R} (I : I^*)$, is the ideal generated by all the test elements.

Huneke introduced the notion of strong test ideals in [**8**]. An ideal J is a *strong test ideal* if $JI^* = JI$ for all ideals I in R. Vraciu [**13**] has shown that the test ideal is a strong test ideal in a complete reduced ring. Hara and Smith [**4**] have shown for a local ring $(R, \mathfrak{m})$, if $\mathfrak{m}$ is the test ideal $\mathfrak{m}$ is a strong test ideal.

The test ideal can be used effectively to compute the tight closure of a parameter ideal in a Gorenstein local ring.

Proposition 2.2. [10]. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d with an $\mathfrak{m}$-primary test ideal τ, and let $x_1, x_2, \dots, x_d$ be a system of parameters. Then*

$$(x_1, x_2, \dots, x_d) : \tau = (x_1, x_2, \dots, x_d)^*.$$

Also, when $(R, \mathfrak{m})$ is a Gorenstein local ring with $\mathfrak{m}$-primary test ideal, one can use a system of parameters to compute the test ideal.

Proposition 2.3. [**7,** Exercise 2.14]. *Let $(R, \mathfrak{m})$ be a Gorenstein ring of dimension d with $\mathfrak{m}$-primary test ideal τ, and let $x_1, x_2, \dots, x_d$ be a system of parameters which are also test elements. Then*

$$(x_1, x_2, \dots, x_d) : (x_1, x_2, \dots, x_d)^* = \tau.$$

The following proposition gives us a nice criterion for computing the tight closure of non-parameter ideals using the test ideal.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a complete Gorenstein local domain of dimension d with $\mathfrak{m}$ primary test ideal τ. Suppose $x_1, x_2, \dots, x_d$ is a system of parameters in R and I is an ideal of R which is the intersection of parameter ideals whose generators are in $k[[x_1, x_2, \dots, x_d]]$. Then $(I : \tau) = I^*$.*

Proof. Let

$$I = \bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd})$$

where $y_{j1}, y_{j2}, \dots, y_{jd}$ are parameters. Note that

$$(y_{j1}, y_{j2}, \dots, y_{jd}) : \tau = (y_{j1}, y_{j2}, \dots, y_{jd})^*.$$

As

$$(I : \tau) = \bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd}) : \tau = \bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd})^*,$$

we see

$$I^* \subseteq \bigcap_{j\geq 1} (y_{j1}, y_{j2}, \ldots, y_{jd})^*.$$

Now as $(y_{j1}, \ldots, y_{jd}) \subseteq (x_1, \ldots, x_d)$, then by Theorem 2.1,

$$\bigcap_{j\geq 1} (y_{j1}, y_{j2}, \ldots, y_{jd})^* \subseteq I^*.$$

$\square$

3. $\mathfrak{m}$-full and basically full ideals in complete domains. As mentioned in the introduction, for non regular rings, there is no criterion for finding $\mathfrak{m}$-full and basically full ideals deep inside the maximal ideal. One effective way for obtaining such ideals is using the known criteria for finding $\mathfrak{m}$-full and basically full ideals deep inside the maximal ideal of a regular local ring and use colon capturing.

Theorem 3.1. *Let R be a complete local domain with coefficient field k, and let $x_1, \ldots, x_d$ be a system of parameters. Let $\mathfrak{n}$ be the maximal ideal of $A = k[[x_1, \ldots, x_d]]$. Suppose $I \subseteq k[[x_1, \ldots, x_d]]$ and $(\mathfrak{n}R)^* = \mathfrak{m}$.*

(1) *If I is $\mathfrak{n}$-full, then $(IR)^*$ is $\mathfrak{m}$-full.*
(2) *If I is basically full in A, then $(IR)^*$ is basically full in R.*

Proof. For (1), note that for some $x \in \mathfrak{n} \setminus \mathfrak{n}^2$, $(\mathfrak{n}I :_A x) = I$ and

$$\begin{aligned}(IR)^* &\subseteq (\mathfrak{m}(IR)^* :_R x)\\ &= ((\mathfrak{n}R)^*(IR)^* :_R x)\\ &\subseteq (((\mathfrak{n}I)R)^* :_R x)\\ &\subseteq ((\mathfrak{n}I :_A x)R)^* = (IR)^*,\end{aligned}$$

where the last containment is by Theorem 2.1. Thus $(IR)^*$ is $\mathfrak{m}$-full.

For (2), $(\mathfrak{n}I :_A \mathfrak{n}) = I$ and

$$\begin{aligned}(IR)^* &\subseteq (\mathfrak{m}(IR)^* :_R \mathfrak{m})\\ &\subseteq ((\mathfrak{n}R)^*(IR)^* :_R \mathfrak{n})\\ &\subseteq (((\mathfrak{n}I)R)^* :_R \mathfrak{n}R)\\ &\subseteq ((\mathfrak{n}I :_A \mathfrak{n})R)^* = (IR)^*,\end{aligned}$$

where the last containment is by Theorem 2.1. Thus $(IR)^*$ is basically full. $\square$

Example 3.2. Let $R = k[[x,y,z]]/(x^2 - y^3 - z^7)$, and denote $\mathfrak{m} = (x,y,z)$. Note that y, z form a system of parameters and $\mathfrak{n} = (y,z)$ is the maximal ideal of $k[[y,z]]$. $(\mathfrak{n}R)^* = \mathfrak{m}$. Note that $(y,z)^n$ is $\mathfrak{n}$-full in $k[[y,z]]$. Thus, $((y,z)^n R)^*$ is $\mathfrak{m}$-full. Now, by Proposition 2.4,

$$((y,z)^n R)^* = \bigcap_{1\le i\le n} ((y^i, z^{n-i+1})R)^* = (x)(y,z)^{n-1} + (y,z)^n$$

since $\mathfrak{m}$ is the test ideal of R and $xy^{i-1}z^{n-i}$ is the socle element of $((y^i, z^{n-i+1})R : \mathfrak{m})$.

$I = (y^n, y^{n-1}z^{n-1}, z^n)$ is basically-full in $k[[y,z]]$. Thus, $(IR)^*$ is basically-full in R. Now by Proposition 2.4,

$$(IR)^* = ((y^n, z^{n-1})R)^* \cap (y^{n-1}, z^n)R)^* = (x(yz)^{n-2})(y,z) + I$$

since $xy(yz)^{n-2}$ is the socle element of $((y^n, z^{n-1})R : \mathfrak{m})$ and $xz(yz)^{n-2}$ is the socle element of $((y^{n-1}, z^n)R : \mathfrak{m})$.

Example 3.3. Let $R = k[[x,y]]/(xy)$. Note that $x^n + by^m$ is a system of parameters in R and $((x^n + by^m)R)^* = (x^n, y^m)$ **[11]**. Note that we can't use $(x^n + by^m)$ in Theorem 3.1 unless $n = m = 1$. The only ideals of $k[[x+by]]$ are of the form $(x+by)^n$ and $(x+by)^n R = (x^n, y^n)$ and these ideals are $\mathfrak{m}$-full by Theorem 3.1. Note that (x^n, y^m) is $\mathfrak{m}$-full in R since $(\mathfrak{m}(x^n, y^m) : x + y) = (x^n, y^m)$, but we cannot obtain this from Theorem 3.1.

4. Tight closure versions of $\mathfrak{m}$-full and basically full. Recall that integrally closed ideals are both $\mathfrak{m}$-full and basically full. By definition, I is $\mathfrak{m}$-full if there exists an $x \in \mathfrak{m}\setminus\mathfrak{m}^2$ such that $(\mathfrak{m}I : x) = I$. There may be an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ where $(\mathfrak{m}I : x)$ properly contains I even if I is integrally closed. Hence, for $\mathfrak{m}$-primary ideals, if there exists an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $(\mathfrak{m}\overline{I} : x) = \overline{I}$, we have the following chain: $I \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : x) \subseteq (\mathfrak{m}\overline{I} : x) = \overline{I}$.

Also, the tight closure of an ideal always lies in the integral closure. If I is not $\mathfrak{m}$-full or basically-full, it may be that $(\mathfrak{m}I : x)$ or $(\mathfrak{m}I : \mathfrak{m})$ are contained in I^*. Hence, it makes sense to define tight closure versions of $\mathfrak{m}$-full and basically full. The containment above yields two different ways of defining tight closure versions of $\mathfrak{m}$-full and basically full. In some cases they will be equivalent, but they may not be in general. We illustrate with some examples in the next section.

Definition. For an ideal I in a local ring $(R, \mathfrak{m})$, we say I is $*$-$\mathfrak{m}$-full if $(\mathfrak{m}I : x) = I^*$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. We say that I is weakly $*$-$\mathfrak{m}$-full if $(\mathfrak{m}I : x)^* = I^*$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.

We will say that I is $*$-$\mathfrak{m}$-full (weakly $*$-$\mathfrak{m}$-full) with respect to x if $(\mathfrak{m}I : x) = I^*$ $((\mathfrak{m}I : x)^* = I^*)$.

Definition. For an ideal I in a local ring $(R, \mathfrak{m})$, we say I is $*$-basically full if $(\mathfrak{m}I : \mathfrak{m}) = I^*$. We say that I is weakly $*$-basically full if

$$(\mathfrak{m}I : \mathfrak{m})^* = I^*.$$

Note that, if I is $*$-$\mathfrak{m}$-full ($*$-basically full), then I is weakly $*$-$\mathfrak{m}$-full (weakly $*$-basically full). Also, if I is $\mathfrak{m}$-full (basically full), I is weakly $*$-$\mathfrak{m}$-full ($*$-basically full). There are ideals I in a local Noetherian ring $(R, \mathfrak{m})$ where I is not $*$-$\mathfrak{m}$-full but is weakly $*$-$\mathfrak{m}$-full. We also have an example exhibiting that $*$-basically full is indeed a separate notion from weakly $*$-basically full. The following proposition sums up what is known.

Proposition 4.1. *Let (R, m) be a local Noetherian ring and I an ideal of R.*

(a) *If I is weakly $*$-$\mathfrak{m}$-full with respect to x and $(\mathfrak{m}I : x)$ is tightly closed, then I is $*$-$\mathfrak{m}$-full.*
(b) *If I^* is $\mathfrak{m}$-full, then I is weakly $*$-$\mathfrak{m}$-full.*
(c) *If I is weakly $*$-basically full and $(\mathfrak{m}I : \mathfrak{m})$ is tightly closed, then I is $*$-basically full.*
(d) *If I^* is basically-full, then I is weakly $*$-basically full.*
(e) *Let $I \subseteq R$ be an ideal which is weakly $*$-$\mathfrak{m}$-full, then I is weakly $*$-basically full.*

Proof. For (a), since $(\mathfrak{m}I : x)$ is tightly closed, then

$$(\mathfrak{m}I : x) = (\mathfrak{m}I : x)^* = I^*.$$

Hence I is $*$-$\mathfrak{m}$-full.

For (b), we know for some $x \in \mathfrak{m}$,

$$I \subseteq (\mathfrak{m}I : x) \subseteq (\mathfrak{m}I^* : x) = I^*.$$

Now take the tight closure of each ideal to observe

$$I^* \subseteq (\mathfrak{m}I : x)^* \subseteq (\mathfrak{m}I^* : x) = I^*.$$

As the ends are equal, we see that there exists and $x \in \mathfrak{m}$ with $(\mathfrak{m}I : x)^* = I^*$. Hence, I is weakly $*$-$\mathfrak{m}$-full.

To see (c), since $(\mathfrak{m}I : \mathfrak{m})$ is tightly closed, then

$$(\mathfrak{m}I : \mathfrak{m}) = (\mathfrak{m}I : \mathfrak{m})^* = I^*.$$

Hence, I is $*$-basically full.

For (d),

$$I \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I^* : \mathfrak{m}) = I^*.$$

Now take the tight closure of each ideal to observe

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m})^* \subseteq (\mathfrak{m}I^* : \mathfrak{m}) = I^*.$$

As the ends are equal we see that $(\mathfrak{m}I : \mathfrak{m})^* = I^*$. Hence, I is weakly $*$-basically full.

Concluding with (e),

$$I \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : x)^* = I^*.$$

Now take the tight closure of each ideal to observe

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m})^* \subseteq (\mathfrak{m}I : \mathfrak{m})^* = I^*.$$

As the ends are equal, we see that $(\mathfrak{m}I : \mathfrak{m})^* = I^*$. Hence, I is weakly $*$-basically full. $\square$

As the Rees property was related to $\mathfrak{m}$-fullness, we would like to define a tight closure notion of Rees property too. Recall, $\mu(I) = \dim_{R/\mathfrak{m}}(I/\mathfrak{m}I)$. If $J \subseteq I$, we know that $J^* \subseteq I^*$. The natural definition of the $*$-Rees property is the following.

Definition. We say I satisfies the $*$-Rees Property if for all $J \supseteq I$, $\mu(J^*) \leq \mu(I^*)$.

Proposition 4.2. *Let $I \subseteq R$ be an ideal such that I^* satisfies the Rees property, then I satisfies the $*$-Rees property.*

Proof. If I^* satisfies the Rees property, then for every $J \supseteq I^*$, $\mu(J) \leq \mu(I^*)$. For all $J \supseteq I$, $J^* \supseteq I^*$. Thus, $\mu(J^*) \leq \mu(I^*)$. Hence, I satisfies the $*$-Rees property. $\square$

In the case that $(R, \mathfrak{m})$ is a local normal isolated singularity with test ideal equal to $\mathfrak{m}$, Hara and Smith [**4**] have shown that $\mathfrak{m}$ is a strong test ideal. In other words, $\mathfrak{m}I = \mathfrak{m}I^*$ for all $I \subseteq R$. This is equivalent to $I^* \subseteq (\mathfrak{m}I : \mathfrak{m})$. We use this containment to show the following.

Proposition 4.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with test ideal equal to $\mathfrak{m}$ and $I \subseteq R$ an ideal.*

(a) *I is $*$-$\mathfrak{m}$-full if and only if I is weakly $*$-$\mathfrak{m}$-full.*
(b) *I $*$-basically full if and only if I is weakly $*$-basically full.*

Proof. To see (a), since $\mathfrak{m}$ is a strong test ideal,

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : x) \subseteq (\mathfrak{m}I : x)^* = I^*.$$

Hence, $(\mathfrak{m}I : x) = I^*$ which implies I is $*$-$\mathfrak{m}$-full.

To see (b), note that if I is $*$-basically full, then I is weakly $*$-basically full since $(\mathfrak{m}I : \mathfrak{m}) = I^*$ is tightly closed; hence,

$$(\mathfrak{m}I : \mathfrak{m})^* = I^*.$$

When $\mathfrak{m}$ is the test ideal of R, $\mathfrak{m}$ is a strong test ideal. Hence,

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m}).$$

If I is weakly $*$-basically full, then

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : \mathfrak{m})^* = I^*.$$

Hence, we have equality throughout and $(\mathfrak{m}I : \mathfrak{m}) = I^*$, and I is $*$-basically full. $\square$

Proposition 4.4. *Let $(R, \mathfrak{m})$ be a local normal isolated singularity with test ideal equal to $\mathfrak{m}$ and $I \subseteq R$ an ideal of R which is $*$-$\mathfrak{m}$-full. The following hold.*

(a) *I^* is $\mathfrak{m}$-full.*
(b) *I is $*$-basically full.*
(c) *I^* is basically full.*
(d) *I^* is full.*
(e) *I satisfies the $*$-Rees Property.*
(f) *Now suppose that I is $\mathfrak{m}$-primary. Then*

$$\mu(I^*) = \lambda(R/I + xR) + \mu(I + xR/xR).$$

Proof. To see (a)–(c), observe that the following inclusions:

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m}) = (\mathfrak{m}I^* : \mathfrak{m}) \subseteq (\mathfrak{m}I^* : x) = (\mathfrak{m}I : x) = I^*$$

become equalities. Hence, $(\mathfrak{m}I^* : \mathfrak{m}) = I^*$ or I^* is $\mathfrak{m}$-full, $(\mathfrak{m}I : \mathfrak{m}) = I^*$ or I is $*$-basically full and $(\mathfrak{m}I^* : x) = I^*$ or I^* is basically full.

For (d), note

$$(I^* : x) \subseteq ((\mathfrak{m}I : \mathfrak{m}) : x) = ((\mathfrak{m}I : x) : \mathfrak{m}) = (I^* : \mathfrak{m}) \subseteq (I^* : x).$$

Hence, I^* is full.

For (e), using the fact that I^* is $\mathfrak{m}$-full, then I^* satisfies the Rees property. So, for every $J \supseteq I^*$, $\mu(J) \leq \mu(I^*)$. Note that, if $J \supseteq I$, then $J^* \supseteq I^*$. Thus $\mu(J^*) \leq \mu(I^*)$.

For (f), we apply [**2,** Lemma 2.2] to obtain

$$\mu(I^*) = \lambda((\mathfrak{m}I : x)/\mathfrak{m}I) = \lambda(R/I + xR) + \mu(I + xR/xR).$$

$\square$

Note that if I is $*$-$\mathfrak{m}$-full, then $I^* = (\mathfrak{m}I : x)$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Also, if τ is a strong test ideal, we have the following containments: $I^* \subseteq (\tau I : \tau) \subseteq (\mathfrak{m}I : \tau) \subseteq (\mathfrak{m}I : y)$ for some $y \in \tau \setminus \tau^2$. If $y \in \mathfrak{m} \setminus \mathfrak{m}^2$, then this implies that $(\tau I : \tau) = I^*$.

5. Examples. To show that the concepts of $*$-$\mathfrak{m}$-fullness and $*$-basically fullness are new we include several examples. First we give some examples of $*$-$\mathfrak{m}$-full ideals which are not $\mathfrak{m}$-full.

Example 5.1. Let $R = k[[x, y, z]]/(x^2 - y^3 - z^7)$, char $(k) = p > 7$ and $I = (y, z)$. Note that, as $\mathfrak{m}$ is the test ideal and y, z form a system of parameters

$$((y,z):\mathfrak{m}) = (x,y,z) = (y,z)^*$$

by Proposition 2.2 and

$$(\mathfrak{m}(y,z):x) = ((xy,xz,y^2,yz,z^2):x) = (x,y,z) = (y,z)^* \neq (y,z),$$

so I is not $\mathfrak{m}$-full, but I is $*$-$\mathfrak{m}$-full. Also $(y,z)^n$ as in Example 3.2 is also $*$-$\mathfrak{m}$-full, but not $\mathfrak{m}$-full.

Note that (y,z) in Example 5.1 satisfies the $*$-Rees property as $(y,z)^* = (x,y,z)$ and the only ideal containing (y,z) is $\mathfrak{m}$ which is three generated.

Example 5.2. Let $R = k[[x,y,z]]/(x^2 - y^3 - z^6)$, $\operatorname{char}(k) = p > 5$ and $I = (y^2, yz, z^3)$. Note that

$$(y^2,yz,z^3)^* = ((y^2,yz,z^3):\mathfrak{m}) = (xy, xz^2, y^2, yz, z^3)$$

and

$$\begin{aligned}(\mathfrak{m}(y^2,yz,z^3):x+z) &= ((xy^2,xyz,xz^3,y^3,y^2z,yz^2,z^4):x+z)\\ &= (xy,xz^2,y^2,yz,z^3) \neq (y^2,yz,z^3).\end{aligned}$$

So I is not $\mathfrak{m}$-full, but I is $*$-$\mathfrak{m}$-full.

The next example offers an ideal which is weakly $*$-$\mathfrak{m}$-full but not $\mathfrak{m}$-full or $*$-$\mathfrak{m}$-full.

Example 5.3. Let $R = k[[t^3,t^5]]$. Since $(\mathfrak{m}(t^8):t^3) = (t^8,t^{10})$,

$$(\mathfrak{m}(t^8):t^5) = (t^6,t^8),$$

and for $a \neq 0$, $(\mathfrak{m}(t^8):t^3+at^5) = (t^8, t^{10} - at^{12})$ and

$$(\mathfrak{m}(t^8):t^5+at^6) = (t^8, t^9 - at^{10})$$

which are all not tightly closed. Hence, (t^8) is not $\mathfrak{m}$-full or $*$-$\mathfrak{m}$-full. However, $(\mathfrak{m}(t^8):t^3)^* = (t^8,t^9,t^{10})$, which is tightly closed. Hence, (t^8) is weakly $*$-$\mathfrak{m}$-full.

The following two examples exhibit that there are ideals which are $*$-$\mathfrak{m}$-full and weakly $*$-basically full, but not $*$-basically full.

Example 5.4. Let $R = k[[x,y,z]]/(x^2 - y^3 - z^{12})$, char $(k) = p > 11$ and $I = (y, z^2)$. The test ideal of R is (x, y, z^2); hence,

$$(y, z^2)^* = ((y, z^2) : (x, y, z^2)) = (x, y, z^2).$$

The ideal (y, z^2) is $*$-$\mathfrak{m}$-full

$$(\mathfrak{m}(y, z^2) : x) = (x, y, z^2).$$

However, (y, z^2) is not basically full nor $*$-basically full, since $(\mathfrak{m}(y, z^2) : \mathfrak{m}) = (xz, y, z^2)$ is not equal to (y, z^2), nor $(y, z^2)^*$. However, (y, z^2) is weakly $*$-basically full.

Example 5.5. Let $R = k[[t^2, t^5]]$. $(\mathfrak{m}(t^4) : \mathfrak{m}) = (t^4, t^7)$ which is not tightly closed. Hence, (t^4) is not basically full or $*$-basically full. However, $(\mathfrak{m}(t^4) : \mathfrak{m})^* = (t^4, t^5)$, which is tightly closed. Hence, (t^4) is weakly $*$-basically full. Note (t^4) is $*$-$\mathfrak{m}$-full since

$$(\mathfrak{m}(t^4) : t^4 + t^5) = (t^4, t^5).$$

This gives an example of an ideal which is $*$-$\mathfrak{m}$-full but not $*$-basically full.

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