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ON THE mod p^7 DETERMINATION OF $\binom{2p-1}{p-1}$

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ABSTRACT. In this paper we prove that for any prime $p \geq 11$,

$$\binom{2p-1}{p-1} \equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} + 4p^2 \sum_{1\leq i<j\leq p-1}\frac{1}{ij} \pmod{p^7}$$

holds. This is a generalization of the famous Wolstenholme's theorem which asserts that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ for all primes $p \geq 5$. Our proof is elementary, and it does not use a standard technique involving the classic formula for power sums in terms of the Bernoulli numbers. Notice that the above congruence reduced modulo p^6, p^5 and p^4 yields related congruences obtained by Tauraso, Zhao and Glaisher, respectively.

1. Introduction and statement of results. *Wolstenholme's theorem* (e.g., see [**4**], [**14**]) asserts that if p is a prime greater than 3, then the binomial coefficient $\binom{2p-1}{p-1}$ satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3} \tag{1.1}$$

for any prime $p \geq 5$. It is well known (e.g., see [**5,** page 89]) that this theorem is equivalent to the assertion that the numerator of the fraction

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is divisible by p^2 for any prime $p \geq 5$.

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Further, by a special case of Glaisher's congruence ([**2,** page 21], [**3,** page 323]; also cf., [**10,** Theorem 2]), for any prime $p \geq 5$ we have

$$\binom{2p-1}{p-1} \equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} \equiv 1 - \frac{2p^3}{3}B_{p-3} \pmod{p^4}, \tag{1.2}$$

where B_k is the kth Bernoulli number. Granville [**4**] established broader generalizations of Wolstenholme's theorem. More recently, Helou and Terjanian [**6**] established many Wolstenholme type congruences modulo p^k with a prime p and $k \in \{4,5,6\}$. One of their main results [**6,** Proposition 2, pages 488–489] is a congruence of the form $\binom{np}{mp} \equiv f(n,m,p)\binom{n}{m} \pmod{p^6}$, where $p \geq 3$ is a prime number, $m,n,\in \mathbf{N}$ with $0 \leq m \leq n$, and f is the function on m,n and p involving Bernoulli numbers B_k $(k \in \mathbf{N})$. In particular, for $p \geq 5$, $m=1$ and $n=2$, using the fact that

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1},$$

this congruence yields [**6,** Corollary 1]

$$\binom{2p-1}{p-1} \equiv 1 - p^3 B_{p^3-p^2-2} + \frac{p^5}{3}B_{p-3} - \frac{6p^5}{5}B_{p-5} \pmod{p^6}. \tag{1.3}$$

Recently, Tauraso [**13,** Theorem 2.4] proved that for any prime $p > 5$

$$\binom{2p-1}{p-1} \equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} + \frac{2p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3} \pmod{p^6}.$$

In this paper we improve the above congruence as follows.

Theorem 1.1. *Let $p \geq 11$ be a prime. Then*

$$\binom{2p-1}{p-1} \equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} + 4p^2 \sum_{1\leq i<j\leq p-1}\frac{1}{ij} \pmod{p^7}. \tag{1.4}$$

Remark 1.2. Note that the congruence (1.4) for $p=3$ and $p=5$ reduces to the identity, while for $p=7$ (1.4) is satisfied modulo 7^6.

Applying a technique of Helou and Terjanian [**6**] based on Kummer type congruences, the congruence (1.4) may be expressed in terms of the Bernoulli numbers as follows.

Corollary 1.3. *Let $p \geq 11$ be a prime. Then*

$$\begin{aligned}\binom{2p-1}{p-1} &\equiv 1 - p^3 B_{p^4-p^3-2} + p^5\left(\frac{1}{2}B_{p^2-p-4} - 2B_{p^4-p^3-4}\right)\\ &\quad + p^6\left(\frac{2}{9}B_{p-3}^2 - \frac{1}{3}B_{p-3} - \frac{1}{10}B_{p-5}\right) \pmod{p^7}.\end{aligned} \tag{1.5}$$

Note that the congruence (1.3) can easily be deduced from the congruence (1.5) by reducing the moduli and using the Kummer congruences.

Corollary 1.4. (cf., [**13,** Theorem 2.4]). *Let $p \geq 7$ be a prime. Then*

$$\begin{aligned}\binom{2p-1}{p-1} &\equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} - 2p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\\ &\equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} + \frac{2p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3} \pmod{p^6}.\end{aligned}$$

Corollary 1.5. ([**16,** Theorem 3.2], [**10,** p. 385]). *Let $p \geq 7$ be a prime. Then*

$$\binom{2p-1}{p-1} \equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} \equiv 1 - p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^5}.$$

A prime p is said to be a *Wolstenholme prime* if it satisfies the congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$. By the congruence (1.2) we see that a prime p is a Wolstenholme prime if and only if p divides the numerator of B_{p-3}. The two known such primes are 16843 and 2124679, and McIntosh and Roettger [**11**] reported that these primes are the only two Wolstenholme primes less than 10^9. However, by using the argument based on the prime number theorem, McIntosh [**10,** page 387] conjectured that there are infinitely many Wolstenholme primes, and that no prime satisfies the congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^5}$.

Remark 1.6. In **[12,** Corollary 1] the author proved that for any Wolstenholme prime p

$$\begin{aligned}\binom{2p-1}{p-1} &\equiv 1-2p\sum_{k=1}^{p-1}\frac{1}{k}-2p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\\ &\equiv 1+2p\sum_{k=1}^{p-1}\frac{1}{k}+\frac{2p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3}\pmod{p^7},\end{aligned}\tag{1.6}$$

holds, and he conjectured **[12,** Remark 1] that any of the previous congruences for a prime p yields that p is necessarily a Wolstenholme prime. Note that this conjecture concerning the first above congruence may be confirmed by using our congruence (1.4). Namely, if a prime p satisfies the first congruence of (1.6), then by (1.4) it must be

$$\begin{aligned}\binom{2p-1}{p-1} &\equiv 1-2p\sum_{k=1}^{p-1}\frac{1}{k}-2p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\\ &\equiv 1-2p\sum_{k=1}^{p-1}\frac{1}{k}+4p^2\sum_{1\le i<j\le p-1}\frac{1}{ij}\pmod{p^7}.\end{aligned}\tag{1.7}$$

Using the identity

$$2\sum_{1\le i<j\le p-1}\frac{1}{ij}=\left(\sum_{k=1}^{p-1}\frac{1}{k}\right)^2-\sum_{k=1}^{p-1}\frac{1}{k^2},$$

the second congruence in (1.7) immediately reduces to

$$2p^2\left(\sum_{k=1}^{p-1}\frac{1}{k}\right)^2\equiv 0\pmod{p^7},$$

whence it follows that

$$\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0\pmod{p^3}.$$

Finally, substituting this into the first Glaisher's congruence in (1.2), we find that

$$\binom{2p-1}{p-1}\equiv 0\pmod{p^4}.$$

Hence, p must be a Wolstenholme prime, and so our conjecture is confirmed related to the first congruence of (1.6).

The situation is more complicated in relation to the conjecture concerning the second congruence of (1.6). Then, comparing this congruence and (1.4), as in the previous case, we obtain

$$2\sum_{k=1}^{p-1}\frac{1}{k}-p\left(\sum_{k=1}^{p-1}\frac{1}{k}\right)^2+p\sum_{k=1}^{p-1}\frac{1}{k^2}+\frac{p^2}{3}\sum_{k=1}^{p-1}\frac{1}{k^3}\equiv 0 \pmod{p^6}.$$

However, from the above congruence we are unable to deduce that p must be a Wolstenholme prime.

Remark 1.7. It follows from Corollary 1.5 that $p^3 \mid \sum_{k=1}^{p-1} 1/k$ and $p^2 \mid \sum_{k=1}^{p-1} 1/k^2$ for any Wolstenholme prime p. This argument together with a technique applied in the proof of Theorem 1.1 suggests the conjecture that such a prime p satisfies the congruence (1.4) modulo p^8. However, a direct calculation shows that this is not true for the Wolstenholme prime 16843.

As noticed in Remark 1.2, the congruence (1.4) for $p = 3$ and $p = 5$ reduces to the identity. However, our computation via `Mathematica` shows that no prime in the range $7 \le p < 500000$ satisfies the congruence (1.4) with the modulus p^8 instead of p^7. Nevertheless, using the heuristic argument for the "probability" that a prime p satisfies (1.4) modulo p^8 about $1/p$, we conjecture that there are infinitely many primes satisfying (1.4) modulo p^8.

2. Proof of Theorem 1.1 and Corollaries 1.4 and 1.5. For the proof of Theorem 1.1, we will need some elementary auxiliary results.

For a prime $p \ge 3$ and a positive integer $n \le p-2$, we denote

$$R_n(p) := \sum_{k=1}^{p-1}\frac{1}{k^n} \quad \text{and} \quad H_n(p) := \sum_{1\le i_1<i_2<\cdots<i_n\le p-1}\frac{1}{i_1 i_2\cdots i_n},$$

with the convention that $H_1(p) = R_1(p)$. In the sequel we shall often write R_n and H_n in the proofs instead of $R_n(p)$ and $H_n(p)$, respectively.

Observe that, by Wolstenholme's theorem, $p^2 \mid R_1(p)$ for any prime $p \ge 5$, which can be generalized as follows.

Lemma 2.1. ([**1,** Theorem 3]; also see [**17**] or [**15,** Theorem 1.6]). *For any prime $p \geq 5$ and a positive integer $n \leq p-3$, we have*

$$R_n(p) \equiv 0 \pmod{p^2} \quad \textit{if } 2 \nmid n,$$

and

$$R_n(p) \equiv 0 \pmod{p} \quad \textit{if } 2 \mid n.$$

Lemma 2.2. *For any prime $p \geq 7$, we have*

$$H_3(p) \equiv \frac{R_3(p)}{3} - \frac{R_1(p)R_2(p)}{2} \pmod{p^6} \tag{2.1}$$

and

$$H_4(p) \equiv -\frac{R_4(p)}{4} + \frac{(R_2(p))^2}{8} \pmod{p^4}. \tag{2.2}$$

In particular, $p^2 \mid H_3(p)$, $p \mid H_2(p)$ and $p \mid H_4(p)$.

Proof. Substituting the shuffle relation $H_2 = (R_1^2 - R_2)/2$ into the identity $3H_3 = R_3 - R_1R_2 + H_2R_1$, we find that $H_3 = (R_3/3) - (R_1R_2)/2 + (R_1^3/6)$. This equality together with the fact that $p^2 \mid R_1$ yields the congruence (2.1), and thus $p^2 \mid H_3$.

Similarly, by Newton's formula [**8**], we have the identity

$$4H_4 = -R_4 + H_1R_3 - H_2R_2 + H_3R_1.$$

Since by Lemma 2.1, $p^4 \mid R_1R_3 = H_1R_3$, and since $p^2 \mid H_3$, we also have $p^4 \mid H_3R_1$. Substituting this and $H_2 = (R_1^2 - R_2)/2$ into the above identity, we obtain

$$4H_4 \equiv -R_4 - \frac{R_1^2R_2}{2} + \frac{R_2^2}{2} \pmod{p^4}.$$

Since by Lemma 2.1, $p^5 \mid R_1^2R_2$, we can exclude the term $R_1^2R_2/2$ in the above congruence to obtain (2.2), and so $p \mid H_4$. This completes the proof. $\square$

Lemma 2.3. *For any prime p and any positive integer r, we have*

$$2R_1 \equiv -\sum_{i=1}^{r} p^i R_{i+1} \pmod{p^{r+1}}. \tag{2.3}$$

Proof. Multiplying the identity

$$1+\frac{p}{i}+\cdots+\frac{p^{r-1}}{i^{r-1}}=\frac{p^r-i^r}{i^{r-1}(p-i)}$$

by $-p/i^2$ $(1\le i\le p-1)$, we obtain

$$-\frac{p}{i^2}\left(1+\frac{p}{i}+\cdots+\frac{p^{r-1}}{i^{r-1}}\right)=\frac{-p^{r+1}+pi^r}{i^{r+1}(p-i)}\equiv\frac{p}{i(p-i)}\pmod{p^{r+1}}.$$

Therefore,

$$\left(\frac{1}{i}+\frac{1}{p-i}\right)\equiv-\left(\frac{p}{i^2}+\frac{p^2}{i^3}+\cdots+\frac{p^r}{i^{r+1}}\right)\pmod{p^{r+1}},$$

whence, after summation over $i=1,\dots,p-1$, we immediately obtain (2.3). This concludes the proof. □

Lemma 2.4. *For any prime $p\ge 7$, we have*

$$2R_1(p)\equiv -pR_2(p)\pmod{p^4},$$

and, for any prime $p\ge 11$,

$$2R_3(p)\equiv -3pR_4(p)\pmod{p^4}$$

holds.

Proof. Note that, by Lemma 2.3,

$$2R_1\equiv -pR_2-p^2R_3-p^3R_4\pmod{p^4}.$$

Since, by Lemma 2.1, $p^2\mid R_3$ and $p\mid R_4$ for any prime $p\ge 7$, the above congruence reduces to the first congruence in our lemma.

Since for each $1\le k\le p-1$

$$\frac{1}{k^3}+\frac{1}{(p-k)^3}=\frac{p^3-3p^2k+3pk^2}{k^3(p-k)^3},$$

it follows that

$$
\begin{aligned}
2R_3 &= \sum_{k=1}^{p-1}\left(\frac{1}{k^3}+\frac{1}{(p-k)^3}\right)\\
&= p^3\sum_{k=1}^{p-1}\frac{1}{k^3(p-k)^3}-3p^2\sum_{k=1}^{p-1}\frac{1}{k^2(p-k)^3}\\
&\quad+3p\sum_{k=1}^{p-1}\frac{1}{k(p-k)^3}.
\end{aligned}\tag{2.4}
$$

First observe that, applying Lemma 2.1, for each prime $p \geq 11$ we have

$$
\sum_{k=1}^{p-1}\frac{1}{k^3(p-k)^3} \equiv -\sum_{k=1}^{p-1}\frac{1}{k^6} \equiv 0 \pmod{p}.\tag{2.5}
$$

Further, in view of the fact that $1/(p-k) \equiv -(p+k)/k^2 \pmod{p^2}$, and that for each prime $p \geq 11$, $p \mid R_6$ and $p^2 \mid R_5$ by Lemma 2.1, we have

$$
\begin{aligned}
\sum_{k=1}^{p-1}\frac{1}{k^2(p-k)^3} &= \sum_{k=1}^{p-1}\frac{1}{(p-k)^2k^3}\\
&\equiv \sum_{k=1}^{p-1}\frac{(p+k)^2}{k^7} \pmod{p^2}\\
&\equiv \sum_{k=1}^{p-1}\frac{2p}{k^6}+\sum_{k=1}^{p-1}\frac{1}{k^5} \equiv 0 \pmod{p^2}.
\end{aligned}\tag{2.6}
$$

Substituting (2.5) and (2.6) into (2.4), we get

$$
2R_3 \equiv 3p\sum_{k=1}^{p-1}\frac{1}{k(p-k)^3} \pmod{p^4}.\tag{2.7}
$$

Next, from the identity

$$
\frac{1}{k(p-k)^3}+\frac{1}{k^4} = \frac{p^3}{k^4(p-k)^3}-\frac{3p^2}{k^3(p-k)^3}+\frac{3p}{k^2(p-k)^3},
$$

for $k = 1, 2, \ldots, p-1$, we obtain

$$\frac{1}{k(p-k)^3} + \frac{1}{k^4} \equiv \frac{3p^2}{k^6} + \frac{3p}{k^2(p-k)^3} \pmod{p^3}.$$

After summation over $k = 1, \ldots, p-1$, the above congruence gives

$$\sum_{k=1}^{p-1} \frac{1}{k(p-k)^3} + R_4 \equiv 3p^2 R_6 + 3p \sum_{k=1}^{p-1} \frac{1}{k^2(p-k)^3} \pmod{p^3}.$$

Since by Lemma 2.1, $p \mid R_6$ for any prime $p \geq 11$, substituting this and (2.6) into the above congruence, we obtain

$$\sum_{k=1}^{p-1} \frac{1}{k(p-k)^3} \equiv -R_4 \pmod{p^3}.$$

Substituting this into (2.7), we finally obtain

$$2R_3 \equiv -3pR_4 \pmod{p^4}.$$

This completes the proof. $\square$

Proof of Theorem 1.1. For any prime $p \geq 11$, we have

$$\begin{aligned}
\binom{2p-1}{p-1} &= \frac{(p+1)(p+2)\cdots(p+k)\cdots(p+(p-1))}{1\cdot 2\cdots k\cdots p-1} \\
&= \left(\frac{p}{1}+1\right)\left(\frac{p}{2}+1\right)\cdots\left(\frac{p}{k}+1\right)\cdots\left(\frac{p}{p-1}+1\right) \\
&= 1 + \sum_{i=1}^{p-1} \frac{p}{i} + \sum_{1\leq i_1<i_2\leq p-1} \frac{p^2}{i_1 i_2} + \cdots \\
&\quad + \sum_{1\leq i_1<i_2<\cdots<i_k\leq p-1} \frac{p^k}{i_1 i_2\cdots i_k} + \cdots + \frac{p^{p-1}}{(p-1)!} \\
&\quad = 1 + \sum_{k=1}^{p-1} p^k H_k = 1 + \sum_{k=1}^{6} p^k H_k + \sum_{k=7}^{p-1} p^k H_k.
\end{aligned}$$

By Lemmas 2.1 and 2.2, we have $R_1 \equiv R_3 \equiv R_5 \equiv H_3 \equiv 0 \pmod{p^2}$ and $R_2 \equiv R_4 \equiv R_6 \equiv H_2 \equiv H_4 \equiv 0 \pmod{p}$ for any prime $p \geq 11$. Since, by Newton's formula, $5H_5 = R_5 + \sum_{i=1}^{4}(-1)^i H_i R_{5-i}$ and $6H_6 = -R_6 - \sum_{i=1}^{5}(-1)^i H_i R_{6-i}$, it follows from the previous congruences that $p^2 \mid H_5$ and $p \mid H_6$. Therefore, $p^7 \mid \sum_{k=5}^{p-1} p^k H_k$ for any prime $p \geq 11$, and so the above expansion yields

$$\binom{2p-1}{p-1} \equiv 1 + pH_1 + p^2H_2 + p^3H_3 + p^4H_4 \pmod{p^7}. \tag{2.8}$$

Recall that $H_1 = R_1$ and $H_2 = (R_1^2 - R_2)/2$. The congruences from Lemma 2.2 yield $H_3 \equiv (R_3/3) - (R_1R_2/2) \pmod{p^4}$ and $H_4 \equiv -(R_4/4) + (R_2^2/8) \pmod{p^3}$. Substituting all the previous expressions for H_i, $i = 1, 2, 3, 4$, into (2.8), we find that

$$\begin{aligned}\binom{2p-1}{p-1} \equiv& 1 + pR_1 + \frac{p^2}{2}(R_1^2 - R_2) \\ &+ \frac{p^3}{6}(2R_3 - 3R_1R_2) + \frac{p^4}{8}(R_2^2 - 2R_4) \pmod{p^7}.\end{aligned} \tag{2.9}$$

Further, by Lemma 2.4, we have

$$2R_1 \equiv -pR_2 \pmod{p^4} \tag{2.10}$$

and

$$2R_3 \equiv -3pR_4 \pmod{p^4}. \tag{2.11}$$

The congruences (2.10) and (2.11) yield $p^4R_2^2 \equiv -2p^3R_1R_2 \pmod{p^7}$ and $p^4R_4 \equiv -(2/3)p^3R_3 \pmod{p^7}$, respectively. Substituting these congruences into the last term on the right hand side of (2.9), we obtain

$$\begin{aligned}\binom{2p-1}{p-1} \equiv& 1 + pR_1 + \frac{p^2}{2}(R_1^2 - R_2) \\ &- \frac{3p^3}{4}R_1R_2 + \frac{p^3}{2}R_3 \pmod{p^7}.\end{aligned} \tag{2.12}$$

It remains to eliminate R_3 from (2.12). Note that, by Lemma 2.3, $2R_1 \equiv -pR_2 - p^2R_3 - p^3R_4 - p^4R_5 - p^5R_6 \pmod{p^6}$. Since, by Lemma 2.2, $p^2 \mid R_5$ and $p \mid R_6$, the previous congruence reduces to

$$2R_1 \equiv -pR_2 - p^2R_3 - p^3R_4 \pmod{p^6}. \tag{2.13}$$

We use again the congruence (2.11) in the form $p^3R_4 \equiv -(2/3)p^2R_3 \pmod{p^6}$, which by inserting in (2.13) yields $2R_1 \equiv -pR_2 - (1/3)p^2R_3 \pmod{p^6}$. Multipying by $3p$, this implies

$$p^3R_3 \equiv -6pR_1 - 3p^2R_2 \pmod{p^7}. \tag{2.14}$$

Substituting this into the last term of (2.12), we immediately get

$$\binom{2p-1}{p-1} \equiv 1-2pR_1-2p^2R_2+\frac{p^2}{4}R_1(2R_1-3pR_2) \pmod{p^7}. \tag{2.15}$$

Now we write (2.10) as

$$2R_1 - 3pR_2 \equiv 8R_1 \pmod{p^4}.$$

Since $p^2 \mid R_1$, and so $p^4 \mid p^2R_1$, multiplying the above congruence by $(1/4)p^2R_1$, we find that

$$\frac{p^2}{4}R_1(2R_1 - 3pR_2) \equiv 2p^2R_1^2 \pmod{p^7}.$$

Replacing this into (2.15), we obtain

$$\binom{2p-1}{p-1} \equiv 1 - 2pR_1 + 2p^2(R_1^2 - R_2) \pmod{p^7}, \tag{2.16}$$

which by the identity $(R_1^2 - R_2)/2 = H_2$ yields the desired congruence. This completes the proof. □

Proof of Corollary 1.4. The first congruence in Corollary 1.4 for $p \geq 11$ is immediate from (2.16), using the fact that $p^2 \mid R_1$, and so $p^6 \mid p^2R_1^2$. Since from (2.14) we have $p^2R_2 \equiv -2pR_1 - (p^3/3)R_3 \pmod{p^6}$, inserting this into the first congruence in Corollary 1.4, we immediately obtain

$$\binom{2p-1}{p-1} \equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} + \frac{2p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3} \pmod{p^6},$$

which is just the second congruence in Corollary 1.4.

A calculation shows that both congruences are also satisfied for $p = 7$, and the proof is completed. □

Proof of Corollary 1.5. Let $p \geq 7$ be any prime. By Corollary 1.4, we have $\binom{2p-1}{p-1} \equiv 1 - 2pR_1 - 2p^2R_2 \pmod{p^5}$. Substituting into this $-pR_2 \equiv 2R_1 \pmod{p^4}$ (Lemma 2.4), we obtain

$$\binom{2p-1}{p-1} \equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^5},$$

as desired. □

3. Proof of Corollary 1.3. As noticed in the Introduction, in the proof of Corollary 1.3, we will apply a method of Helou and Terjanian [**6**] based on Kummer type congruences.

Lemma 3.1. *Let p be a prime, and let m be any even positive integer. Then the denominator d_m of the Bernoulli number B_m, written in reduced form, is given by*

$$d_m = \prod_{p-1|m} p,$$

where the product is taken over those primes p such that $p-1$ divides m.

Proof. The assertion is an immediate consequence of the von Staudt-Clausen theorem (e.g., see [**7,** page 233, Theorem 3]) which asserts that $B_m + \sum_{p-1|m} 1/p$ is an integer for all even m, where the summation is over all primes p such that $p-1$ divides m. □

For a prime p and a positive integer n, we denote

$$R_n(p) = R_n = \sum_{k=1}^{p-1} \frac{1}{k^n} \quad \text{and} \quad P_n(p) = P_n = \sum_{k=1}^{p-1} k^n.$$

Lemma 3.2. ([**6,** p. 8]). *Let p be a prime greater than 5, and let n, r be positive integers. Then*

$$P_n(p) \equiv \sum_{s-\mathrm{ord}_p(s)\le r} \frac{1}{s}\binom{n}{s-1} p^s B_{n+1-s} \pmod{p^r}, \tag{3.1}$$

where $\mathrm{ord}_p(s)$ is the largest power of p dividing s, and the summation is taken over all integers $1 \le s \le n+1$ such that $s - \mathrm{ord}_p(s) \le r$.

The following result is well known as the Kummer congruences.

Lemma 3.3. ([**7**]). *Suppose that $p \ge 3$ is a prime and m, n, r are positive integers such that m and n are even, $r \le n-1 \le m-1$ and $m \not\equiv 0 \pmod{p-1}$. If $n \equiv m \pmod{\varphi(p^r)}$, where $\varphi(p^r) = p^{r-1}(p-1)$ is Euler's totient function, then*

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \pmod{p^r}. \tag{3.2}$$

The following congruences are also due to Kummer.

Lemma 3.4. ([**9**]; also see [**6,** p. 20]). *Let $p\ge 3$ be a prime and let m, r be positive integers such that m is even, $r\le m-1$ and $m\not\equiv 0 \pmod{p-1}$. Then*

$$\sum_{k=0}^{r}(-1)^k\binom{m}{k}\frac{B_{m+k(p-1)}}{m+k(p-1)}\equiv 0 \pmod{p^r}. \tag{3.3}$$

Lemma 3.5. *For any prime $p\ge 11$, we have*

(i) $R_1(p) \equiv -\frac{p^2}{2}B_{p^4-p^3-2}-\frac{p^4}{4}B_{p^2-p-4}+\frac{p^5}{6}B_{p-3}+\frac{p^5}{20}B_{p-5} \pmod{p^6}$.

(ii) $R_1^2(p)\equiv \frac{p^4}{9}B_{p-3}^2 \pmod{p^5}$.

(iii) $R_2(p)\equiv pB_{p^4-p^3-2}+p^3B_{p^4-p^3-4} \pmod{p^5}$.

Proof. If s is a positive integer such that $\mathrm{ord}_p(s)=e\ge 1$, then for $p\ge 11$, it holds that $s-e\ge p^e-e\ge 10$. This shows that the condition $s-\mathrm{ord}_p(s)\le 6$ implies that $\mathrm{ord}_p(s)=0$, and thus, for such a s must be $s\le 6$. Therefore,

$$P_n(p)\equiv\sum_{s=1}^{6}\frac{1}{s}\binom{n}{s-1}p^sB_{n+1-s} \pmod{p^6} \quad \text{for } n=1,2,\dots . \tag{3.4}$$

By Euler's theorem [**5**], for $1\le k\le p-1$, and positive integers n, e, we have $1/k^{\varphi(p^e)-n}\equiv k^n \pmod{p^e}$, where $\varphi(p^e)=p^{e-1}(p-1)$ is the Euler's totient function. Hence, $R_{\varphi(p^e)-n}(p)\equiv P_n(p) \pmod{p^e}$. In particular, if $n=\varphi(p^6)-1=p^5(p-1)-1$, then by Lemma 3.1, $p^6\mid p^6B_{p^5(p-1)-6}$ for each prime $p\ge 11$. Therefore, using the fact that $B_{p^5(p-1)-1}=B_{p^5(p-1)-3}=B_{p^5(p-1)-5}=0$, (3.4) yields

$$\begin{aligned}R_1(p) &\equiv P_{p^5(p-1)-1}(p)\equiv\frac{1}{2}(p^5(p-1)-1)p^2B_{p^5(p-1)-2}\\ &\quad+\frac{1}{4}\frac{(p^5(p-1)-1)(p^5(p-1)-2)(p^5(p-1)-3)}{6}\\ &\qquad\times p^4B_{p^5(p-1)-4} \pmod{p^6},\end{aligned}$$

whence we have

$$R_1(p) \equiv -\frac{p^2}{2}B_{p^6-p^5-2} - \frac{p^4}{4}B_{p^6-p^5-4} \pmod{p^6}. \tag{3.5}$$

By the Kummer congruences (3.2) from Lemma 3.3, we have

$$\begin{aligned} B_{p^6-p^5-2} &\equiv \frac{p^6-p^5-2}{p^4-p^3-2}B_{p^4-p^3-2} \equiv \frac{2B_{p^4-p^3-2}}{p^3+2} \\ &\equiv \left(1-\frac{p^3}{2}\right)B_{p^4-p^3-2} \pmod{p^4}. \end{aligned}$$

Substituting this into (3.5), we obtain

$$R_1(p) \equiv -\frac{p^2}{2}B_{p^4-p^3-2} + \frac{p^5}{4}B_{p^4-p^3-2} - \frac{p^4}{4}B_{p^6-p^5-4} \pmod{p^6}. \tag{3.6}$$

Similarly, we have

$$B_{p^4-p^3-2} \equiv \frac{p^4-p^3-2}{p-3}B_{p-3} \equiv \frac{2}{3}B_{p-3} \pmod{p}$$

and

$$\begin{aligned} B_{p^6-p^5-4} &\equiv \frac{p^6-p^5-4}{p^2-p-4}B_{p^2-p-4} \equiv \frac{4B_{p^2-p-4}}{p+4} \\ &\equiv \left(1-\frac{p}{4}\right)B_{p^2-p-4} \pmod{p^2}. \end{aligned}$$

Substituting the above two congruences into (3.6), we get

$$\begin{aligned} R_1(p) \equiv {} & -\frac{p^2}{2}B_{p^4-p^3-2} + \frac{p^5}{6}B_{p-3} - \frac{p^4}{4}B_{p^2-p-4} \\ & + \frac{p^5}{16}B_{p^2-p-4} \pmod{p^6}. \end{aligned} \tag{3.7}$$

Finally, since

$$B_{p^2-p-4} \equiv \frac{p^2-p-4}{p-5}B_{p-5} \equiv \frac{4}{5}B_{p-5} \pmod{p},$$

the substitution of the above congruence into (3.7) immediately gives the congruence (i).

Further, (3.7) immediately gives

$$R_1^2(p) \equiv \frac{p^4}{4}B_{p^4-p^3-2}^2 \pmod{p^5}. \tag{3.8}$$

Again by the Kummer congruences (3.2) from Lemma 3.3, we have

$$B_{p^4-p^3-2} \equiv \frac{p^4-p^3-2}{p-3} B_{p-3} \equiv \frac{2}{3} B_{p-3} \pmod{p}.$$

Substituting this into (3.8), we immediately obtain the congruence (ii).

In order to prove the congruence (iii), note that if $n-3 \not\equiv 0 \pmod{p-1}$, then by Lemma 3.1, for even $n \geq 6$, $p^5 \mid p^5 B_{n-4}$ holds, and we know that $B_{n-1} = B_{n-3} = 0$ for such a n. Therefore, reducing the modulus in (3.4) to p^5, and using the same argument as in the beginning of the proof of (i), for all even $n \geq 2$,

$$P_n(p) \equiv pB_n + \frac{p^3}{6} n(n-1) B_{n-2} \pmod{p^5} \tag{3.9}$$

holds. In particular, for $n = p^4 - p^3 - 2$ and using $P_{\varphi(p^4)-2}(p) \equiv R_2(p) \pmod{p^4}$, (3.9) reduces to

$$R_2(p) \equiv P_{p^4-p^3-2}(p) \equiv pB_{p^4-p^3-2} + p^3 B_{p^4-p^3-4} \pmod{p^5}. \tag{3.10}$$

This completes the proof. □

Proof of Corollary 1.3. The congruence (1.5) of Corollary 1.3 follows directly by substituting congruences (i), (ii) and (iii) of Lemma 3.5 into the congruence (1.4) of Theorem 1.1. □

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