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ALGEBRAIC POLYNOMIALS WITH SYMMETRIC RANDOM COEFFICIENTS

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ABSTRACT. This paper provides an asymptotic estimate for the expected number of real zeros of algebraic polynomials $P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$, where a_j's $(j = 0, 1, 2, \cdots, n-1)$ are a sequence of normal standard independent random variables with the symmetric property $a_j \equiv a_{n-1-j}$. It is shown that the expected number of real zeros in this case still remains asymptotic to $(2/\pi)\log n$. In the previous study, it was shown for the case of random trigonometric polynomials this expected number of real zeros is halved when we assume the above symmetric properties.

1. Introduction. Let a_j, $j = 0, 1, 2, \ldots n-1$, be a sequence of random variables, and denote $N_n(a, b)$ as the number of real zeros of $P_n(x)$ in the interval (a, b) where

$$P_n(x) = \sum_{j=0}^{n-1} a_j x^j. \tag{1.1}$$

It is well known that, for independent, normally distributed coefficients a_j's with mean $\mu = 0$ and n large $EN_n(-\infty, \infty)$, the expected value of $N_n(-\infty, \infty)$, is asymptotic to $(2/\pi)\log n$. However, this asymptotic value simply reduces by half when the mean of the coefficients becomes non-zero, see for example [**5**] or [**9**]. On the other hand, there is no reduction in the expected number of real zeros if we consider the

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random trigonometric polynomial defined as

$$T_n(x) = \sum_{j=0}^{n-1} a_j \cos jx. \tag{1.2}$$

In this case the expected number of real zeros in the interval $(0, 2\pi)$ remains $2n/\sqrt{3}$, for both cases of μ zero and nonzero constant. Earlier results concerning the above and other types of polynomials are reviewed in [**1**] and [**3**].

Motivated by its application in number theory and random matrix theory, see for example [**11**], [**12**] and the more recent work of [**8**], there has been interest in random algebraic polynomials with complex random coefficients and self-reciprocal properties. Using the definition developed in [**13**] for reciprocal polynomials with deterministic coefficients, in [**4**] it is shown that this type of random polynomial, with a simple transformation, will yield to the random trigonometric polynomials (1.2) with real coefficients. Different types of random trigonometric polynomials with self-reciprocal properties are studied in [**6, 7**] which show again that the expected number of real zeros reduces by half to $n/\sqrt{3}$.

Therefore, it is of interest to study the algebraic polynomial with real random coefficients which possess this self-reciprocal property and ask whether or not any reduction to the expected number of real zeros occurs. To this end we assume the coefficients in (1.1) are random with the symmetric property $a_j = a_{n-j-1}$, $j = 0, 1, \ldots, [n/2]$. Here we show that the expected number of real zeros in the algebraic case with these symmetric properties remains the same as $(2/\pi)\log n$. Therefore, the symmetric assumption has an opposite influence on the mathematical behavior of random polynomials than that of the means stated above. That is, in the trigonometric case with symmetric properties, the number of real zeros reduces by half, while for the algebraic case this expected number remains the same. We prove:

Theorem 1.1. *If the random variables a_j, $j = 0, 1, 2, \ldots, [n/2]$, are independent, identically normally distributed and symmetric with finite variance σ^2, then for sufficiently large n, the expected number of real* *zeros of* $P_n(x)$ *in* (1.1) *satisfies*

$$EN_n(-\infty,\infty) \sim \frac{2}{\pi}\log n.$$

2. Proof of the Theorem. It is easy to note that, see also [**3,** page 31], for the classical random algebraic polynomial $P_n(x)$ defined in (1.1), without symmetric properties of the coefficients

$$\begin{aligned} A_o^2(\sigma^2,n) &\equiv \operatorname{var}\{P_n(x)\} = \sigma^2\sum_{j=0}^{n-1} x^{2j}, \\ B_o^2(\sigma^2,n) &\equiv \operatorname{var}\{P_n'(x)\} = \sigma^2\sum_{j=0}^{n-1} j^2x^{2j-2}, \end{aligned}$$

and

$$C_o(\sigma^2,n) \equiv \operatorname{cov}\{P_n(x),P_n'(x)\} = \sigma^2\sum_{j=0}^{n-1} jx^{2j-1}, \tag{2.1}$$

where $P_n'(x)$ is the derivative of $P_n(x)$ with respect to x. For this case the Kac-Rice formula [**10, 14**] gives the expected number of real zeros in the interval (a,b) as, see also [**2**],

$$EN(a,b) = \left(\frac{1}{\pi}\right)\int_a^b \left(\frac{\Delta_o}{A_o^2}\right)dx, \tag{2.2}$$

where

$$\Delta_o = \sqrt{A_o^2B_o^2 - C_o^2}. \tag{2.3}$$

Without loss of generality, we assume n is even. For n odd, the upper limit of $n/2$ in the second sum in (2.4) will be replace by $(n-1)/2-1$, as well as having an additional term of $a_{(n-1)/2}x^{(n-1)/2}$. As a result, the variance of $P_n(x)$ obtained in (2.5) will include a term σ^2x^{n-1}. For sufficiently large n this is significantly smaller than the existing σ^2nx^{n-1} and hence can be ignored. A similar argument is valid for the evaluation of B_N^2 and C_N in (2.6) and (2.7). Using the above assumption of the symmetric coefficients, a_j, we have

$$P_n(x) = \sum_{j=0}^{n-1} a_j x^j = \sum_{j=0}^{n/2} a_j (x^j + x^{n-j-1}). \tag{2.4}$$

Then the new properties can lead to the following identities in term of A_o^2, B_o^2 and C_o, where the subscript $_N$ refers to the polynomial of the form (2.4).

$$\begin{aligned} A_N^2(\sigma^2, n) &\equiv A_N^2 \\ &= \sigma^2 \sum_{j=0}^{n/2} (x^j + x^{n-j-1})^2 \;= A_o^2 + \sigma^2 \sum_{j=0}^{n/2} 2x^{n-1} \\ &= \sigma^2 \frac{1-x^{2n}}{1-x^2} + \sigma^2 n x^{n-1}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} B_N^2(\sigma^2, n) &\equiv B_N^2 \\ &= \sigma^2 \sum_{j=0}^{n/2} (jx^{j-1} + (n-j-1)x^{n-j-2})^2, \\ &= B_o^2 + 2\sigma^2 x^{n-3} \sum_{j=0}^{n/2} \{(n-1)j - j^2\} \\ &= \sigma^2(-x^{2n+2} - x^{2n} + x^2 + 1 - n^2 x^{2n+2} \\ &\quad +2n^2 x^{2n} - n^2 x^{2n-2} + 2nx^{2n+2} \\ &\quad -2nx^{2n})/(1-x^2)^3 \\ &\quad +\sigma^2 x^{n-3} \frac{n(n+2)(n-2)}{6}, \end{aligned} \tag{2.6}$$

and finally, since $E(a_i a_j) = 0$ for all $i \neq j$,

$$\begin{aligned} C_N(\sigma^2, n) \equiv C_N = \operatorname{cov}\left(P(x), P'(x)\right) \\ = E\left(\sum_{j=0}^{n/2} (x^j + x^{n-j+1}) a_j \sum_{j=0}^{n/2} (jx^{j-1} + (n-j-1)x^{n-j-2}) a_j \right) \\ = E\left(\sum_{j=0}^{n/2} (x^j j x^{j-1} + x^{n-j-1}(n-j-1)x^{n-j-2}) a_j^2 \right. \end{aligned}$$

$$+\sum_{j=0}^{n/2}(x^j(n-j-1)x^{n-j-2}+x^{n-j-1}jx^{j-1})a_j^2\Big)$$
$$=C_o+\sigma^2x^{n-2}\frac{n(n-1)}{2}$$
$$=\sigma^2\left\{\frac{x(1-x^{2n})}{(1-x^2)^2}-\frac{nx^{2n-1}}{1-x^2}\right\}$$
$$+\sigma^2x^{n-2}\frac{n(n-1)}{2}. \tag{2.7}$$

We now notice that if we change x to $-x$ the distribution of the coefficients in $P_n(x)$ remain invariant. Therefore, $EN_n(-\infty,0)\sim EN_n(0,\infty)$. Also, for $y=1/x$, we can write $P_n(x)$ as

$$y^{-n+1}(a_0y^{n-1}+a_1y^{n-2}+a_2y^{n-3}+\cdots+a_{n-3}y^2+a_{n-2}y+a_{n-1}).$$

Since the polynomial $a_0y^{n-1}+a_1y^{n-2}+a_2y^{n-3}+\cdots+a_{n-3}y^2+a_{n-2}y+a_{n-1}$ has the same self- reciprocal properties as $P_n(x)$ given in (1.1), corresponding to every zero in $(0,1)$ there is a zero in $(1,\infty)$. Hence, $EN_n(0,1)\sim EN_n(1,\infty)$, and hence $EN_n(-\infty,\infty)\sim 4EN_n(0,1)$. Therefore, in order to obtain $EN_n(-\infty,\infty)$ using (2.2), it is sufficient to consider only $(0,1)$. We separate this interval into two subintervals $(0,1-\varepsilon)$ and $(1-\varepsilon,1)$. We assume $\varepsilon_n\equiv\varepsilon=n^{-a}$, where a is any value in (0, 1), to be defined later. When $0<x<1-\varepsilon$, for sufficiently large n, we have

$$x^n<(1-\varepsilon)^n=(1-n^{-a})^n\le e^{-n^{1-a}}.$$

If $a=1-\log\log n^{10}/\log n$ is selected, then for sufficiently large n,

$$e^{-n^{1-a}}=n^{-10}.$$

Therefore,

$$x^n<n^{-10}. \tag{2.8}$$

Hence for $0<x<1-\varepsilon$, we can rewrite (2.5)–(2.7) as

$$A_N^2=\frac{\sigma^2}{1-x^2}+O\left(\sigma^2n^{-9}\right), \tag{2.9}$$

$$B_N^2=\sigma^2\left(\frac{1+x^2}{(1-x^2)^3}+O\left(n^{-7}\right)\right), \tag{2.10}$$

$$C_N = \sigma^2\left(\frac{x}{(1-x^2)^2} + O\left(n^{-8}\right)\right). \tag{2.11}$$

Then, from (2.9)–(2.11), we have

$$\begin{aligned} \Delta_N^2 &= A_N^2 B_N^2 - C_N^2 \\ &\sim \frac{\sigma^4}{(1-x^2)^4}. \end{aligned} \tag{2.12}$$

Now (2.9) and (2.12) gives Kac's asymptotic formula for this case

$$\begin{aligned} EN_n(0, 1-\varepsilon) &= \frac{1}{\pi}\int_0^{1-\varepsilon} \frac{\Delta_N}{A_N^2}\,dx \sim \frac{1}{\pi}\int_0^{1-\varepsilon} \frac{1}{1-x^2}\,dx \\ &= \frac{1}{2\pi}\left\{\log(2-n^{-a}) + a\log n\right\} \\ &\sim \frac{1}{2\pi}\log n, \end{aligned} \tag{2.13}$$

since $a \to 1$ as $n \to \infty$. Now we show that (2.13) is the main contributor to $EN_n(0,1)$. To this end, for the interval $(1-\varepsilon, 1)$, we have

$$\begin{aligned} &\sum_{j=0}^{n-1} x^{2j} - \sum_{j=0}^{n-1} x^{n-1} \\ &= (1-x^{n-1}) + (x^2 - x^{n-1}) + (x^4 - x^{n-1}) + \cdots + (x^{n-2} - x^{n-1}) \\ &\quad - x^{n-1}(1-x^{n-1}) - x^{n-3}(x^2 - x^{n-1}) \\ &\quad - \cdots - x^3(x^{n-4} - x^{n-1}) - x(x^{n-2} - x^{n-1}) \\ &= (1-x^{n-1})(1-x^{n-1}) + (1-x^{n-3})(x^2 - x^{n-1}) + (1-x^{n-5})(x^4 - x^{n-1}) \\ &\quad + \cdots + (1-x^3)(x^{n-4} - x^{n-1}) + (1-x)(x^{n-2} - x^{n-1}) \geq 0. \end{aligned}$$

This shows $A_o^2 \geq \sigma^2 n x^{n-1}$, that is, from (2.5), $A_N^2 \leq 2A_o^2$. Similarly, from (2.6) we can write

$$\begin{aligned} \frac{B_o^2}{\sigma^2} &= \sum_{j=0}^{n-1} j^2 x^{2j-2} \\ &= 1 + 4x^2 + 9x^4 + \cdots + \left(\frac{n}{2} - 1\right)^2 x^{n-4} \\ &\quad + \left(\frac{n}{2}\right)^2 x^{n-2} + \left(\frac{n}{2} + 1\right)^2 x^n + \cdots + (n-1)^2 x^{2n-4} \end{aligned}$$

$$
\begin{aligned}
&> 1 + 4x^2 + 9x^4 + \cdots + \left(\frac{n}{2} - 1\right)^2 x^{n-4} \\
&> x^{n-3} \sum_{j=0}^{n/2-1} j^2 = x^{n-3} \frac{(n/2)((n/2)-1)(n-1)}{6}. \qquad (2.14)
\end{aligned}
$$

Therefore, we have
(2.15)

$$
\frac{\sigma^2 x^{n-3}[n(n^2-4)]/6}{B_o^2} < \frac{\sigma^2 x^{n-3}[n(n^2-4)]/6}{\sigma^2 x^{n-3}[(n/2)((n/2)-1)(n-1)]/6} \sim 4.
$$

Thus, we obtain $\sigma^2 x^{n-3}[n(n^2-4)]/6 < 4B_o^2$ and the estimate $B_N^2 < 5B_o^2$. Since $\Delta_o^2 = A_o^2 B_o^2 - C_o^2$ is the same as the one from Farahmand's work [**3,** page 33], then we calculate Δ_N^2 as

$$
\begin{aligned}
\Delta_N^2 &= A_N^2 B_N^2 - C_N^2 \\
&= 10\Delta_o^2 + 9C_o^2 \\
&< 10\sigma^4 \frac{(1-x^{2n})^2}{(1-x^2)^4} \left\{ 1 - \frac{n^2 x^{4n-2}(1-x^2)^2}{(1-x^{2n})^2} \right. \\
&\qquad \left. + \left[x - \frac{nx^{2n-1}(1-x^2)}{1-x^{2n}} \right]^2 \right\} \\
&\leq 10\sigma^4 \frac{(1-x^{2n})^2}{(1-x^2)^4} \left\{ \frac{(1-x^2)(2n-2nx^{2n})}{1-x^{2n}} \right\} \\
&= 20\sigma^4 n \frac{(1-x^{2n})^2}{(1-x^2)^3}. \qquad (2.16)
\end{aligned}
$$

Consequently, Δ_N is given as

$$
\begin{aligned}
\Delta_N &< \sqrt{20}\sigma^2 \frac{1-x^{2n}}{1-x^2} \sqrt{\frac{n}{1-x^2}} \\
&< \sqrt{20}\sigma^2 \frac{1-x^{2n}}{1-x^2} \sqrt{\frac{n}{1-x}}. \qquad (2.17)
\end{aligned}
$$

Also, since

$$
\sqrt{n\varepsilon} = o\left(n^{\log\log n^{10}/2\log n}\right) = o\left(\sqrt{\log n}\right),
$$

from (2.2), (2.5) and (2.17) it is easy to show that

$$\begin{aligned} EN_n(1-\varepsilon,1) &= \frac{1}{\pi}\int_{1-\varepsilon}^{1}\frac{\Delta_N}{A_N^2}\,dx \le \frac{1}{\pi}\int_{1-\varepsilon}^{1}\frac{\Delta_N}{A_o^2}\,dx \\ &< \frac{2\sqrt{5n}}{\pi}\int_{1-\varepsilon}^{1}\frac{1}{\sqrt{1-x}}\,dx \\ &= \frac{4\sqrt{5}}{\pi}\sqrt{n\varepsilon} \\ &\sim o(\sqrt{\log n}). \end{aligned} \tag{2.18}$$

Hence, we have the proof for the theorem.

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