

RADFORD'S FORMULA FOR GENERALIZED WEAK BIFROBENIUS ALGEBRAS

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ABSTRACT. In this paper we introduce the notion of a generalized weak biFrobenius algebra (or briefly GWBF-algebra) which is designed to cover both weak Hopf algebras and bi-Frobenius algebras. Then we study properties of various modular elements of GWBF-algebras. As an application we mainly show Radford's formula for S^4 in the setting of GWBF-algebras.

1. Introduction. Radford's formula for the fourth power of the antipode is an important result in the theory of finite-dimensional Hopf algebras. Let H be a finite-dimensional Hopf algebra over a field $\mathbf{K}$. Then it says that

$$S^4(h) = g(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})g^{-1}$$

for any element $h \in H$, where g and α are the so-called distinguished group-like elements in H and its dual H^* , respectively. This formula was initially proved by Larson in [8] for finite-dimensional unimodular Hopf algebras and extended by Radford in [11] to any finite-dimensional Hopf algebra H .

Moreover, in the last few years, this formula has been adapted for several types of generalized Hopf algebras. For example, Radford's formula was recently generalized for weak Hopf algebras by Nikshych in [9], for bi-Frobenius algebras (introduced in [5] as a generalization of finite-dimensional Hopf algebras) by Ferrer Santors and Haim in [6], for

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any co-Frobenius Hopf algebra by Beattie et al. in [1], for any algebraic quantum groups by Delvaux et al. in [4] and for bornological quantum hypergroups by Van Daele and Wang in [13]. From this point of view, it is worthwhile to know the minimal set of conditions under which this formula holds. This will form one of the motivations of the paper.

The main purpose of this paper is to introduce the notion of a generalized weak biFrobenius algebra (or simply GWBF-algebra) as a generalization of both weak Hopf algebras and biFrobenius algebras, the comultiplication and counit in biFrobenius algebras are weakened, the counit is not algebraic homomorphism and $\Delta(1) \neq 1 \otimes 1$ (see Definition 1.1). Then we study properties of various modular elements of GWBF-algebras. As an application of our theory, we mainly show Radford's formula for S^4 in the setting of GWBF-algebras.

The paper is organized as follows. In Section 1 we present axioms for what we call a *generalized weak biFrobenius algebra* (or briefly *GWBF algebra*). The concept is designed to cover both weak Hopf algebras and bi-Frobenius algebras. Also, a lot of new examples are given in this section. In Section 2, we study modular properties of Nakayama automorphisms related to integrals. As an application we prove in Section 3 Radford's formula for the fourth power of the antipode of GWBF-algebras still holds in the setting of GWBF-algebras (see Theorem 3.2).

Throughout, $\mathbf{K}$ is the fixed field. Unless otherwise stated, all vector spaces are over $\mathbf{K}$ and all maps are $\mathbf{K}$ -linear. By the symbol id , we mean the identity map and $\mathbf{K}^\times$ the multiplication group of $\mathbf{K}$. Let C be a coalgebra with a coproduct Δ . We will use the Heyneman-Sweedler's notation, $\Delta(c) = \sum c_1 \omega c_2$ for all $c \in C$, for coproduct (summation understood). We will write Δ^2 for $(id\omega\Delta)\Delta = (\Delta\omega id)\Delta$ for the coassociative axiom.

Let H be both an algebra and a coalgebra. Then the dual algebra $H^* = \text{Hom}(H, \mathbf{K})$ has a two-sided H -module structure

$$(h \rightharpoonup f)(x) = f(xh) \quad \text{and} \quad (f \leftharpoonup h)(x) = f(hx)$$

for all $h, x \in H$ and $f \in H^*$, and H has a two-sided H^* -module structure

$$f \rightharpoonup h = h_1 f(h_2) \quad \text{and} \quad h \leftharpoonup f = f(h_1) h_2$$

for all $f \in H^*$ and $h \in H$.

1. Definition and examples. In this section, we will introduce the concept of a generalized weak biFrobenius algebra and provide a list of examples as well.

Definition 1.1. Let H be a finite dimensional algebra with unit 1 and coalgebra with counit ε , $0 \neq \psi \in H^*$. There is a bijective map $S : H \rightarrow H$ satisfying, for all $h, g \in H$,

$$(1.1) \quad \psi(hg_1)S(g_2) = \psi(h_1g)h_2,$$

The data (H, ψ, S) is called a *generalized weak biFrobenius algebra* (or *GWBF-algebra*) if the following conditions hold:

(GWBF1) Unit 1 satisfies the following identities:

$$(\Delta \otimes \text{id}) \Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1);$$

(GWBF2) Counit ε satisfies the following identities:

$$\varepsilon(fgh) = \varepsilon(fg_1)\varepsilon(g_2h) = \varepsilon(fg_2)\varepsilon(g_1h);$$

(GWBF3) The pair (H, ψ) is a Frobenius algebra, i.e.,

$$\psi \leftarrow H = H^*;$$

(GWBF4) The map S is an anti-algebra map, i.e.,

$$S(hg) = S(g)S(h), \quad S(1) = 1;$$

(GWBF5) The map S is also an anti-coalgebra map, i.e.,

$$\Delta(S(h)) = \sum S(h_2) \otimes S(h_1), \quad \varepsilon(S(h)) = \varepsilon(h).$$

for all $f, g, h \in H$.

Remark 1.2. (i) We notice that Δ and ε are not necessarily algebra homomorphisms and $\Delta(1) \neq 1 \otimes 1$.

(ii) Since S is invertible, its inverse will be denoted by S^{-1} .

(iii) From (GWBFB3), we have the following k -linear isomorphisms:

$$\tau : H \simeq H^*, \quad h \mapsto \psi \leftarrow h,$$

$$\theta : H \simeq H^*, \quad h \mapsto h \rightarrow \psi.$$

Let (H, ψ, S) be GWBF-algebra. We define the maps $\varepsilon_t, \varepsilon_s : H \rightarrow H$ by the formulas

$$\varepsilon_t(x) = \sum \varepsilon(1_1 x) 1_2, \quad \text{and} \quad \varepsilon_s(x) = \sum 1_1 \varepsilon(x 1_2)$$

and denote the image $\varepsilon_t(H)$ and $\varepsilon_s(H)$ by H_t and H_s , respectively.

By (GWBFB2), one immediately obtains the following identities:

$$(1.2) \quad \varepsilon(x \varepsilon_t(y)) = \varepsilon(xy),$$

$$(1.3) \quad \varepsilon(\varepsilon_s(x)y) = \varepsilon(xy),$$

$$(1.4) \quad \varepsilon_t \circ \varepsilon_t = \varepsilon_t,$$

$$(1.5) \quad \varepsilon_s \circ \varepsilon_s = \varepsilon_s.$$

Example 1.3 (Finite dimensional weak Hopf algebra). Let H be a weak Hopf algebra with an antipode S in the sense of [2, 3, 8]. Let $\psi \in \int_{H^*}^r$ be a right integral on H . Then by [10, equation (17)], we get $\psi(hg_1)S(g_2) = \psi(h_1g)h_2$ for all $h, g \in H$. Thus, H is a GWBF-algebra.

Example 1.4. Let H and L be GWBF-algebras. Then the tensor product algebra $H \otimes L$ with the tensor coproduct coalgebra is also a GWBF-algebra. In fact, we define the antipode $S_{H \otimes L} = S_H \otimes S_L$ and the integral $\psi_{H \otimes L} = \psi_H \otimes \psi_L$. It is now clear that $(H \otimes L, S_{H \otimes L}, \psi_{H \otimes L})$ is a GWBF-algebra.

Example 1.5. Fix a natural number $n \in \mathbf{N}$ and set $\mathbf{N}_{\leq n} = \{0, 1, 2, \dots, n\}$. We define $A^{\leq n}$ as the complex algebra generated by a set $\{X_{i,0}, X_{i,1}, Y_{j,0}, Y_{j,1} \mid i, j \in \mathbf{N}_{\leq n}\}$ with the following relations:

(i) $X_{0,0} = Y_{0,0}$ is the unit;

(ii) $X_{s,t} = Y_{s,t} = 0$, for all $s > n, t = 0, 1$;

(iii) $X_{i,1} X_{j,1} = X_{i+j,1}$, $Y_{i,1} Y_{j,1} = Y_{i+j,1}$, $X_{i,1} Y_{j,1} = 0 = Y_{j,1} X_{i,1}$, and $X_{i,0} = Y_{i,0} = X_{i,1} + Y_{i,1}$ for all $i, j \in \mathbf{N}_{\leq n}$.

- The coproduct on $A^{\leq n}$ is given by:

$$\begin{aligned}\Delta(X_{0,1}) &= X_{0,1} \otimes X_{0,1}, \\ \Delta(Y_{0,1}) &= Y_{0,1} \otimes Y_{0,1}, \\ \Delta(X_{i,1}) &= X_{0,1} \otimes X_{i,1} + X_{i,1} \otimes X_{0,1}, \\ \Delta(X_{n,1}) &= \sum_{s=0}^n X_{s,1} \otimes X_{n-s,1}, \\ \Delta(Y_{i,1}) &= Y_{0,1} \otimes Y_{i,1} + Y_{i,1} \otimes Y_{0,1}, \\ \Delta(Y_{n,1}) &= \sum_{s=0}^n Y_{s,1} \otimes Y_{n-s,1}\end{aligned}$$

for all $0 < i < n$.

It is straightforward to check that Δ as above is coassociative. If $n \geq 2$, it is straightforward to check that Δ is not an algebra map. For example, take $n = 5$; we have $X_{5,1} = X_{4,1}X_{1,1}$ and

$$\begin{aligned}\Delta(X_{4,1})\Delta(X_{1,1}) &= (X_{0,1} \otimes X_{4,1} + X_{4,1} \otimes X_{0,1})(X_{0,1} \otimes X_{1,1} + X_{1,1} \otimes X_{0,1}) \\ &= X_{0,1} \otimes X_{5,1} + X_{1,1} \otimes X_{4,1} + X_{4,1} \otimes X_{1,1} + X_{5,1} \otimes X_{0,1},\end{aligned}$$

and this is not equal to $\Delta(X_{5,1})$.

We note that $\Delta(X_{0,0}) = \Delta(X_{0,1}) + \Delta(Y_{0,1}) \neq X_{0,0} \otimes X_{0,0}$.

- The counit on $A^{\leq n}$ is given by:

$$\varepsilon(X_{i,1}) = \delta_{i,0} \quad \text{and} \quad \varepsilon(Y_{i,1}) = \delta_{i,0}$$

for all $i \in \mathbf{N}_{\leq n}$.

- The left and right integral ψ are the same and given by:

$$\psi(X_{i,1}) = \delta_{i,n}, \quad \psi(Y_{i,1}) = \delta_{i,n}$$

for all $i \in \mathbf{N}_{\leq n}$.

- Define a linear map S on $A^{\leq n}$ as the identity map from $A^{\leq n}$ to $A^{\leq n}$.

It is straightforward to check that $A^{\leq n}$ is a GWBF-algebra.

Example 1.6. Fix an odd natural number $n \in \mathbf{N}$. Let $\mathbf{N}_{\leq n} = \{0, 1, 2, \dots, n\}$. Denote the set of even numbers in $\mathbf{N}_{\leq n}$ by $\mathbf{N}_{\leq n(ev)}$, and denote the set of odd numbers in $\mathbf{N}_{\leq n-1}$ by $\mathbf{N}_{\leq n(od)}$. Let $A^{\leq n(od)}$ be the algebra as in Example 1.5 with the same counit as in Example 1.5, but with a different coproduct structure given by:

$$\begin{aligned}
\Delta(X_{0,1}) &= X_{0,1} \otimes X_{0,1}, \\
\Delta(Y_{0,1}) &= Y_{0,1} \otimes Y_{0,1}, \\
\Delta(X_{i,1}) &= X_{0,1} \otimes X_{i,1} + X_{i,1} \otimes X_{0,1} \quad \text{for all } i \in \mathbf{N}_{\leq n(od)}, \\
\Delta(X_{i,1}) &= X_{0,1} \otimes X_{i,1} + X_{1,1} \otimes X_{i-1,1} \\
&\quad + X_{i-1,1} \otimes X_{1,1} + X_{i,1} \otimes X_{0,1} \quad \text{for all } i \in \mathbf{N}_{\leq n(ev)}, \\
\Delta(X_{n,1}) &= \sum_{s=0}^n X_{s,1} \otimes X_{n-s,1}, \\
\Delta(Y_{j,1}) &= Y_{0,1} \otimes Y_{j,1} + Y_{j,1} \otimes Y_{0,1} \quad \text{for all } j \in \mathbf{N}_{\leq n(od)}, \\
\Delta(Y_{j,1}) &= Y_{0,1} \otimes Y_{j,1} + Y_{1,1} \otimes Y_{j-1,1} \\
&\quad + Y_{j-1,1} \otimes Y_{1,1} + X_{j,1} \otimes Y_{0,1} \quad \text{for all } j \in \mathbf{N}_{\leq n(ev)}, \\
\Delta(Y_{n,1}) &= \sum_{s=0}^n Y_{s,1} \otimes Y_{n-s,1}.
\end{aligned}$$

It is straightforward to check that $A^{\leq n(od)}$ has the same integrals and antipode as in Example 1.5, and that $A^{\leq n(od)}$ is a GWBF-algebra.

Remark. From Examples 1.5–1.6, we can get a lot of GWBF-algebras which are neither weak Hopf algebras nor bi-Frobenius algebras.

2. Modular properties of the integrals. In this section, let (H, ψ, S) be a GWBF-algebra. Then we will introduce the notion of an integral and study modular properties of the integrals.

Definition 2.1. A nonzero element φ (respectively, ψ) $\in H^*$ is called a left (respectively, right) *integral*, if it satisfies the following condition:

$$\varphi(h_2)h_1 = \varphi(1_2h)S(1_1) \quad (\text{respectively } \psi(h_1)h_2 = \psi(1_1h)S(1_2)),$$

for all $h \in H$.

Remark. (i) Let $g = 1$ in equation (1.1). Then we have that ψ is a right integral in H^* .

(ii) $\varphi = \psi \circ S$ is a left integral and

$$\varphi(h_2y)S(h_1) = \varphi(hy_2)y_1,$$

for all $h, y \in H$.

Let A be an algebra over $\mathbf{K}$. Recall that A is Frobenius, if there is a Frobenius system (φ, e) for A , which in turn consists of a $\mathbf{K}$ -linear map $\varphi : A \rightarrow \mathbf{K}$ and an element $e = \sum e_1 \otimes e_2 \in A \otimes A$ such that, for all $a \in A$,

$$\sum \varphi(ae_1)e_2 = a = \sum e_1\varphi(e_2a).$$

Equivalently, A is finite dimensional, and there is a $\mathbf{K}$ -linear map $\varphi : A \rightarrow \mathbf{K}$ such that the bilinear form $B_\varphi : A \otimes A \rightarrow \mathbf{K}$ given by $B_\varphi(a, a') = \varphi(aa')$ is nondegenerate. It follows that $e \in A \otimes A$ is a Casimir element in the sense that $(a \otimes 1)e = e(1 \otimes a)$.

We see that, for all $a, a' \in A$, the formula $\varphi(aa') = \varphi(a'\eta(a))$ defines an automorphism η of A , which has the alternative definition

$$\eta(a) = \sum e_1\varphi(ae_2),$$

called the *Nakayama automorphism*.

Given a GWBF-algebra (H, ψ, S) , we notice that θ defined in Remark 1.2 is known as a *Frobenius isomorphism* and, from θ , we can get a Frobenius system $(\psi, b^i = \theta^{-1}(f^i), e^i)$, where θ^{-1} is the inverse of θ and (e^i, f^i) a dual base of H . We can have the Nakayama automorphism

$$\mathbf{N}(h) = \sum_i \theta^{-1}(f^i)\psi(he^i),$$

for all $h \in H$.

Since τ is a linear isomorphism, there exists an element $\kappa \in H$ such that

$$\psi \leftarrow \kappa = \varepsilon.$$

Set $\alpha = \kappa \rightarrow \psi$. Then we define a k -linear map

$$\chi(f) = S^{-2}(h \leftarrow \alpha),$$

for all $f \in H^*$, where h is uniquely determined by τ in Remark 1.2 such that $\psi \leftarrow h = f$.

Lemma 2.2. *With χ defined above, then $\chi = \theta^{-1}$.*

Proof. It is sufficient to check that χ is a right inverse of θ . In fact, for all $h \in H$,

$$\begin{aligned}
 \chi \circ \theta(h) &= \chi(h \rightarrow \psi) \\
 &= S^{-2}(h' \leftarrow \alpha) \quad (\text{since } h \rightarrow \psi = \psi \leftarrow h') \\
 &= S^{-2}(h'_2)\psi(h'_1\kappa) \\
 &\stackrel{(1.1)}{=} S^{-1}(\kappa_2)\psi(h'\kappa_1) \\
 &= S^{-1}(\kappa_2)\psi(\kappa_1h) \\
 &\stackrel{(1.1)}{=} h_2\psi(\kappa h_1) = h.
 \end{aligned}$$

The proof of the lemma is completed. $\square$

Remark. (i) From Lemma 2.2, the Nakayama automorphism $\mathbf{N}$ is

$$\mathbf{N}(h) = \sum_i \chi(f^i)\psi(h e^i) = \chi(\psi \leftarrow h) = S^{-2}(h \leftarrow \alpha).$$

Moreover, the map

$$\xi = S^2 \circ \mathbf{N} : H \longrightarrow H, \quad h \mapsto h \leftarrow \alpha$$

is an algebraic automorphism. Hence, α is $*$ -invertible, and its inverse is denoted by α^{-1} . Using ξ as an algebraic automorphism, we can follow that $\alpha_1 \otimes \alpha_2 = \alpha \varepsilon_1 \otimes \alpha \varepsilon_2$.

(ii) From Lemma 2.2, we have an extra bonus as follows: since $\chi = \theta^{-1}$, we have $\theta \circ \chi = \text{id}_{H^*}$. Hence, for all $f \in H^*$, it follows that $\theta \circ \chi(f) = f$. For $a \in H$, we have

$$\begin{aligned}
 \theta \circ \chi(f)(a) &= (S^{-2}(h \leftarrow \alpha) \rightarrow \psi)(a) \quad (\psi \leftarrow h = f) \\
 &= \psi(a S^{-2}(h_2))\psi(h_1\kappa) \\
 &= \psi(a S^{-1}(\kappa_2))\psi(h\kappa_1)
 \end{aligned}$$

$$\begin{aligned}
&= \psi(aS^{-1}(\kappa_2))f(\kappa_1) \\
&= f(\psi(aS^{-1}(\kappa_2))\kappa_1),
\end{aligned}$$

i.e., $\psi(aS^{-1}(\kappa_2))\kappa_1 = a$. Then $(S^{-1}(\kappa_2), \kappa_1)$ is a dual base for ψ .

Given a GWBF-algebra (H, ψ, S) , it follows from (2) in the remark above that we can have such a Frobenius system $(\psi, S^{-1}(\kappa_2), \kappa_1)$. Using the anti-automorphism $S^{-1} : H \rightarrow H$, we have another Frobenius system as follows:

$$(\varphi = \psi \circ S, S^{-1}(\kappa_1), S^{-2}(\kappa_2)).$$

The Nakayama automorphism of φ is $\mathfrak{N} = S^{-1} \circ \mathbf{N}^{-1} \circ S$. By the uniqueness of Frobenius systems, we have the following result.

Proposition 2.3. *Let (H, ψ, S) be a GWBF-algebra. Then there exists an invertible element δ such that*

$$(2.1) \quad \psi(S(g)) = \psi(\delta g),$$

$$(2.2) \quad \psi(h_2)h_1 = \psi(hS(1_1))\delta 1_2,$$

$$(2.3) \quad \varepsilon_s(\delta) = 1,$$

$$(2.4) \quad \Delta(\delta) = \delta 1_1 \otimes \delta 1_2,$$

where $\delta = \sum_i \psi(\kappa_2)\kappa_1 = \psi \rightharpoonup \kappa$.

Proof. Let $\sigma = \psi(S^{-1}(\kappa_1))S^{-2}(\kappa_2)$. It is straightforward that σ is the inverse of δ .

For equation (2.2), for all $h, g \in H$, applying ψ to equation (1.1), we deduce that

$$\psi(hg_1)\psi(S(g_2)) = \psi(h_1g)\psi(h_2).$$

Since $\psi \circ S$ is a left integral, so we have

$$\psi(hS(1_1))\psi(S(1_2g)) = \psi(h_1g)\psi(h_2).$$

Thus,

$$\begin{aligned}
\psi(hS(1_1))\psi(S(1_2g)) &\stackrel{(2.1)}{=} \psi(hS(1_1))\psi(\delta 1_2g) \\
&= \psi(\psi(hS(1_1))\delta 1_2g) \\
&= \psi(\psi(h_2)h_1g).
\end{aligned}$$

From the nondegeneracy of ψ , it is implied that

$$\psi(hS(1_1))\delta 1_2 = \psi(h_2)h_1.$$

Finally, applying ε to both sides of equation (2.2), we have

$$\psi(h) = \psi(hS(1_1))\varepsilon(\delta 1_2) = \psi(hS(\varepsilon_s(\delta))).$$

By the nondegeneracy of ψ , we get $\varepsilon_s(\delta) = 1$.

We are left to check the equation (2.4). Since $(S^{-1}(\kappa_2), \kappa_1)$ is a dual base for ψ , we have $\sum \kappa_1 \otimes \kappa_2 S(h) = \sum \kappa_1 h \otimes \kappa_2$, for all $h \in H$. Since

$$\begin{aligned} \Delta(\delta) &= \kappa_1 \otimes k_2 \psi(\kappa_3) = \kappa_1 \otimes \psi(\kappa_2 S(1_1))\delta 1_2 \\ &= \psi(\kappa_2)\kappa_1 1_1 \otimes \delta 1_2. \end{aligned}$$

This finishes the proof of the proposition. $\square$

Now, we say that the elements $\delta = \psi \rightharpoonup \kappa$ and $\alpha = \kappa \rightharpoonup \psi$ are *modular* elements of H and H^* , respectively.

Let $\Omega = S^{-1}\mathbf{N}S$. Immediately, we have $\varphi(hg) = \varphi(\Omega(g)h)$ (also called the Nakayama automorphism of φ , for all $h, g \in H$). Next, we shall list more properties associated to Ω and $\mathbf{N}$.

Proposition 2.4. *Let (H, ψ, S) be a GWBF-algebra. Then, for all $h, x \in H$,*

- (i) $\varphi(h_1)h_2 = \varphi(1_1h)S^{-1}(\delta 1_2)$
- (ii) $\varphi(S^{-1}(\delta 1_2)x)1_1 = \psi(x1_1)\Omega(S(1_2))$,
- (iii) $\varphi(S^{-1}(\delta)x) = \psi((\alpha \rightharpoonup 1)x)$.

Proof. (i) From the equation $\psi(h_2)h_1 = \psi(hS(1_1))\delta 1_2$, we deduce that

$$\varphi(h_1)h_2 = \varphi(1_1h)S^{-1}(\delta 1_2),$$

for all $h \in H$.

(ii) For all $x \in H$, we have $\varphi(h_1)h_2x = \varphi(1_1h)S^{-1}(\delta 1_2)x$, then applying φ to it, we get

$$\varphi(h_1)\varphi(h_2x) = \varphi(1_1h)\varphi(S^{-1}(\delta 1_2)x).$$

Since

$$\begin{aligned}
 \varphi(h_1)\varphi(h_2x) &= \varphi(S^{-1}(\varphi(h_2x)S(h_1))) \\
 &= \varphi(S^{-1}(\varphi(hx_2)x_1)) \text{ (}\varphi \text{ a left integral)} \\
 &= \psi(x_1)\varphi(hx_2) \text{ (}\psi \text{ a right integral)} \\
 &= \psi(x1_1)\varphi(hS(1_2)),
 \end{aligned}$$

we get

$$\begin{aligned}
 \varphi(h_1)\varphi(h_2x) &= \varphi(\varphi(S^{-1}(\delta 1_2)x)1_1h) \\
 &= \varphi(\psi(x1_1)hS(1_2)) \\
 &= \varphi(\psi(x1_1)\Omega(S(1_2))h).
 \end{aligned}$$

(iii) Applying ε to $\varphi(S^{-1}(\delta 1_2)x)1_1 = \psi(x1_1)\Omega S(1_2)$, we get

$$\begin{aligned}
 \varphi(S^{-1}(\delta)x) &= \psi(x1_1)\varepsilon \circ \Omega(S(1_2)) \\
 &= \psi(x1_1)\psi(tNS^2(1_2)) \\
 &= \psi(x1_1)\psi(S^2(1_2)t) \\
 &= \psi(x1_1)\alpha(S^2(1_2)) \\
 &= \psi(x_1)\alpha(S(x_2)) \\
 &= \psi(1_1x)\alpha(1_2) \\
 &= \psi((\alpha \rightharpoonup 1)x).
 \end{aligned}$$

Thus,

$$\varphi(S^{-1}(\delta)x) = \psi((\alpha \rightharpoonup 1)x).$$

This finishes the proof of the proposition. $\square$

By equation (2.1) and the third equation in Proposition 2.4, we have $\psi(\delta S^{-1}(\delta)x) = \psi((\alpha \rightharpoonup 1)x)$. Thus, $\delta S^{-1}(\delta) = \alpha \rightharpoonup 1$; since δ is invertible, it follows that $\omega = \alpha \rightharpoonup 1$ is invertible, and $S(\delta) = \delta^{-1}S(w)$.

Proposition 2.5. *Let (H, ψ, S) be a GWBF-algebra. Then, for all $h \in H$,*

- (i) $\triangle(\mathbf{N}(h)) = (\mathbf{N} \otimes S^{-2}) \triangle(h)$ and $\triangle(\mathbf{N}^{-1}(h)) = (\mathbf{N}^{-1} \otimes S^2) \triangle(h)$,
- (ii) $\triangle(\Omega^{-1}(h)) = (S^2 \otimes \Omega^{-1}) \triangle(h)$ and $\triangle(\Omega(h)) = (S^{-2} \otimes \Omega) \triangle(h)$,

$$(iii) \mathbf{N}^{-1}(h) = \delta\Omega(h)\delta^{-1}.$$

Proof. (i) For all $h \in H$, we compute

$$\begin{aligned} S^{-2}(h_2)\psi(g\mathbf{N}(h_1)) &= S^{-2}(h_2)\psi(h_1g) \\ &= S^{-1}(g_2)\psi(hg_1) \\ &= S^{-1}(g_2)\psi(g_1\mathbf{N}(h)) \\ &= \mathbf{N}(h)_2\psi(g\mathbf{N}(h)_1). \end{aligned}$$

It follows from the nondegeneracy of ψ that $\triangle(\mathbf{N}(h)) = (\mathbf{N} \otimes S^{-2})\triangle(h)$. By the equation $\psi(g\mathbf{N}(h)) = \psi(\mathbf{N}^{-1}(h)g)$, we obtain the second formula.

(ii) By a similar way as that of part (i) and noticing that $\varphi(hg) = \varphi(g\Omega^{-1}(h))$, we finish the proof of part (ii).

(iii) For all $g, h \in H$, since $\psi(h) = \varphi(\delta^{-1}h)$, we have

$$\psi(hg) = \varphi(\delta^{-1}hg) = \varphi(\Omega(g)\delta^{-1}h)$$

and

$$\psi(hg) = \psi(\mathbf{N}^{-1}(g)h) = \varphi(\delta^{-1}\mathbf{N}^{-1}(g)h).$$

By the nondegeneracy of φ , we get the equation of part (3). $\square$

3. Radford's formula. In this section, we will prove Radford's formula for S^4 still holds in the setting of GWBF-algebras as an application of our theory established as in Section 2.

Let $\mathbf{Ad}_u$ denote conjugation by a unit $u \in H$, where $\mathbf{Ad}_u(x) = uxu^{-1}$, for all $x \in H$. First, we recall an important result as follows.

Lemma 3.1 [7, Lemma 3.1]. *If H is a Frobenius algebra with Nakayama automorphism η and anti-automorphism $S : H \rightarrow H$, then there is an invertible element $d \in H$ such that*

$$\mathbf{Ad}_{d^{-1}} = S \circ \eta \circ S^{-1} \circ \eta.$$

Remark. Lemma 3.1 can be viewed as a simplification of equation (3) in Proposition 2.5 as follows: for all $h \in H$, we apply equation (3) to h and yield

$$\mathbf{N}^{-1}(h) = \delta(S^{-1} \circ \mathbf{N} \circ S \circ (h))\delta^{-1}.$$

Hence, we deduce that

$$\delta^{-1}h\delta = S^{-1} \circ \mathbf{N} \circ S \circ \mathbf{N}(h).$$

As we know, the Nakayama automorphism $\mathbf{N}$ of ψ is

$$\mathbf{N}(h) = \sum_i \chi(f^i)\psi(he^i) = \chi(\psi \leftarrow h) = S^{-2}(h \leftarrow \alpha).$$

Applying the last equation in the remark above to $h \in H$ yields

$$S^{-1}(S^{-2}(S^{-1}(h \leftarrow \alpha)) \leftarrow \alpha \circ S^2) = \delta^{-1}h\delta.$$

Since $S^{-1}(h \leftarrow \alpha) = \alpha \circ S \rightarrow S^{-1}(h)$ for $h \in H$, this last equation simplifies to

$$\alpha \circ S^3 \rightarrow S^{-4}(h) \leftarrow \alpha \circ S^4 = \delta^{-1}h\delta.$$

So we can get Radford's formula for the fourth power of S .

Theorem 3.2. *Take the notations in Section 2. Then, for all $h \in H$,*

$$S^4(h) = \delta(\beta \rightarrow h \leftarrow \beta \circ S)\delta^{-1},$$

where $\beta = \alpha \circ S^3$.

Given a GWBF-algebra (H, ψ, S) , we have $S(\delta) = \delta^{-1}S(w)$, where $w = \alpha \rightarrow 1$. Then $S(\delta) = \delta^{-1} \Leftrightarrow \alpha \rightarrow 1 = 1$. If we make the following assumption that κ is S -fixed, i.e., $S(\kappa) = \kappa$. Since

$$\begin{aligned} S^{-1}(\delta) &= \psi(\kappa_2)S^{-1}(\kappa_1) = \psi(S^{-1}(\kappa)_2)S^{-1}(S^{-1}(\kappa)_1) \\ &= \psi(S^{-1}(\kappa_1))S^{-2}(\kappa_2) = \sigma = \delta^{-1}. \end{aligned}$$

It follows that $S(\delta) = \delta^{-1}$. Also, we make the following assumption $\alpha \rightarrow 1 = 1$ which holds in weak Hopf algebras (cf. [9]).

With the different assumptions, we always have $S(\delta) = \delta^{-1}$. Next, we shall check that $\alpha \circ S = \alpha^{-1}$; several lemmas are needed.

Lemma 3.3. *Let $\Omega = S^{-1}\mathbf{N}S$. Then*

$$(3.1) \quad \alpha(S(h)) = \psi(\mathbf{N}(b)h).$$

Proof. Applying ε to equation (iii) in Proposition 2.5, we have

$$\begin{aligned}\varepsilon(\delta S^{-1} \mathbf{N} S(h) \delta^{-1}) &= \varepsilon(\varepsilon_s(\delta) S^{-1} \mathbf{N} S(h) \delta^{-1}) \\ &= \varepsilon(S^{-1}(\delta \mathbf{N} S(h))) = \varepsilon(\delta \mathbf{N} S(h)) \\ &= \varepsilon(\varepsilon_s(\delta) \mathbf{N} S(h)) = \varepsilon(\mathbf{N} S(h)) \\ &= \varepsilon(\mathbf{N}^{-1}(h)).\end{aligned}$$

Since

$$\varepsilon(\mathbf{N} S(h)) = \psi(t \mathbf{N} S(h)) = \psi(S(h)t) = \alpha(S(h))$$

and

$$\varepsilon(\mathbf{N}^{-1}(h)) = \psi(t \mathbf{N}^{-1}(h)) = \psi(\mathbf{N}^{-1}(h) \mathbf{N}(t)) = \psi(\mathbf{N}(t)h).$$

Thus,

$$\alpha(S(h)) = \psi(\mathbf{N}(b)h). \quad \square$$

Lemma 3.4. $\alpha \circ S \circ \mathbf{N} = \varepsilon$.

Proof. For all $h \in H$, we compute

$$\begin{aligned}\alpha \circ S \circ \mathbf{N}(h) &= \psi(S(\mathbf{N}(h))t) \stackrel{(3.1)}{=} \psi(\mathbf{N}(t) \mathbf{N}(h)) \\ &= \psi(\mathbf{N}(th)) = \psi(th) = \varepsilon(h). \quad \square\end{aligned}$$

Lemma 3.5. $\alpha * (\alpha \circ S^{-1}) = \varepsilon$ and $\alpha^{-1} = \alpha \circ S = \alpha \circ S^{-1}$.

Proof. For all $h \in H$, we compute

$$\begin{aligned}\alpha * (\alpha \circ S^{-1})(h) &= \alpha(h_1) \alpha(S^{-1}(h_2)) \\ &= (\alpha \circ S^{-1})(h \leftarrow \alpha) \\ &= (\alpha \circ S \circ \mathbf{N})(h) \\ &= \varepsilon(h). \quad \square\end{aligned}$$

From Lemma 3.5 and Theorem 3.2, we have

Corollary 3.3. *Take the notations in Section 2. Then, for all $h \in H$,*

$$S^4(h) = \delta(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)\delta^{-1}.$$

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