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A NOTE ON CHARACTERIZING COMPLETELY MULTIPLICATIVE FUNCTIONS USING GENERALIZED MÖBIUS FUNCTIONS

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Dedicated to the memory of Professor Temba Shonhiwa

ABSTRACT. In 1997, Haukkanen showed that, if f is a completely multiplicative arithmetic function, then $f^\alpha = \mu_{-\alpha} f$ for any real number α, where μ_α is the Souriau-Hsu-Möbius function. In 2002, Laohakosol, Pabhapote and Wechwiriyakul established that the converse holds (under certain conditions). In the present paper, we provide a very short and simple proof for this converse result using the ideas of Haukkanen from 1997.

1. Introduction. An *arithmetic function* is a complex-valued function defined on the set of positive integers $\mathbf{Z}^+$. Let $\mathbf{A}$ denote the set of arithmetic functions. The *Dirichlet product* (*or convolution*) of $f, g \in \mathbf{A}$, denoted by $f * g \in \mathbf{A}$, is defined by

$$(f * g)(n) = \sum_{ab=n} f(a)g(b) \tag{1.1}$$

for all $n \in \mathbf{Z}^+$, where the sum on the right-hand side of (1.1) runs over all ordered pairs (a, b) of positive integers satisfying $ab = n$. The *usual product* of $f, g \in \mathbf{A}$, denoted by $fg \in \mathbf{A}$, is defined by

$$(fg)(n) = f(n)g(n) \tag{1.2}$$

for all $n \in \mathbf{Z}^+$.

An arithmetic function f with $f(1) = 1$ is said to be *multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbf{Z}^+$ with $(m, n) = 1$. Here, as usual, (m, n) denotes the greatest common divisor of m and n.

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A multiplicative function f is said to be *completely multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbf{Z}^+$.

In 1966, Lambek [**14**] showed that $f \in \mathbf{A}$ with $f(1) \neq 0$ is completely multiplicative if and only if f *distributes* over every Dirichlet product; that is, $f \in \mathbf{A}$ with $f(1) \neq 0$ is completely multiplicative if and only if $f(g * h) = fg * fh$ for all $g, h \in \mathbf{A}$.

Subsequently, several mathematicians showed interest in characterizing completely multiplicative functions involving distributivity. In 1970, Sivaramakrishnan [**27**] showed that a multiplicative function f is completely multiplicative if and only if f distributes over $\varphi * 1$, where φ is the Euler totient function and 1 denotes the constant function 1 defined by $1(n) = 1$ for all $n \in \mathbf{Z}^+$. In 1971, Carlitz [**3**] showed that $f \in \mathbf{A}$ with $f(1) \neq 0$ is completely multiplicative if and only if f distributes over $1 * 1$. In the same year, Apostol [**1**, Theorem 2] proved, among other results, that a multiplicative function f is completely multiplicative if and only if $f^{-1} = \mu f$, where μ is the Möbius function and f^{-1} denotes the inverse of f with respect to the Dirichlet convolution $*$, or equivalently, a multiplicative function f is completely multiplicative if and only if f distributes over $\mu * 1$.

In 1973, Langford [**19**] obtained sufficient conditions on distributivity, which ensure the complete multiplicativity of an arithmetic function. His [**19**] results contain the results of Lambek [**14**], Sivaramakrishnan [**27**], Carlitz [**3**] and Apostol [**1**] as particular cases.

It is also interesting to note that there are certain other characterizations of completely multiplicative functions that appear in the literature. For example, Carroll [**5**] characterized completely multiplicative functions by using Rearick's [**21**] logarithmic operator; Haukkanen [**10**] characterized these functions by using Ivić's [**12**] extended von Mongoldt's function. It may be noted that Haukkanen [**8**] also characterized completely multiplicative functions by using the logarithmic operator of Carlitz and Subbarao [**4**]. Further, Ivić [**13**] found necessary and sufficient conditions for a multiplicative function to be completely multiplicative in terms of Ramanujan's sum. The results of Ivić [**13**] have been generalized by Redmond [**23**] and Haukkanen [**7**]. For further, recent papers on characterizations of completely multiplicative functions involving distributivity, logarithmic operators and extended von Mongoldt's function, we refer to [**15, 16, 25, 26, 29**].

For any real number α, let μ_α denote the multiplicative function defined by $\mu_\alpha(1) = 1$ and

$$\mu_\alpha(p^k) = (-1)^k \binom{\alpha}{k} \tag{1.3}$$

for all primes p and $k \in \mathbf{Z}^+$, where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}. \tag{1.4}$$

The function μ_α has been studied by the authors in [**2, 11**], and it is called the generalized Möbius function. The authors in [**18**] refer to this Möbius function as the *Souriau-Hsu-Möbius function*, see [**28**]. Clearly, $\mu_1 = \mu$, the usual Möbius function, $\mu_{-1} = 1$, the constant function 1, and $\mu_0 = e$, the identity under the Dirichlet convolution, where $e \in \mathbf{A}$ is defined by

$$e(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1.5}$$

A basic property of the function μ_α is

$$\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta} \tag{1.6}$$

for all real numbers α and β (see [**2**]). It appears that Popovici [**20**] introduced and studied μ_α when α is a positive integer (see also Sándor and Bege [**24**]).

In [**6**], among other results, Carroll and Gioia explicitly evaluated the rth roots of completely multiplicative functions. Surprisingly, Haukkanen [**9**] found in a simple way the value of f^α for any completely multiplicative function f. He [**9**] proved that

$$f^\alpha = \mu_{-\alpha} f \tag{1.7}$$

for all real numbers α (for the definition of f^α we refer to Section 2 of the present paper). In 2002, the authors in [**17**] investigated the converse of (1.7). They [**17**] established the following interesting results.

Theorem. *Let f be a multiplicative function.*

(a) *Let $\alpha \in \mathbf{R} \setminus \{0\}$ be fixed. Then f is completely multiplicative if and only if $(\mu_\alpha f)^{-1} = \mu_{-\alpha} f$, the inverse being under the Dirichlet convolution.*

(b) *Let $\alpha \in \mathbf{R} \setminus \{0, 1\}$ be fixed. If $f^\alpha = \mu_{-\alpha} f$ and if, in addition, $f(p^{-\alpha-1}) = (f(p))^{-\alpha-1}$ for all primes p in the case α is an even integer ≤ -4, then f is completely multiplicative.*

The proofs of parts (a) and (b) in [**17**] involve large calculations in binomial coefficients and Rearick's [**21**] logarithmic operator. Also, the proof of part (b) is divided into many cases: integral and nonintegral etc.

As an easy consequence of the above theorem and the property $(f^\alpha)^{-1} = f^{-\alpha}$ (see (2.13)), we note that for a fixed $\alpha \in \mathbf{R} \setminus \{-1, 0\}$, a multiplicative function f is completely multiplicative if and only if $f^\alpha = (\mu_\alpha f)^{-1}$ and, in addition, $f(p^{\alpha-1}) = (f(p))^{\alpha-1}$ for all primes p in the case where α is an even integer ≥ 4. This property has not previously been presented in the literature.

The purpose this note is to provide a more simple proof for the above theorem (see Section 3) without tedious calculations. This is accomplished by invoking a result of Langford [**19**] and an idea of Haukkanen [**9**].

In Section 2 we present preliminaries and in Section 3 the proof of the theorem.

2. Preliminaries. Let

$$\mathbf{A}_1 = \{f \in \mathbf{A} : \ f(1) \in \mathbf{R}\} \tag{2.1}$$

and

$$P' = \{f \in \mathbf{A}_1 : f(1) > 0\}. \tag{2.2}$$

Let $L : (P', *) \to (\mathbf{A}_1, +)$ be defined by

$$(Lf)(1) = \log f(1), \tag{2.3}$$

and, for $n > 1$,

$$(Lf)(n) = \sum_{ab=n} f(a)(\log a) f^{-1}(b), \tag{2.4}$$

the inverse of f being with respect to the Dirichlet convolution $*$. In (2.3), log denotes the usual real logarithm function.

Rearick [**21**] showed that the operator L is an isomorphism. Let E denote the inverse of L. For $f \in P'$ and $\alpha \in \mathbf{R}$, Rearick [**21**] defined f^α by

$$f^\alpha = E(\alpha L f). \tag{2.5}$$

On the other hand, for $r \in \mathbf{Z}^+$, f^r is defined by $f^1 = f$ and $f^r = f^{r-1} * f$ if $r \geq 2$. As usual, $f^0 = e$. Further, $f^{-r} = (f^{-1})^r$ if $r \geq 1$. It is not difficult to see that, for any integer r, f^r coincides with f^α defined by (2.5) when $\alpha = r$.

Let $\mathrm{Log} : (P', *) \to (\mathbf{A}_1, +)$ be defined by

$$(\mathrm{Log}\, f)(1) = \log f(1), \tag{2.6}$$

and, for $n > 1$,

$$(\mathrm{Log}\, f)(n) = \frac{1}{\log n} \sum_{ab=n} f(a)(\log a) f^{-1}(b). \tag{2.7}$$

From (2.4) it is clear that $Lf(n) = (\log n)\mathrm{Log}\, f(n)$ if $n > 1$. Thus, Log is also an isomorphism. Let Exp denote the inverse of the isomorphism Log. Then f^α can also be defined by

$$f^\alpha = \mathrm{Exp}\,(\alpha \mathrm{Log}\, f), \tag{2.8}$$

where $\alpha \in \mathbf{R}$ and $f \in P'$ (see [**22**]).

It has been shown by Rearick [**22**, Theorem 6] that $\mathrm{Exp}\,(\alpha \mathrm{Log}\, f)$ and $E(\alpha L f)$ are the same. Therefore, f^α can either be defined by (2.5) or (2.8). The latter definition (that is, (2.8)) has been used by Haukkanen [**9**] to establish (1.7). The advantage in using (2.8) lies in the fact that the Exp operator can be written inductively (see [**22**]):

For $f \in \mathbf{A}_1$,

$$\mathrm{Exp}\, f(1) = \exp f(1), \tag{2.9}$$

and, for $n > 1$,

$$\mathrm{Exp}\, f(n) = \frac{1}{\log n} \sum_{\substack{ab=n \\ a<n}} \mathrm{Exp}\, f(a)(\log b) f(b). \tag{2.10}$$

In (2.9), exp denotes the usual real exponential function.

We have

$$f^\alpha(1) = (f(1))^\alpha, \tag{2.11}$$

and, for $n > 1$,

$$f^\alpha(n) = \frac{\alpha}{\log n} \sum_{\substack{ab=n \\ a<n}} f^\alpha(a)(\log b)\mathrm{Log}\, f(b) \tag{2.12}$$

for all $f \in P'$ and $\alpha \in \mathbf{R}$. Note that (2.12) is obtained by replacing f with $\alpha \mathrm{Log}\, f$ in (2.10). From the property $(\mathrm{Exp}\, g) * (\mathrm{Exp}\, h) = \mathrm{Exp}\,(g+h)$ for all $g, h \in \mathbf{A}_1$, it follows that $f^\alpha * f^{-\alpha} = e$, that is,

$$(f^\alpha)^{-1} = f^{-\alpha} \tag{2.13}$$

for all $f \in P'$ and $\alpha \in \mathbf{R}$.

Definition 2.1 (Langford [**19**]). The Dirichlet product $g * h$ of $g, h \in \mathbf{A}$ is said to be *partially discriminative* if, for any prime power $p^k > 1$, $(g * h)(p^k) = g(1)h(p^k) + h(1)g(p^k)$ implies that $k = 1$.

Proposition 2.1 (Langford [**19**, Theorem 2]). *A multiplicative function $f \in \mathcal{A}$ is completely multiplicative if and only if f distributes over a partially discriminative product.*

3. Proof of Theorem. (a) Let $\alpha \in \mathbf{R} \setminus \{0\}$ be fixed. We have $(\mu_\alpha f)^{-1} = \mu_{-\alpha} f$ if and only if $\mu_\alpha f * \mu_{-\alpha} f = e$. The second equality is equivalent to saying that f distributes over the Dirichlet product $\mu_\alpha * \mu_{-\alpha}$. This is a partially discriminative product according to Definition 2.1. In fact, for any prime p and positive integer $k \geq 2$,

$$(\mu_\alpha * \mu_{-\alpha})(p^k) = 0$$

and

$$\mu_\alpha(p^k) + \mu_{-\alpha}(p^k) = (-1)^k \left(\binom{\alpha}{k} + \binom{-\alpha}{k} \right) \neq 0,$$

since $\alpha \in \mathbf{R} \setminus \{0\}$. Therefore, by Proposition 2.1, f is completely multiplicative if and only if $(\mu_\alpha f)^{-1} = \mu_{-\alpha} f$.

(b) Let $\alpha \in \mathbf{R} \setminus \{0, 1\}$ be fixed. We assume that $f^\alpha = \mu_{-\alpha} f$ and, in addition, that $f(p^{-\alpha-1}) = (f(p))^{-\alpha-1}$ for all primes p in the case α is an even integer ≤ -4, and we show that f is completely multiplicative.

Since f is multiplicative, it suffices to show that

$$f(p^s) = (f(p))^s \tag{3.1}$$

for all primes p and $s \in \mathbf{Z}^+$.

Let p be an arbitrary but fixed prime. We proceed by induction on s to show that (3.1) holds. Clearly (3.1) holds when $s = 1$. We assume that (3.1) holds for all s with $1 \leq s \leq k-1$, where $k \geq 2$, and we show that (3.1) holds for $s = k$.

Let r be an integer ≥ 2. From (1.1) it is clear that

$$f^r(p^k) = \sum_{a_1+a_2+\cdots+a_r=k} f(p^{a_1}) f(p^{a_2}) \cdots f(p^{a_r}), \tag{3.2}$$

where the summation on the right-hand side of (3.2) is over all r-tuples $(a_1, a_2, \ldots, a_r)$ of nonnegative integers, whose sum is k. Using the induction hypothesis, we obtain

$$\begin{aligned} f^r(p^k) &= r f(p^k) + (f(p))^k \sum_{\substack{a_1+a_2+\cdots+a_r=k \\ a_i<k \\ 1\leq i\leq r}} 1 \\ &= r f(p^k) + (f(p))^k \, \Sigma', \end{aligned} \tag{3.3}$$

say. (Compare with [**16**, equation (3.3)].) In particular (for $f \equiv 1$),

$$\mu_{-r}(p^k) = 1^r(p^k) = \sum_{a_1+a_2+\cdots+a_r=k} 1 = r + \Sigma'. \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain

$$f^r(p^k) = r f(p^k) + (f(p))^k \left\{ \mu_{-r}(p^k) - r \right\}. \tag{3.5}$$

We now prove that, for any real number β,

$$f^\beta(p^k) = \beta f(p^k) + (f(p))^k \left\{ \mu_{-\beta}(p^k) - \beta \right\}. \tag{3.6}$$

We use the idea of Haukkanen [**9**]. From (2.11) and (2.12), it follows by induction on n that $f^{\beta}(n)$ is a polynomial in β. Thus, for each fixed prime p and $k \in \mathbf{Z}^+$, both sides of (3.6) are polynomials in β. By (3.5) it follows that these polynomials are satisfied by all integers $r \geq 2$. Thus (3.6) holds for all real β.

By assumption, $f^{\alpha} = \mu_{-\alpha} f$. Therefore, using (3.6) with $\beta = \alpha$ we obtain

$$\left\{\mu_{-\alpha}(p^k) - \alpha\right\} f(p^k) = \left\{\mu_{-\alpha}(p^k) - \alpha\right\} (f(p))^k$$

or

$$\left\{(-1)^k \binom{-\alpha}{k} - \alpha\right\} f(p^k) = \left\{(-1)^k \binom{-\alpha}{k} - \alpha\right\} (f(p))^k.$$

It is not difficult to show that, for $\alpha \neq 0, 1$,

$$(-1)^k \binom{-\alpha}{k} - \alpha = 0$$

if and only if k is odd and $k = -\alpha - 1$. Thus $f(p^k) = (f(p))^k$, that is, (3.1) holds for $s = k$. Note that if k is odd and $k = -\alpha - 1$, then $f(p^k) = (f(p))^k$ is the additional assumption in (b). This completes the proof of part (b).

The proof of the theorem is complete.

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