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SOME q-SERIES IDENTITIES RELATED TO THE q-TRIPLICATE INVERSE

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ABSTRACT. The q-triplicate inverse series relations are obtained by Zhang and Chen, who used the q-finite difference method. By applying the (f,g)-inversion formula by Ma in [**19**], the purpose of this paper is to give a simple proof of the q-triplicate inverse and derive some q-series identities. These results generalize Zhang and Chen's results [**20**].

1. Notation and introduction. We employ the notation and terminology in [**10**], and we always assume that $0 < |q| < 1$. The q-shifted factorial is defined by

$$(a;q)_0 = 1, (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k). \tag{1}$$

For brevity, we adopt the usual notation

$$(a_1, a_2, \dots , a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n. \tag{2}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{3}$$

As usual, the basic hypergeometric series ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots , a_{r+1} \\ b_1, b_2, \dots , b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots , a_{r+1}; q)_n}{(q, b_1, b_2, \dots , b_r; q)_n} z^n. \tag{4}$$

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Recall that an inversion formula or a reciprocal relation in the context of combinatorics is defined to be a pair of infinite dimensional lower-triangular matrices $F = (f_{n,k})_{n,k\in\mathbf{Z}}$ and $G = (g_{n,k})_{n,k\in\mathbf{Z}}$ over the complex field $\mathbf{C}$ such that

$$\sum_{n\geq i\geq k} f_{n,i}g_{i,k} = \delta_{n,k}$$

for all $n, k \in \mathbf{Z}$, where δ denotes the usual Kronecker delta, $\mathbf{Z}$ denotes the set of integers. Assume that $(f_{n,k})$ and $(g_{n,k})$ are two lower-triangular matrices that are inverse of each other; then the system of equations

$$\sum_{k=0}^{n} f_{n,k}A(k) = B(n), \tag{5}$$

is equivalent to

$$\sum_{k=0}^{n} g_{n,k}B(k) = A(n). \tag{6}$$

As many facts display, the inversion formulas, just as the celebrated Lagrange inversion formula and its q-analogues, have already been proved to be a powerful tool in finding general or basic hypergeometric summation and transformation formulas. After Gould and Hsu discovered the very general matrix inversions [**13**] and Carlitz found the q-analogue of Gould-Hsu inversions [**9**], Gessel and Stanton [**11, 12**] used matrix inversion to derive a number of basic hypergeometric summations and transformations. Chu used the inversion formulas to get many identities in his papers [**5**–**9**]. The reader may refer to them for more details. For other relevant research, see [**1**–**4, 14**–**18**].

For the open question presented by Chu [**6**], Zhang and Chen established and proved the q-triplicate inversion formula with the q-finite difference method in [**20**]. The objective of this paper is to use the (f, g)-inversion in Ma [**19**] to prove the q-triplicate inverse formula which is simpler than the q-finite difference method used by Zhang and Chen in [**20**]. Then we apply the q-triplicate inverse series to Watson's transformation formula on ${}_8\phi_7$ series in [**10**, (III.17)] and get some formulas which generalize the results of Zhang and Chen [**20**].

The following two lemmas play important roles throughout the paper.

Lemma 1.1 [**19**]. *Let $f(x,y)$ and $g(x,y)$ be two arbitrary functions over the complex field $\mathbf{C}$ in variables x, y. Suppose that $g(x,y)$ is antisymmetric, i.e., $g(x,y)=-g(y,x)$. Let $F=(f_{n,k})_{n,k\in\mathbf{Z}}$ and $G=(g_{n,k})_{n,k\in\mathbf{Z}}$ be two matrices with entries given by*

$$f_{n,k}=\frac{\prod_{i=k}^{n-1}f(x_i,b_k)}{\prod_{i=k+1}^{n}g(b_i,b_k)} \tag{7}$$

and

$$g_{n,k}=\frac{f(x_k,b_k)}{f(x_n,b_n)}\frac{\prod_{i=k+1}^{n}f(x_i,b_n)}{\prod_{i=k}^{n-1}g(b_i,b_n)}, \tag{8}$$

respectively. Then $F=(f_{n,k})_{n,k\in\mathbf{Z}}$ and $G=(g_{n,k})_{n,k\in\mathbf{Z}}$ is matrix inversion if and only if $f(x,y)\in\Lambda_3^{(\phi(g))}$ or $f(x,y)\in\Lambda_3$, where $\Lambda_3^{(\phi(g))}$ and Λ_3 denote the sets of functions $f(x,y)$ such that for all a, b, c, $x\in\mathbf{C}$,

$$g(a,b)f(x,c)-g(a,c)f(x,b)+g(b,c)f(x,a)=0 \tag{9}$$

and

$$f(a,b)f(x,c)-f(a,c)f(x,b)+f(b,c)f(x,a)=0, \tag{10}$$

respectively.

Lemma 1.2 (Watson's transformation formula on ${}_8\phi_7$ series [**10**, (III.17)]):

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a & qa^{1/2} & -qa^{1/2} & b & c & d & e & f \\ & a^{1/2} & -a^{1/2} & aq/b & aq/c & aq/d & aq/e & aq/f \end{matrix};\ q,\ \frac{a^2q^2}{bcdef}\right] \\ &\quad=\frac{(aq,aq/de,aq/ef,aq/df;q)_\infty}{(aq/d,aq/e,aq/f,aq/def;q)_\infty}\,{}_4\phi_3\left[\begin{matrix} aq/bc & d & e & f \\ & aq/b & aq/c & def/a \end{matrix};\ q,q\right]. \end{aligned} \tag{11}$$

2. A simple proof of the (f,g)-inversion for the q-triplicate inverse formula. In this section, we will use a simpler method to prove the q-triplicate inverse formula which was obtained by Zhang and Chen in [**20**]. As defined in [**20**],

$$\phi_i(q^x;0) \equiv 1, \qquad \phi_i(q^x;n) = \prod_{k=0}^{n-1}(A_{ik} + q^x B_{ik}),$$

$i = 0,1,2$; $n = 1,2,\ldots$, where $\{A_{ij}|B_{ij}\}$, $i = 0,1,2$; $j = 0,1,2,\ldots$ are two complex sequences. For convenience, products of the form $\prod_{j=u}^{u-1}$ are defined to be equal to 1, and when $u > v-1$, a product $\prod_{j=u}^{v-1}$ by definition is equal to 0. They obtained

Theorem 2.1 (The q-triplicate inverse formula). *Suppose ϕ_i-polynomial is defined as above. Then the system of equations*

(12)
$$\begin{aligned}\Omega(n) = \sum_{k\ge 0}(-1)^{3k}\begin{bmatrix} n \\ 3k\end{bmatrix}&\frac{A_{2k}+q^{3k}B_{2k}}{\phi_0(q^n;k)\phi_1(q^n;k)\phi_2(q^n;k+1)}f(x)\\ -\sum_{k\ge 0}(-1)^{3k}\begin{bmatrix} n \\ 1+3k\end{bmatrix}&\frac{A_{1k}+q^{3k+1}B_{1k}}{\phi_0(q^n;k)\phi_1(q^n;k+1)\phi_2(q^n;k+1)}g(x)\\ +\sum_{k\ge 0}(-1)^{3k}\begin{bmatrix} n \\ 2+3k\end{bmatrix}&\frac{A_{0k}+q^{3k+2}B_{0k}}{\phi_0(q^n;k+1)\phi_1(q^n;k+1)\phi_2(q^n;k+1)}h(x)\end{aligned}$$

is equivalent to the system of equations

$$f(n) = \sum_{k=0}^{3n}(-1)^{3k}\begin{bmatrix} 3n \\ k\end{bmatrix} q^{\binom{3n-k}{2}}\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n)\Omega(k), \tag{13}$$

(14)
$$g(n)=\sum_{k=0}^{3n+1}(-1)^{3k}\begin{bmatrix} 3n+1 \\ k\end{bmatrix}q^{\binom{1+3n-k}{2}}\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n+1)\Omega(k),$$

(15)
$$h(n)=\sum_{k=0}^{3n+2}(-1)^{3k}\begin{bmatrix} 3n+2 \\ k\end{bmatrix}q^{\binom{2+3n-k}{2}}\phi_0(q^k;n)\phi_1(q^k;n+1)\phi_2(q^k;n+1)\Omega(k).$$

Proof. By Lemma 1.1 and the inverse formulas (5), (6) we get the following

$$F(n)=\sum_{k=0}^{n}\frac{\prod_{i=k}^{n-1}f(x_{i,b_k})}{\prod_{i=k+1}^{n}g(b_i,b_k)}G(k)$$

is equivalent to

$$G(n)=\sum_{k=0}^{n}\frac{f(x_k,b_k)\prod_{i=k+1}^{n}f(x_i,b_n)}{f(x_n,b_n)\prod_{i=k}^{n-1}g(b_i,b_n)}F(k).$$

In this case, we suppose $f(x,y)=x+y, g(x,y)=x-y$, obviously, $f(x,y)\in\Lambda_3^{(\phi(g))}$. Giving the following sequences: $\{x_i\},\{y_i\}$, where $x_i=(a_i/b_i), b_i=q^i$, respectively, then

$$f_{n,k}=(-1)^{n-k}q^{\binom{n-k}{2}}\frac{\prod_{i=k}^{n-1}(a_i+q^kb_i)}{(q;q)_{n-k}}$$

and

$$g_{n,k}=\frac{(a_k+q^kb_k)}{(a_n+q^nb_n)}\frac{\prod_{i=k+1}^{n}(a_i+q^kb_i)}{(q;q)_{n-k}}.$$

Using $(x+y,x-y)$-inversion we get a pair of equivalent formulas:

$$\frac{(-1)^nG(n)}{(q;q)_n}=\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\frac{\prod_{i=k}^{n-1}(a_i+q^kb_i)}{(q;q)_{n-k}}\left\{\frac{\prod_{i=0}^{k-1}(a_i+q^kb_i)}{(q;q)_k}F(k)\right\}$$

if and only if

$$\frac{\prod_{i=0}^{n-1}(a_i+q^nb_i)}{(q;q)_n}F(n)=\sum_{k=0}^{n}\frac{(a_k+q^kb_k)\prod_{i=k+1}^{n}(a_i+q^nb_i)}{(a_n+q^nb_n)(q;q)_{n-k}}\frac{(-1)^kG(k)}{(q;q)_k}.$$

Rewriting them as

$$G(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\prod_{i=0}^{n-1}(a_i+q^kb_i)F(k) \tag{16}$$

if and only if

$$F(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{a_k+q^kb_k}{\prod_{i=0}^{k}(a_i+q^nb_i)}G(k). \tag{17}$$

Then we split the identity (17) into three sums:

$$\begin{aligned}F(n)=&\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\3k\end{bmatrix}\frac{a_{3k}+q^{3k}b_{3k}}{\prod_{i=0}^{3k}(a_i+q^nb_i)}G(3k)\\&-\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\3k+1\end{bmatrix}\frac{a_{3k+1}+q^{3k+1}b_{3k+1}}{\prod_{i=0}^{3k+1}(a_i+q^nb_i)}G(3k+1)\\&+\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\3k+2\end{bmatrix}\frac{a_{3k+2}+q^{3k+2}b_{3k+2}}{\prod_{i=0}^{3k+2}(a_i+q^nb_i)}G(3k+2).\end{aligned} \tag{18}$$

Substituting $a_{3k}=A_{2k}$, $b_{3k}=B_{2k}$, $a_{3k+1}=A_{1k}$, $b_{3k+1}=B_{1k}$, $a_{3k+2}=A_{0k}$, $b_{3k+2}=B_{0k}$, $G(3k)=f(k)$, $G(3k+1)=g(k)$, $G(3k+2)=h(k)$, $F(n)=\Omega(n)$ into (18), after some easy transformations and simplification, we get the following identity:

$$\begin{aligned}\Omega(n)=&\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\3k\end{bmatrix}\frac{A_{2k}+q^{3k}B_{2k}}{\phi_0(q^n;k)\phi_1(q^n;k)\phi_2(q^n;k+1)}f(k)\\&-\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\1+3k\end{bmatrix}\frac{A_{1k}+q^{3k+1}B_{1k}}{\phi_0(q^n;k)\phi_1(q^n;k+1)\phi_2(q^n;k+1)}g(k)\\&+\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\2+3k\end{bmatrix}\frac{A_{0k}+q^{3k+2}B_{0k}}{\phi_0(q^n;k+1)\phi_1(q^n;k+1)\phi_2(q^n;k+1)}h(k).\end{aligned}$$

By (16), we have

$$\begin{aligned}G(3n)&=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}3n\\k\end{bmatrix}q^{\binom{3n-k}{2}}\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n)F(k),\\G(3n+1)&=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}3n+1\\k\end{bmatrix}q^{\binom{1+3n-k}{2}}\\&\qquad\times\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n+1)F(k),\\G(3n+2)&=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}3n+2\\k\end{bmatrix}q^{\binom{2+3n-k}{2}}\\&\qquad\times\phi_0(q^k;n)\phi_1(q^k;n+1)\phi_2(q^k;n+1)F(k).\end{aligned}$$

Substituting $G(3n)$, $G(3n+1)$, $G(3n+2)$, $F(k)$ into $f(n)$, $g(n)$, $h(n)$, $\Omega(k)$ respectively, we can easily derive that

$$f(n)=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}3n\\ k\end{bmatrix}q^{\binom{3n-k}{2}}\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n)\Omega(k),$$

$$g(n)=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}3n+1\\ k\end{bmatrix}q^{\binom{1+3n-k}{2}} \times \phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n+1)\Omega(k),$$

$$h(n)=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}3n+2\\ k\end{bmatrix}q^{\binom{2+3n-k}{2}} \times \phi_0(q^k;n)\phi_1(q^k;n+1)\phi_2(q^k;n+1)\Omega(k).$$

The proof is completed. $\square$

In [**20**], Zhang and Chen chose parameters α, β, $\gamma \in \{0,-1,-2\}$, and α, β, γ are distinct. They got the following two corollaries which play important roles. Similarly, they are very important in Section 3 of our paper.

Corollary 2.2. *The system of equations*

(19)
$$\Omega(n)=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\ 3k\end{bmatrix}\frac{1-aq^{6k+\gamma}}{(aq^{n+3+\alpha};q^3)_k(aq^{n+3+\beta};q^3)_k(aq^{n+\gamma};q^3)_{k+1}}f(k)$$
$$-\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\ 1+3k\end{bmatrix} \times\frac{1-aq^{6k+4+\beta}}{(aq^{n+3+\alpha};q^3)_k(aq^{n+3+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}g(k)$$
$$+\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\ 2+3k\end{bmatrix} \times\frac{1-aq^{6k+5+\alpha}}{(aq^{n+3+\alpha};q^3)_{k+1}(aq^{n+3+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}f(k)$$

is equivalent to the system of equations

(20)
$$f(n)=q^{\binom{3n}{2}}\sum_{k=0}^{3n}\frac{(q^{-3n};q)_kq^k}{(q;q)_k}\frac{(aq^{k+1};q)_{3n}(aq^{k+\gamma};q^3)_n}{(aq^{k+3+\gamma};q^3)_n}\Omega(k),$$

(21)
$$g(n)=q^{\binom{1+3n}{2}}\sum_{k=0}^{3n+1}\frac{(q^{-3n-1};q)_kq^k}{(q;q)_k}(aq^{k+1};q)_{3n}(1-aq^{k+\gamma})\Omega(k),$$

(22)
$$h(n)=q^{\binom{2+3n}{2}}\sum_{k=0}^{3n+2}\frac{(q^{-3n-2};q)_kq^k}{(q;q)_k} \times(aq^{k+1};q)_{3n}(1-aq^{3n+k+3+\beta})(1-aq^{k+\gamma})\Omega(k).$$

Corollary 2.3. *Then the system of equations*

(23)
$$\begin{aligned}\Omega(n)=&\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\3k\end{bmatrix}\frac{1-aq^{6k+\gamma}}{(aq^{n+\alpha};q^3)_k(aq^{n+\beta};q^3)_k(aq^{n+\gamma};q^3)_{k+1}}f(k)\\&-\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\1+3k\end{bmatrix}\\&\times\frac{1-aq^{6k+1+\beta}}{(aq^{n+\alpha};q^3)_k(aq^{n+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}g(k)\\&+\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix}n\\2+3k\end{bmatrix}\\&\times\frac{1-aq^{6k+2+\alpha}}{(aq^{n+\alpha};q^3)_{k+1}(aq^{n+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}f(k)\end{aligned}$$

is equivalent to the system of equations

$$f(n)=q^{\binom{3n}{2}}\sum_{k=0}^{3n}\frac{(q^{-3n};q)_kq^k}{(q;q)_k}(aq^{k-2};q)_{3n}\Omega(k), \tag{24}$$

(25)
$$g(n)=q^{\binom{1+3n}{2}}\sum_{k=0}^{3n+1}\frac{(q^{-3n-1};q)_kq^k}{(q;q)_k}(aq^{k-2};q)_{3n}(1-aq^{3n+k+\gamma})\Omega(k),$$
(26)
$$h(n)=q^{\binom{2+3n}{2}}\sum_{k=0}^{3n+2}\frac{(q^{-3n-2};q)_kq^k}{(q;q)_k}\times(aq^{k-2};q)_{3n}(1-aq^{3n+k+\beta})(1-aq^{3n+k+\gamma})\Omega(k).$$

3. Some applications of q-triplicate inverse relations. Replacing q by q^3, then setting $d=q^{-n}$, $e=q^{-n+1}$, $f=q^{-n+2}$ in Lemma 1.2, we establish that

$$\sum_{k\geq 0}\frac{(a,q^3\sqrt{a},-q^3\sqrt{a},b,c,q^{-n},q^{-n+1},q^{-n+2};q^3)_k}{(q^3,\sqrt{a},-\sqrt{a},aq^3/b,aq^3/c,aq^{n+1},aq^{n+2},aq^{n+3};q^3)_k}\left(\frac{a^2q^{3+3n}}{bc}\right)^k$$
$$=\frac{(a;q^3)_n(aq;q)_n}{(a;q)_{2n}}{}_4\phi_3\left[\begin{matrix}aq^3/bc & q^{-n} & q^{-n+1} & q^{-n+2}\\ & aq^3/c & aq^3/c & q^{-3n+3}/a\end{matrix};q^3,q^3\right].$$

Letting

(27)
$$\omega(n)=\frac{(a;q^3)_n(aq;q)_n}{(a;q)_{2n}}{}_4\phi_3\left[\begin{matrix}aq^3/bc & q^{-n} & q^{-n+1} & q^{-n+2}\\ & aq^3/c & aq^3/c & q^{-3n+3}/a\end{matrix};q^3,q^3\right],$$
(28)
$$T(n)=\frac{(a,b,c,q,q^2;q^3)_n}{(1-a)(aq^3/b,aq^3/b;q^3)_n}\left(\frac{a^2q^3}{bc}\right)^n,$$

then we can get

$$\omega(n)=\sum_{k\geq 0}\begin{bmatrix}n\\3k\end{bmatrix}\frac{(-1)^{3k}}{(aq^{n+1},aq^{n+2},aq^{n+3};q^3)_k}(1-aq^{6k})q^{\binom{3k}{2}}T(k).\tag{29}$$

Equation (29) can also be reformulated as

$$\begin{aligned}\frac{\omega(n)}{(1-aq^{n+\gamma})}\\ &= \sum_{k\geq 0}\begin{bmatrix} n\\ 3k\end{bmatrix}\frac{(-1)^{3k}(1-aq^{6k+\gamma})}{(aq^{n+3+\alpha},aq^{n+3+\beta};q^3)_k(aq^{n+\gamma};q^3)_{k+1}}\\ &\qquad\times\frac{(1-aq^{6k})}{(1-aq^{6k+\gamma})}q^{\binom{3k}{2}}T(k),\end{aligned}$$

then through Corollary 2.2, the following summations can be derived

$$\sum_{k=0}^{3n}\frac{(q^{-3n};q)_k q^k}{(q;q)_k}\frac{(aq^{k+1};q)_{3n}(aq^{k+3+\gamma};q^3)_{n-1}}{(aq^{k+3+\gamma};q^3)_n}\omega(k) = \frac{(1-aq^{6n})}{(1-aq^{6n+\gamma})}T(n), \tag{30}$$

$$\sum_{k=0}^{3n+1}\frac{(q^{-3n-1};q)_k q^k}{(q;q)_k}(aq^{k+1};q)_{3n}\omega(k)=0, \tag{31}$$

$$\sum_{k=0}^{3n+2}\frac{(q^{-3n-2};q)_k q^k}{(q;q)_k}(aq^{k+1};q)_{3n}(1-aq^{3n+k+3+\beta})\omega(k)=0. \tag{32}$$

Considering that

$$\begin{aligned}(a;q^3)_k &= (a^{1/2},\omega^{1/2},\omega^2a^{1/2};q)_k,\\ (a;q)_{2k} &= (\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_k,\end{aligned}$$

where $\omega := e^{2\pi i/3}$, the above identities (30), (31), (32) can be simplified to give Theorem 3.1.

Theorem 3.1.

$$\begin{aligned}&\sum_{k=0}^{3n}\frac{(q^{-3n},aq^{3n+1},aq^{3n+\gamma},a^{1/3},\omega a^{1/3},\omega^2a^{1/3};q)_k}{(q,aq^{1+\gamma+3n},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_k}q^k\\ &\qquad\times{}_4\phi_3\left[\begin{matrix} aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2}\\ aq^3/c & aq^3/c & q^{-3k+3}/a & \end{matrix};q^3,q^3\right]\\ &=\frac{(1-aq^{6n})(1-aq^{3n+\gamma})(a,b,c,q,q^2;q^3)_n}{(1-aq^{3n})(1-aq^{6n+\gamma})(a;q)_{3n}(aq^3/b,aq^3/c;q^3)_n}\left(\frac{a^2q^3}{bc}\right)^n,\end{aligned}\tag{33}$$

(34)
$$\sum_{k=0}^{3n+1}\frac{(q^{-3n-1},aq^{3n+1},a^{1/3},\omega a^{1/3},\omega^2a^{1/3};q)_k}{(q,\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_k}q^k \times {}_4\phi_3\left[\begin{matrix}aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2}\\ & aq^3/c & aq^3/c & q^{-3k+3}/a\end{matrix};q^3,q^3\right]=0,$$

(35)
$$\sum_{k=0}^{3n+2}\frac{(q^{-3n-2},aq^{3n+1},aq^{3n+4+\beta},a^{1/3},\omega a^{1/3},\omega^2a^{1/3};q)_k}{(q,aq^{3+\beta+3n},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_k}q^k \times {}_4\phi_3\left[\begin{matrix}aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2}\\ & aq^3/c & aq^3/c & q^{-3k+3}/a\end{matrix};q^3,q^3\right]=0.$$

When $\gamma=0$, $\beta=-2$, the identity (33) can be reduced to

$$\sum_{k=0}^{n}\frac{(q^{-n},aq^{n},a^{1/3},\omega a^{1/3},\omega^2a^{1/3};q)_k}{(q,\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_k}q^k \times {}_4\phi_3\left[\begin{matrix}aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2}\\ & aq^3/c & aq^3/c & q^{-3k+3}/a\end{matrix};q^3,q^3\right] = \begin{cases}\frac{(a,b,c,q,q^2;q^3)_{n/3}}{(a;q)_n(aq^3/b,aq^3/c;q^3)_{n/3}}\left(\frac{a^2q^3}{bc}\right)^{n/3} & \text{if } n=0 \bmod 3\\ 0 & \text{if } n\neq 0 \bmod 3.\end{cases} \tag{36}$$

In particular, for $bc=aq^3$, the identities (33), (34), (35), (36) can be simplified to the results in Zhang and Chen [**20**, (3.2)–(3.5)] as follows.

Corollary 3.2.

(37)
$$ {}_6\phi_5\left[\begin{matrix}q^{-3n} & aq^{3n+1} & aq^{3n+\gamma} & a^{1/3} & \omega a^{1/3} & \omega^2a^{1/3}\\ & aq^{3n+1+\gamma} & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq}\end{matrix};q,q\right] = \frac{(1-aq^{6n})(1-aq^{3n+\gamma})}{(1-aq^{6n+\gamma})(1-aq^{3n})}\frac{a^n(q,q^2;q^3)_n}{(a;q)_{3n}},7$$

(38)
$$ {}_5\phi_4\left[\begin{matrix}q^{-3n-1} & aq^{3n+1} & a^{1/3} & \omega a^{1/3} & \omega^2a^{1/3}\\ & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq}\end{matrix};q,q\right]=0,$$

(39)
$$
{}_6\phi_5\left[\begin{matrix} q^{-3n-2} & aq^{3n+1} & aq^{3n++4+\beta} & a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ & aq^{3n+3+\beta} & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q\right] = 0,
$$

(40)
$$
{}_5\phi_4\left[\begin{matrix} q^{-n} & aq^n & a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q\right]
= \begin{cases} \frac{a^{n/3}(a,q,q^2;q^3)_{n/3}}{(a;q)_n} & \textit{if } n = 0 \bmod 3 \\ 0 & \textit{if } n \neq 0 \bmod 3. \end{cases}
$$

According to the factorization technique, see [**20**, (4.2)],

$$
\begin{aligned}
1 - aq^{2n+\alpha} = {} & \frac{1}{1-aq^{6k+\beta}}(1-aq^{n+3k+\alpha})(1-aq^{n+3k+\beta}) \\
& + \frac{aq^{6k+\alpha+1}(1-q^{\beta-\alpha-1})}{(1-aq^{6k+\beta})(1-aq^{6k+\alpha+1})}(1-q^{n-3k}) \\
& \times (1-aq^{n+3k+\alpha}) \\
& - \frac{aq^{6k+\alpha+1}}{1-aq^{6k+\alpha+1}}(1-q^{n-3k})(1-q^{n-3k-1}).
\end{aligned}
$$

Using the above identity along with (29), we get

$$
\begin{aligned}
& \frac{(1-aq^{2n+\alpha})\omega(n)}{(1-aq^{n+\alpha})(1-aq^{n+\beta})(1-aq^{n+\gamma})} \\
& \quad = \sum_{k\geq 0} \begin{bmatrix} n \\ 3k \end{bmatrix} \frac{(-1)^{3k}(1-aq^{6k+\gamma})}{(aq^{n+\alpha}, aq^{n+\beta}; q^3)_k (aq^{n+\gamma}; q^3)_{k+1}} \\
& \qquad\qquad \times \frac{(1-aq^{6k})q^{\binom{3k}{2}}T(k)}{(1-aq^{6k+\beta})(1-aq^{6k+\gamma})} \\
& \quad - \sum_{k\geq 0} \begin{bmatrix} n \\ 3k+1 \end{bmatrix} \frac{(-1)^{3k}(1-aq^{6k+1+\beta})}{(aq^{n+\alpha}; q^3)_k (aq^{n+\beta}, aq^{n+\gamma}; q^3)_{k+1}} \\
& \quad \times \frac{-aq^{6k+\alpha+1}(1-aq^{6k})(1-q^{3k+1})(1-q^{\beta-\alpha-1})}{(1-aq^{6k+\beta})(1-aq^{6k+1+\beta})(1-aq^{6k+\alpha+1})} q^{\binom{3k}{2}}T(k) \\
& \quad + \sum_{k\geq 0} \begin{bmatrix} n \\ 3k+2 \end{bmatrix} \frac{(-1)^{3k}(1-aq^{6k+2+\alpha})}{(aq^{n+\alpha}, aq^{n+\beta}, aq^{n+\gamma}; q^3)_{k+1}}
\end{aligned}
$$

$$\times \frac{-aq^{6k+\alpha+1}(1-aq^{6k})(1-q^{3k+1})(1-q^{3k+2})}{(1-aq^{6k+\alpha+1})(1-aq^{6k+2+\alpha})} q^{\binom{3k}{2}} T(k).$$

By Corollary 2.3, we get the terminating summation formulas

$$\sum_{k=0}^{3n} \frac{(q^{-3n}, aq^{3n-2}; q)_k q^k}{(q;q)_k} \frac{(aq^{2+\alpha}; q^2)_k}{(aq^{\alpha}; q^2)_k} \frac{\omega(k)}{(aq;q)_k}$$
$$= \frac{(1-aq^{6n})}{(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+\gamma})} \frac{T(n)}{(aq;q)_{3n-3}},$$
$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, aq^{3n-2}, aq^{3n+1+\gamma}; q)_k q^k}{(q, aq^{3n+\gamma}; q)_k} \frac{(aq^{2+\alpha}; q^2)_k}{(aq^{\alpha}; q^2)_k} \frac{\omega(k)}{(aq;q)_k}$$
$$= \frac{(1-aq^{6n})(1-q^{3n+1})(1-q^{1+\alpha-\beta})}{(1-aq^{3n+\gamma})(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})(1-aq^{6n+\alpha+1})} \frac{aq^{3n+\beta}T(n)}{(aq;q)_{3n-3}},$$
$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, aq^{3n-2}, aq^{3n+1+\beta}; q)_k q^k}{(q, aq^{3n+\beta}; q)_k} \frac{(aq^{3n+1+\gamma}; q)_k}{(aq^{3n+\gamma}; q)_k} \frac{(aq^{2+\alpha}; q^2)_k}{(aq^{\alpha}; q^2)_k} \frac{\omega(k)}{(aq;q)_k}$$
$$= \frac{-aq^{\alpha}(1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+2+\alpha})} \frac{T(n)}{(aq;q)_{3n-3}}.$$

Through reducing, we have

Theorem 3.3.

(41)
$$\sum_{k=0}^{3n} \frac{(q^{-3n}, aq^{3n-2}, q\sqrt{aq^{\alpha}}, -q\sqrt{aq^{\alpha}}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}; q)_k}{(q, \sqrt{aq^{\alpha}}, -\sqrt{aq^{\alpha}}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_k} q^k$$
$$\times {}_4\phi_3 \left[\begin{matrix} aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2} \\ & aq^3/c & aq^3/c & q^{-3k+3}/a \end{matrix} ; q^3, q^3 \right]$$
$$= \frac{(1+aq^{3n})(q, q^2, aq^3, b, c; q^3)_n \left(\frac{a^2q^3}{bc}\right)^n}{(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+\gamma})(aq^3/b, aq^3/c; q^3)_n (a;q)_{3n-2}},$$

(42)
$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, aq^{3n-2}, aq^{3n+\gamma+1}, q\sqrt{aq^{\alpha}}, -q\sqrt{aq^{\alpha}}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}; q)_k}{(q, aq^{3n+\gamma}, \sqrt{aq^{\alpha}}, -\sqrt{aq^{\alpha}}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_k} q^k$$
$$\times {}_4\phi_3 \left[\begin{matrix} aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2} \\ & aq^3/c & aq^3/c & q^{-3k+3}/a \end{matrix} ; q^3, q^3 \right]$$

$$= \frac{a^{n+1}q^{3n+\beta}(1+aq^{3n})(1-q^{3n+1})(1-q^{1+\alpha-\beta})}{(1-aq^{3n+\gamma})(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})(1-aq^{6n+1+\alpha})} \times \frac{(q,q^2,aq^3,b,c;q^3)_n(\frac{aq^3}{bc})^n}{(aq^3/b,aq^3/c;q^3)_n(a;q)_{3n-2}},$$

(43)

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2},aq^{3n-2},aq^{3n+\beta+1},aq^{3n+\gamma+1},q\sqrt{aq^{\alpha}},-q\sqrt{aq^{\alpha}},a^{1/3},\omega a^{1/3},\omega^2 a^{1/3};q)_k}{(q,aq^{3n+\beta},aq^{3n+\gamma},\sqrt{aq^{\alpha}},-\sqrt{aq^{\alpha}},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_k} q^k \times {}_4\phi_3\left[\begin{matrix} aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2} \\ & aq^3/c & aq^3/c & q^{-3k+3}/a \end{matrix}; q^3, q^3\right]$$

$$= \frac{-a^{n+1}q^{\alpha}(1+aq^{3n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{\alpha})} \times (1-aq^{6n+1+\alpha})(1-aq^{6n+\alpha+2}) \times \frac{(q,q^2,aq^3,b,c;q^3)_n(aq^3/bc)^n}{(aq^3/b,aq^3/c;q^3)_n(a;q)_{3n-2}}.$$

In fact, setting $aq^3 = bc$ in (41), (42), (43), we obtain the results of Zhang and Chen [**20**, (4.4)–(4.6)] as follows.

Corollary 3.4.

(44)

$${}_7\phi_6\left[\begin{matrix} q^{-3n} & aq^{3n-2} & q\sqrt{aq^{\alpha}} & -q\sqrt{aq^{\alpha}} & a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ & \sqrt{aq^{\alpha}} & -\sqrt{aq^{\alpha}} & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q\right] = \frac{(1-aq^{6n})}{(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+\gamma})} \frac{a^n(a,q,q^2;q^3)_n}{(a;q)3n-2},$$

(45)

$${}_8\phi_7\left[\begin{matrix} q^{-3n-1} & aq^{3n-2} & aq^{3n+\gamma+1} & q\sqrt{aq^{\alpha}} & -q\sqrt{aq^{\alpha}} & a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ & aq^{3n+\gamma} & \sqrt{aq^{\alpha}} & -\sqrt{aq^{\alpha}} & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q\right]$$

$$= \frac{a^{n+1}q^{3n+\beta}(1-aq^{6n})(1-q^{3n+1})(1-q^{1+\alpha-\beta})}{(1-aq^{3n+\gamma})(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})(1-aq^{6n+1+\alpha})} \frac{(a,q,q^2;q^3)_n}{(a;q)_{3n-2}},$$

(46)

$$
{}_9\phi_8\left[\begin{matrix} q^{-3n-2} & aq^{3n-2} & aq^{3n+\beta+1} & aq^{3n+\gamma+1} & q\sqrt{aq^{\alpha}} & -q\sqrt{aq^{\alpha}} \\ & aq^{3n+\beta} & aq^{3n+\gamma} & \sqrt{aq^{\alpha}} & -\sqrt{aq^{\alpha}} & \sqrt{a} \end{matrix}\right.
$$

$$
\left.\begin{matrix} a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q\right]
$$

$$
= \frac{-a^{n+1}q^{\alpha}(1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{\alpha})(1-aq^{6n+1+\alpha})(1-aq^{6n+2+\alpha})}\frac{(q,q^2,a;q^3)_n}{(a;q)_{3n-2}}.
$$

Employing the factorization technique, see [**20**, (4.7)],

$$
\begin{aligned}
q^n = {} & \frac{q^{3k}(1-aq^{n+3k+\alpha})(1-aq^{n+3k+\beta})}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})} \\
& - \frac{q^{3k}(1-a^2q^{12k+\beta+\alpha+1})(1-q^{n-3k})(1-aq^{n+3k+\alpha})}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})(1-aq^{6k+\alpha+1})} \\
& + \frac{aq^{9k+\alpha+1}(1-q^{n-3k})(1-q^{n-3k-1})}{(1-aq^{6k+\alpha})(1-aq^{6k+\alpha+1})}.
\end{aligned}
$$

Applying the above identity to (29), we get

$$
\begin{aligned}
& \frac{q^n\omega(n)}{(1-aq^{n+\alpha})(1-aq^{n+\beta})(1-aq^{n+\gamma})} \\
& = \sum_{k\ge 0}\begin{bmatrix} n \\ 3k \end{bmatrix}\frac{(-1)^{3k}(1-aq^{6k+\gamma})}{(aq^{n+\alpha},aq^{n+\beta};q^3)_k(aq^{n+\gamma};q^3)_{k+1}} \\
& \qquad \times \frac{(1-aq^{6k})q^{\binom{3k}{2}+3k}T(k)}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})(1-aq^{6k+\gamma})} \\
& \quad - \sum_{k\ge 0}\begin{bmatrix} n \\ 3k+1 \end{bmatrix}\frac{(-1)^{3k}(1-aq^{6k+1+\beta})}{(aq^{n+\alpha};q^3)_k(aq^{n+\beta},aq^{n+\gamma};q^3)_{k+1}} \\
& \qquad \times \frac{(1-aq^{6k})(1-q^{3k+1})(1-a^2q^{12k+\beta+\alpha+1})q^{\binom{3k}{2}+3k}T(k)}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})(1-aq^{6k+1+\beta})(1-aq^{6k+1+\alpha})} \\
& \quad + \sum_{k\ge 0}\begin{bmatrix} n \\ 3k+2 \end{bmatrix}\frac{(-1)^{3k}(1-aq^{6k+2+\alpha})}{(aq^{n+\alpha},aq^{n+\beta},aq^{n+\gamma};q^3)_{k+1}} \\
& \qquad \times \frac{aq^{9k+\alpha+1}(1-aq^{6k})(1-q^{3k+1})(1-q^{3k+2})q^{\binom{3k}{2}}T(k)}{(1-aq^{6k+\alpha})(1-aq^{6k+1+\alpha})(1-aq^{6k+2+\alpha})}.
\end{aligned}
$$

By Corollary 2.3, the following three summation formulas are obtained:

$$\begin{aligned}
&\sum_{k=0}^{3n} \frac{(q^{-3n}, aq^{3n-2}; q)_k}{(q;q)_k} \frac{q^{2k}\omega(k)}{(aq;q)_k} \\
&\qquad\qquad = \frac{(1-aq^{6n})}{(1-aq^{6n+\alpha})(1-aq^{6n+\beta})(1-aq^{6n+\gamma})} \frac{q^{3n}T(n)}{(aq;q)_{3n-3}} \\
&\qquad\qquad = \frac{q^{3n}T(n)}{(1-aq^{6n-1})(1-aq^{6n-2})(aq;q)_{3n-3}},
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, aq^{3n-2}; q)_k}{(q;q)_k} \frac{(aq^{3n+1+\gamma};q)_k}{(aq^{3n+\gamma};q)_k} \frac{q^{2k}\omega(k)}{(aq;q)_k} \\
&= \frac{(1-aq^{6n})(1-q^{3n+1})(1-a^2q^{12n+\beta+\alpha+1})T(n)}{(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})(1-aq^{6n+1+\alpha})(aq;q)_{3n-3}},
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, aq^{3n-2}; q)_k}{(q;q)_k} \frac{(aq^{3n+1+\beta}, aq^{3n+1+\gamma};q)_k}{(aq^{3n+\beta}, aq^{3n+\gamma};q)_k} \frac{q^{2k}\omega(k)}{(aq;q)_k} \\
&= \frac{(1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})aq^{3n+\alpha}T(n)}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+1+\alpha})(1-aq^{6n+2+\alpha})(aq;q)_{3n-3}}.
\end{aligned}$$

By arrangement and reduction, we have

Theorem 3.5.

$$\begin{aligned}
\sum_{k=0}^{3n} & \frac{(q^{-3n}, aq^{3n-2}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}; q)_k q^{2k}}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_k} \\
&\times {}_4\phi_3 \left[\begin{matrix} aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2} \\ & aq^3/c & aq^3/c & q^{-3k+3}/a \end{matrix} ; q^3, q^3 \right] \\
&= \frac{(q, q^2, a, b, c; q^3)_n}{(1-a)(1-aq^{6n-1})(1-aq^{6n-2})(aq;q)_{3n-2}(aq^3/b, aq^3/c; q^3)_n} \left(\frac{a^2q^6}{bc}\right)^n,
\end{aligned} \tag{47}$$

$$\begin{aligned}
\sum_{k=0}^{3n+1} & \frac{(q^{-3n-1}, aq^{3n-2}, aq^{3n+\gamma+1}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}; q)_k q^{2k}}{(q, aq^{3n+\gamma}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_k} \\
&\times {}_4\phi_3 \left[\begin{matrix} aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2} \\ & aq^3/c & aq^3/c & q^{-3k+3}/a \end{matrix} ; q^3, q^3 \right] \\
&= \frac{(1+aq^{3n})(1-q^{3n+1})(1-a^2q^{12n+\beta+\alpha+1})}{(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+\beta})(1-aq^{6n+\beta+1})} \\
&\quad \times \frac{(q, q^2, aq^3, b, c; q^3)_n}{(a;q)_{3n-2}(aq^3/b, aq^3/c; q^3)_n} \left(\frac{a^2q^3}{bc}\right)^n,
\end{aligned} \tag{48}$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2},aq^{3n-2},aq^{3n+\beta+1},aq^{3n+\gamma+1},a^{1/3},\omega a^{1/3},\omega^2 a^{1/3};q)_k q^{2k}}{(q,aq^{3n+\beta},aq^{3n+\gamma},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_k} \tag{49}$$
$$\times {}_4\phi_3\left[\begin{matrix} aq^3/bc & q^{-k} & q^{-k+1} & q^{-k+2} \\ & aq^3/c & aq^3/c & q^{-3k+3}/a \end{matrix}; q^3, q^3\right]$$
$$= \frac{aq^{\alpha}(1+aq^{3n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+\alpha+2})}$$
$$\times \frac{(q,q^2,aq^3,b,c;q^3)_n}{(a;q)_{3n-2}(aq^3/b,aq^3/c;q^3)_n}\left(\frac{a^2q^6}{bc}\right)^n.$$

Putting $aq^3 = bc$ and reformulating the identities in Theorem 3.5, we obtain the results of Zhang and Chen [**20**, (4.8)–(4.10)] as the following corollary.

Corollary 3.6.

$${}_5\phi_4\left[\begin{matrix} q^{-3n} & aq^{3n-2} & a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q^2\right] = \frac{a^n q^{3n}(q,q^2,a;q^3)_n}{(1-aq^{6n-1})(1-aq^{6n-2})(a;q)_{3n-2}}, \tag{50}$$

(51)
$${}_6\phi_5\left[\begin{matrix} q^{-3n-1} & aq^{3n-2} & aq^{3n+\gamma+1} & a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ & aq^{3n+\gamma} & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q^2\right]$$
$$= \frac{(1-aq^{6n})(1-q^{3n+1})(1-a^2q^{12n+\beta+\alpha+1})}{(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+\beta})(1-aq^{6n+\beta+1})}$$
$$\times \frac{a^n(q,q^2,a;q^3)_n}{(a;q)_{3n-2}},$$

$${}_7\phi_6\left[\begin{matrix} q^{-3n-2} & aq^{3n-2} & aq^{3n+\beta+1} & aq^{3n+\gamma+1} & a^{1/3} & \omega a^{1/3} & \omega^2 a^{1/3} \\ & aq^{3n+\beta} & aq^{3n+\gamma} & \sqrt{a} & -\sqrt{a} & \sqrt{aq} & -\sqrt{aq} \end{matrix}; q, q^2\right] \tag{52}$$
$$= \frac{(1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+\alpha+2})}$$
$$\times \frac{a^{n+1}q^{3n+\alpha}(q,q^2,a;q^3)_n}{(a;q)_{3n-2}}.$$

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