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FACTORIZATIONS AND REPRESENTATIONS OF BINARY POLYNOMIAL RECURRENCES BY MATRIX METHODS

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ABSTRACT. In this paper we derive factorizations and representations of a polynomial analogue of an arbitrary binary sequence by matrix methods. It generalizes various results on Fibonacci, Lucas, Chebyshev and Morgan-Voyce polynomials.

1. Introduction. In [**10**], the divisibility properties of the Fibonacci polynomial sequence $\{f_n(x)\}$ was studied. The Fibonacci polynomial sequence is defined by the recursion

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x); \qquad f_0(x) = 0, \ f_1(x) = 1.$$

Five years later, Hoggatt and Long [**2**] considered the general Fibonacci type polynomial sequence $\{u_n(x,y)\}$ of two variables. This sequence is defined by

$$u_{n+2}(x,y) = xu_{n+1}(x,y) + yu_n(x,y) \tag{1.1}$$

where $u_0(x,y) = 0$ and $u_1(x,y) = 1$.

The authors of [**2, 10**] found the roots of these polynomials and then obtained the factorizations of their polynomials. In [**2**], the authors found that

$$u_n(x,y) = y^{(n-1)/2} \prod_{k=1}^{n-1} \left(\frac{x}{\sqrt{y}} - 2i \cos \frac{k\pi}{n} \right).$$

Further, in [**1**], the authors use the relationships between the determinants of certain tridiagonal matrices and the Fibonacci and Lucas

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numbers, and then by matrix methods, they obtained the factorizations and representations of these sequences. The factorization of Fibonacci numbers was initially proposed in [**6**], and the factorization of Lucas numbers was obtained in [**11**].

The (companion) generalized Lucas polynomial sequence $v_n(x,y)$ is defined by

$$v_{n+2}(x,y) = xv_{n+1}(x,y) + yv_n(x,y)$$

where $v_0(x,y) = 2$ and $v_1(x,y) = 1$.

Recently, in [**5**], the binary sequential analogues of the generalized Fibonacci and Lucas polynomial sequences was considered and factorizations and representations of these sequences was obtained. These sequences are defined by

$$\begin{aligned} U_{n+1} &= AU_n + BU_{n-1} \\ V_{n+1} &= AV_n + BV_{n-1} \end{aligned}$$

where $U_0 = 0, U_1 = 1$ and $V_0 = 2, V_1 = A$, respectively. Also, in [**4**], we gave the more general factorizations of second order linear recurrences $\{U_n\}$ and $\{V_n\}$ with indices in arithmetic progressions. Furthermore, we obtained the factorization of these general sequences by the matrix methods considering how these recurrences are related to the determinants of certain tridiagonal matrices.

As can be seen from the above-mentioned results, the most general cases of these polynomial and binary sequences have not been studied. Consequently, we define $\{A_n(a,b;p,q)(x)\}$ (we shall often drop the argument $(a,b;p,q)$ and simply write $\{A_n(x)\}$) to be a polynomial sequence satisfying

$$A_{n+1}(x) = p(x)A_n(x) + q(x)A_{n-1}(x),$$

and initial conditions $A_0 = a(x)$, $A_1 = b(x)$, where a, b, p, q are polynomials of an indeterminate x with real coefficients. For easy notation, we shall sometimes write A_n, p, q, a, b for $A_n(x)$, $p(x)$, $q(x)$, $a(x)$ and $b(x)$, respectively. We display some special cases of the sequence $\{A_n\}$ in the following table.

TABLE 1.

$p(x)$	$q(x)$	$a(x)$	$b(x)$	$A_n(a,b;p,q)(x)$	Polynomial Type
$2x$	-1	1	x	$T_n(x)$	1st kind Chebyshev
$2x$	-1	1	$2x$	$U_n(x)$	2nd kind Chebyshev
$x+1$	-1	1	$x+1$	$b_n(x)$	Morgan-Voyce
x	y	0	1	$u_n(x,y)$	generalized Fibonacci
x	y	2	1	$v_n(x,y)$	generalized Lucas
$2x$	-1	1	$2x+1$	$W_n(x)$	4th kind Chebyshev
$2x$	-1	1	$2x-1$	$V_n(x)$	3rd kind Chebyshev

In Section 2 we present a recurrence, and define a tridiagonal matrix whose determinant is precisely $A_n(a,b;p,q(x)$, with n in an arithmetic progression, $n \equiv c \pmod{k}$. In our main Section 3 we derive the factorization and representations of the sequence $\{A_n(a,b;p,q)\}$ (under some assumptions), by matrix methods, thus generalizing some results of [**1**–**5**] and others. As a consequence, we obtain the factorizations for the Chebyshev's and generalized Lucas polynomials, among others.

2. A recurrence for $A_n(a,b;p,q)(x)$, where $n \equiv c \pmod{k}$. We start this section with the Binet formulas of both positively and negatively indexed terms of the sequence $\{A_n\}$, namely

$$A_{\pm n} = \left(\frac{b-a\beta}{\alpha-\beta}\right)\alpha^{\pm n} + \left(\frac{a\alpha-b}{\alpha-\beta}\right)\beta^{\pm n} \tag{2.1}$$

where

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \quad \beta = \frac{p-\sqrt{p^2+4q}}{2}. \tag{2.2}$$

First, we prove the following lemma.

Lemma 1. *For $k > 0$, $n > 0$, the sequence $\{A_n\}$ satisfies the following recursion*

$$A_{p(n+1,\pm k,c)} = y_{\pm k} A_{p(n,\pm k,c)} - z_{\pm k} A_{p(n-1,\pm k,c)} \tag{2.3}$$

where $y_{\pm k} = \alpha^{\pm k}+\beta^{\pm k}$, $z_{\pm k} = q^{\pm k}$ and $p(n,k,c) = nk+c$ (c constant).

Proof. From the definition of the sequence $A_n\}$, considering the case $A_0 = 2$ and $A_1 = p$, we write

$$y_{\pm k} = A_{\pm k}(2, p; p, q) = \alpha^{\pm k} + \beta^{\pm k}$$

where α and β are given by (2.2). Further, for the positive case, we note that

$$y_k = A_k(2, p; p, q) = \alpha^k + \beta^k.$$

Since the positively and negatively indexed terms cases are similar, we only consider the positively indexed terms case. By the Binet formula of the sequence $\{A_n\}$ and since $z_k = q^k = (\alpha\beta)^k$,

$$\begin{aligned}
y_k A_{p(n,k,c)} - z_k A_{p(n-1,k,c)} \\
&= \left(\alpha^k + \beta^k\right)\left[\left(\tfrac{b-a\beta}{\alpha-\beta}\right)\alpha^{kn+c} + \left(\tfrac{a\alpha-b}{\alpha-\beta}\right)\beta^{kn+c}\right] \\
&\quad - (\alpha\beta)^k\left[\left(\tfrac{b-a\beta}{\alpha-\beta}\right)\alpha^{k(n-1)+c} + \left(\tfrac{a\alpha-b}{\alpha-\beta}\right)\beta^{k(n-1)+c}\right] \\
&= \tfrac{1}{\alpha-\beta}((b-a\beta)\,\alpha^{k(n+1)+c} + (a\alpha-b)\,\beta^{k(n+1)+c} \\
&\quad + (b-a\beta)\,\alpha^{kn+c}\beta^k + (a\alpha-b)\,\beta^{kn+c}\alpha^k \\
&\quad - (b-a\beta)\,\alpha^{kn+c}\beta^k - (a\alpha-b)\,\beta^{kn+c}\alpha^k) \\
&= \left(\tfrac{b-a\beta}{\alpha-\beta}\right)\alpha^{k(n+1)+c} + \left(\tfrac{a\alpha-b}{\alpha-\beta}\right)\beta^{k(n+1)+c} \\
&= A_{p(n+1,k,c)},
\end{aligned}$$

which proves the lemma. $\square$

Now we present a relationship between the terms $A_{p(\pm(n+1),k,c)}$ and the determinant of a certain tridiagonal matrix.

Define the $n \times n$ tridiagonal matrix $M_{n,k}$ as shown:

(2.4)

$$M_{n,\pm k} = \begin{bmatrix} A_{p(1,\pm k,c)} & A_{p(0,\pm k,c)}\sqrt{z_{\pm k}} & & & \\ \sqrt{z_{\pm k}} & y_{\pm k} & \sqrt{z_{\pm k}} & & \\ & \sqrt{z_{\pm k}} & y_{\pm k} & \ddots & \\ & & \ddots & \ddots & \sqrt{z_{\pm k}} \\ & & & \sqrt{z_{\pm k}} & y_{\pm k} \end{bmatrix}.$$

Theorem 1. *For $n > 1$, we have*

$$\det M_{n,\pm k} = A_{p(n,\pm k,c)}.$$

Proof. As before, we only consider the positively indexed terms. The other case can be similarly derived. Expanding $\det M_{n,k}$ using the cofactor expansion of a determinant with respect to the last column, we obtain

$$\det M_{n,k} = y_k \det M_{n-1,k} - z_k \det M_{n-2,k}.$$

Replacing $n = 1$ in equation (2.3), we obtain $A_{p(2,k,c)} = y_k A_{p(1,k,c)} - z_k A_{p(0,k,c)}$ and $\det M_{2,k} = y_k A_{p(1,k,c)} - z_k A_{p(0,k,c)}$, and so, detillas at la Contendre $M_{2,k} = A_{p(2,k,c)}$. Obviously, $\det M_{1,k} = A_{p(1,k,c)}$. Since the recurrence relations (and initial conditions) of $\det M_{n,k}$ and the sequence $\{A_{p(n,k,c)}\}$ are the same, the conclusion follows from Lemma 1. □

3. Factorizations for $A_n(a,b;p,q)(x)$**.** Now we investigate all possible cases of our main general considerations and then we give their factorizations and representations. However, a few special cases could be treated separately. For compactness, we do not consider all these cases, but we shall point out whenever appropriate some hints in treating those cases.

First, we consider the case $A_{p(1,\pm k,c)} = y_{\pm k}$ and $A_{p(0,\pm k,c)} = 1$. Under these assumptions, we label the matrix $M_{n,\pm k}$ and the sequence $A_{p(\pm n,k,c)}$ by $M^{(1)}_{n,\pm k}$ and $A^{(1)}_{p(n,\pm k,c)}$, respectively. By Theorem 1, the matrix $M^{(1)}_{n,\pm k}$ takes the following form:

$$M^{(1)}_{n,\pm k} = \begin{bmatrix} y_{\pm k} & \sqrt{z_{\pm k}} & & \\ \sqrt{z_{\pm k}} & \sqrt{z_{\pm k}} & \ddots & \\ & \ddots & \ddots & \sqrt{z_{\pm k}} \\ & & \sqrt{z_{\pm k}} & y_{\pm k} \end{bmatrix}$$

and we have that

$$\det M^{(1)}_{n,\pm k} = A^{(1)}_{p(n,\pm k,c)}. \tag{3.1}$$

Let Q_1 be the following $(n \times n)$ tridiagonal Toeplitz matrix

$$Q_1 = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}$$

The characteristic equation of the matrix Q_1 satisfies the following recursion, for $n > 2$

$$f_n(\lambda) = -\lambda f_{n-1}(\lambda) - f_{n-2}(\lambda),$$

where $f_1(\lambda) = -\lambda$ and $f_2(\lambda) = \lambda^2 - 1$.

Taking $\lambda \equiv -2x$, the family $\{f_n(\lambda)\}$ is reduced to the family $\{U_n(x)\}$ (Chebyshev polynomial of the second kind). From [**7**–**9**], the zeros of the Chebyshev polynomials are known, and so, the eigenvalues of Q_1 are of the form:

$$\lambda_r = -2\cos\frac{\pi r}{n+1}, \quad r = 1, 2, \ldots, n. \tag{3.2}$$

We can also write $M^{(1)}_{n,\pm k} = y_{\pm k} I_n + \sqrt{z_{\pm k}}\, Q_1$, where I_n is the $n \times n$ unit matrix.

Theorem 2. *Let $A_{p(1,\pm k,c)} = y_{\pm k}$ and $A_{p(0,\pm k,c)} = 1$. Then for $n > 1$,*

$$A^{(1)}_{p(n,\pm k,c)} = \prod_{r=1}^{n} \left(y_{\pm k} - 2\sqrt{z_{\pm k}} \cos\frac{\pi r}{n+1} \right).$$

Proof. Assume that the λ_r's are eigenvalues of the matrix Q_1 with respect to the eigenvectors w_r. Since

$$\begin{aligned} M^{(1)}_{n,\pm k} w_r &= \left(y_{\pm k} I_n + \sqrt{z_{\pm k}}\, Q_1 \right) w_r \\ &= y_{\pm k} I_n w_r + \sqrt{z_{\pm k}}\, Q_1 w_r \\ &= \left(y_{\pm k} + \sqrt{z_{\pm k}}\, \lambda_r \right) w_r, \end{aligned}$$

the $(y_{\pm k} + \sqrt{z_{\pm k}}\lambda_r)$'s are the eigenvalues of the matrix $M^{(1)}_{n,\pm k}$ with respect to the eigenvectors w_r. Thus, by (3.1) and (3.2), we have the conclusion:

$$\det M^{(1)}_{n,\pm k} = A^{(1)}_{p(n,\pm k,c)} = \prod_{r=1}^{n} \left(y_{\pm k} + \sqrt{z_{\pm k}}\lambda_r\right)$$
$$= \prod_{r=1}^{n} \left(y_{\pm k} - 2\sqrt{z_{\pm k}} \cos \tfrac{\pi r}{n+1}\right). \qquad \square$$

Corollary 1. *For $n > 1$,*

$$A^{(1)}_{p(n,\pm k,c)} = \begin{cases} y_{\pm k} \prod\limits_{r=1}^{n/2} \left(y^2_{\pm k} - 4z_{\pm k} \cos^2 \tfrac{\pi r}{n+1}\right) & \textit{if } n \textit{ is even}, \\ \prod\limits_{r=1}^{(n-1)/2} \left(y^2_{\pm k} - 4z_{\pm k} \cos^2 \tfrac{\pi r}{n+1}\right) & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. Since $\cos(k\pi/n) = -\cos((n-k)\pi/n)$ for $1 \le k \le n/2$, the conclusion follows from Theorem 2. $\square$

Now we give some applications of Theorem 2 in the following corollaries.

Corollary 2. *Let U_n be the nth Chebyshev polynomial of the second kind. For $n, k > 1$, then*

$$U_{\pm(n+1)k-1} = 2^n U_{\pm k-1} \prod_{r=1}^{n} \left(T_{\pm k} - \cos \tfrac{\pi r}{n+1}\right). \tag{3.3}$$

where T_n is the nth Chebyshev polynomial of the first kind.

Proof. When $p = 2x$, $q = -1$, then the first coefficient $y_{\pm k} = 2T_{\pm k}$, where T_k is the kth Chebyshev polynomial of the first kind (see Table 1). According to the first case, to satisfy $A^{(1)}_{p(1,\pm k,c)} = y_{\pm k} = 2T_{\pm k}$ and

$A^{(1)}_{p(0,\pm k,c)} = 1$, we shall choose $a = 1$ and $b = 2x$. We see that for $c = -1$,

$$A^{(1)}_{p(1,\pm k,-1)} = \frac{U_{\pm 2k-1}}{U_{\pm k-1}} = 2T_{\pm k}$$

and

$$A^{(1)}_{p(0,\pm k,-1)} = \frac{U_{\pm k-1}}{U_{\pm k-1}} = 1,$$

and in general,

$$A^{(1)}_{p(n,\pm k,-1)} = \frac{U_{\pm (n+1)k-1}}{U_{\pm k-1}}.$$

Then by Theorem 2, we get for $y_k = 2T_k$ and $z_k = 1$

$$A^{(1)}_{p(n,\pm k,c)} = \frac{U_{\pm (n+1)k-1}}{U_{\pm k-1}} = \prod_{r=1}^{n} \left(2T_{\pm k} - 2\cos \frac{\pi r}{n+1} \right). \tag{3.4}$$

The required equation (3.3) follows immediately. □

Corollary 3. *For $n, k > 1$, then*

$$U_{\pm k(n+1)-1} = \begin{cases} 2^{n-1} U_{\pm 2k-1} \prod\limits_{r=1}^{(n-1)/2} \left[T^2_{\pm k} - \cos^2\left(\frac{\pi r}{n+1}\right) \right] & \textit{if } n \textit{ is odd,} \\ 2^{n} U_{\pm k-1} \prod\limits_{r=1}^{n/2} \left[T^2_{\pm k} - \cos^2\left(\frac{\pi r}{n+1}\right) \right] & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. The proof follows from Corollary 1 and equation (3.4). □

Corollary 4. *Let $u_n(x,t)$ be the generalized Fibonacci polynomial. For $n > 1$ and $k > 0$,*

$$u_{\pm (n+1)k}(x,t) = u_{\pm k}(x,t) \prod_{r=1}^{n} \left(v_{\pm k}(x,t) - 2\sqrt{(-t)^{\pm k}} \cos \frac{\pi r}{n+1} \right). \tag{3.5}$$

Proof. When $p = x$, $q = t$, then the first coefficient $y_{\pm k} = v_{\pm k}(x,t)$ where v_k is the kth term of the generalized Lucas polynomial sequence (see Table 1). For our first case, to satisfy $A^{(1)}_{p(1,\pm k,c)} = y_{\pm k} = v_{\pm k}(x,t)$ and $A^{(1)}_{p(0,\pm k,c)} = 1$, we let $a = 0$ and $b = 1$. For $c = 0$, we get

$$A^{(1)}_{p(1,\pm k,0)} = \frac{u_{\pm 2k}\left(x,t\right)}{u_{\pm k}\left(x,t\right)} = v_{\pm k}\left(x,t\right)$$

and

$$A^{(1)}_{p(0,\pm k,0)} = \frac{u_{\pm k}\left(x,t\right)}{u_{\pm k}\left(x,t\right)} = 1$$

and in general

$$A^{(1)}_{p(n,\pm k,0)} = \frac{u_{\pm k(n+1)}}{u_{\pm k}}.$$

Then by Theorem 2, we obtain for $y_{\pm k} = v_{\pm k}(x,t)$ and $z_k = (-t)^k$,

$$A^{(1)}_{p(n,\pm k,0)} = \frac{u_{\pm(n+1)k}}{u_{\pm k}} = \prod_{r=1}^{n}\left(v_{\pm k} - 2\sqrt{(-t)^{\pm k}}\cos\frac{\pi r}{n+1}\right) \tag{3.6}$$

and so, we have the required equation (3.5). $\square$

Indeed, when $k = 1$, then by the above corollary, we get that $u_1 = 1$, $v_1 = x$ and so

$$u_n\left(x,t\right) = t^{(n-1)/2}\prod_{r=1}^{n-1}\left(\frac{x}{\sqrt{t}} - 2i\cos\frac{r\pi}{n}\right)$$

which is given in [**2**]. Thus it can be seen that our result generalizes earlier work.

Corollary 5. *For $n > 1$ and $k > 0$,*

$$u_{(n+1)k} = \begin{cases} u_k \prod\limits_{r=1}^{n/2}\left(v_k^2 - 4y\cos^2\frac{\pi r}{n+1}\right) & \text{if } n \text{ is even,} \\ u_{2k} \prod\limits_{r=1}^{(n-1)/2}\left(v_k^2 - 4y\cos^2\frac{\pi r}{n+1}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $u_{2n} = u_n v_n$, by Corollary 1 and equation (3.6), we have the conclusion. $\square$

Next, we consider both binary and polynomial sequences satisfying $A_{p(1,\pm k,c)} = y_{\pm k}$ and $A_{p(0,\pm k,c)} = 2$. In this case, we label the matrix $M_{n,\pm k}$ and the general sequence $A_{p(n,\pm k,c)}$ by $M^{(2)}_{n,\pm k}$ and $A^{(2)}_{p(n,\pm k,c)}$, respectively.

Theorem 1 implies the following facts. For

$$M^{(2)}_{n,\pm k} = \begin{bmatrix} y_{\pm k} & 2\sqrt{z_{\pm k}} & & & \\ \sqrt{z_{\pm k}} & y_{\pm k} & \sqrt{z_{\pm k}} & & \\ & \sqrt{z_{\pm k}} & y_{\pm k} & \ddots & \\ & & \ddots & \ddots & \sqrt{z_{\pm k}} \\ & & & \sqrt{z_{\pm k}} & y_{\pm k} \end{bmatrix},$$

we have

$$\det M^{(2)}_{n,\pm k} = A^{(2)}_{p(n,\pm k,c)}. \tag{3.7}$$

The $(n \times n)$ tridiagonal matrix Q_2 as shown:

$$Q_2 = \begin{bmatrix} 0 & 2 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of matrix Q_n satisfies the following recurrence relation: for $n > 2$

$$t_n(\delta) = -\delta t_{n-1}(\delta) - t_{n-2}(\delta)$$

where $t_1(\delta) = -\delta$, $t_2(\delta) = \delta^2 - 2$.

Taking $\delta \equiv -2x$, the family $\{t_n(\delta) : n = 1, 2, \dots\}$ is reduced to the family of Chebyshev polynomials of the first kind $\{2T_n(x) : n =$

$1, 2, \dots\}$. From [**7**–**9**], since the zeros of Chebyshev polynomials of the first kind are known, we obtain

$$\delta_r = -2\cos\frac{(2r-1)\,\pi}{2n}, \quad r = 1, 2, \dots, n. \tag{3.8}$$

We can also write $M_{n,k}^{(2)} = y_k I_n + \sqrt{z_k} Q_2$, where I_n is the unit matrix of order n.

Theorem 3. *Let $A_{p(1,\pm k,c)} = y_{\pm k}$ and $A_{p(0,\pm k,c)} = 2$. For $n > 1$, $k > 0$,*

$$A_{p(n,\pm k,c)}^{(2)} = \prod_{r=1}^{n}\left[y_{\pm k} - 2\sqrt{z_{\pm k}}\cos\frac{(2r-1)\,\pi}{2n}\right].$$

Proof. If the eigenvalues of Q_2 are δ_r's with respect to the eigenvectors w_r, then

$$\begin{aligned} M_{n,k}^{(2)} w_r &= (y_k I_n + \sqrt{z_k} Q_2)\, w_r = y_k I_n w_r + \sqrt{z_k} Q_2 w_r \\ &= (y_k + \sqrt{z_k}\delta_r)\, w_r. \end{aligned}$$

Thus the eigenvalues of $M_{n,\pm k}^{(2)}$ are the $y_{\pm k} + \sqrt{z_{\pm k}}\delta_r$'s with respect to the eigenvectors w_r. By (3.8),

$$\det M_{n,\pm k}^{(2)} = \prod_{r=1}^{n}\left(y_{\pm k} + \sqrt{z_{\pm k}}\delta_r\right) = \prod_{r=1}^{n}\left[y_{\pm k} - 2\sqrt{z_{\pm k}}\cos\frac{(2r-1)\,\pi}{2n}\right].$$

By (3.7), the theorem is proven. $\square$

Corollary 6. *For $n > 1$, $k > 0$,*

$$A_{p(n,k,c)}^{(2)} = \begin{cases} y_k \displaystyle\prod_{r=1}^{(n-1)/2}\left[y_k^2 - 4z_k\cos^2\left(\tfrac{(2r-1)\pi}{2n}\right)\right] & \textit{if } n \textit{ is odd,} \\ \displaystyle\prod_{r=1}^{n/2}\left[y_k^2 - 4z_k\cos^2\left(\tfrac{(2r-1)\pi}{2n}\right)\right] & \textit{if } n \textit{ is even.} \end{cases}$$

Now we give some applications of the above results for some well-known sequences.

Corollary 7. *Let $v_n(x,t)$ be the generalized Lucas polynomial sequence. Then for $n > 1$, $k > 0$,*

$$v_{\pm kn}(x,t) = \prod_{r=1}^{n} \left(v_{\pm k}(x,t) - 2\sqrt{(-t)^{\pm k}} \cos \frac{(2r-1)\pi}{2n} \right).$$

Proof. When $p = x$, $q = t$, then the first coefficient $y_{\pm k} = v_{\pm k}(x,t)$. For our second case and satisfying $A^{(2)}_{p(1,\pm k,c)} = y_{\pm k} = v_{\pm k}$ and $A^{(2)}_{p(0,\pm k,c)} = 2$, letting $a = 2$ and $b = x$, we see that for $c = 0$

$$A^{(2)}_{p(1,\pm k,0)} = v_{\pm k} \quad \text{and} \quad A^{(2)}_{p(0,\pm k,0)} = x,$$

and in general $A^{(2)}_{p(n,\pm k,)} = v_{\pm kn}$. By Theorem 3, we have the conclusion: for $y_{\pm k} = v_{\pm k}$ and $z_{\pm k} = (-t)^{\pm k}$,

$$\begin{aligned} A^{(2)}_{p(n,\pm k,0)} &= v_{\pm kn}(x,t) \\ &= \prod_{r=1}^{n} \left(v_{\pm k}(x,t) - 2\sqrt{(-t)^{\pm k}} \cos \frac{(2r-1)\pi}{2n} \right). \qquad \square \end{aligned}$$

Corollary 8. *Let $v_n(x,t)$ be the generalized Lucas polynomial sequence. Then for $n > 1$, $k > 0$,*

$$v_{\pm kn}(x,t) = \begin{cases} v_{\pm k}(x,t) \prod\limits_{r=1}^{(n-1)/2} \left[v_{\pm k}(x,t)^2 - 4(-t)^{\pm k} \cos^2\left(\frac{(2r-1)\pi}{2n}\right) \right] & \text{if } n \text{ is odd,} \\ \prod\limits_{r=1}^{n/2} \left[v_{\pm k}(x,t)^2 - 4(-t)^{\pm k} \cos^2\left(\frac{(2r-1)\pi}{2n}\right) \right] & \text{if } n \text{ is even.} \end{cases}$$

For the Chebyshev polynomial of the first kind, if we consider $v_n(2x,-1)$ in the above two corollaries, then we see that $v_n(2x,-1) = 2T_n$, so

$$2T_{\pm kn} = v_{\pm kn}(2x,-1) = \prod_{r=1}^{n} \left(2T_{\pm k} - 2\cos \frac{(2r-1)\pi}{2n} \right)$$

and clearly

$$T_{\pm kn} = 2^{n-1} \prod_{r=1}^{n} \left(T_{\pm k} - \cos \frac{(2r-1)\,\pi}{2n} \right).$$

In this third case, we let $A_{p(1,\pm k,c)} - y_{\pm k} = -1$ and $A_{p(0,\pm k,c)} = 1$. Further, we label the matrix $M_{n,\pm k}$ and the sequence $A_{p(n,\pm k,c)}$ by $M^{(3)}_{n,\pm k}$ and $A^{(3)}_{p(n,\pm k,c)}$, respectively.

Theorem 1 implies the following facts. For the $n \times n$ tridiagonal matrix $M^{(3)}_{n,k}$:

$$M^{(3)}_{n,\pm k} = \begin{bmatrix} y_{\pm k} - 1 & \sqrt{z_{\pm k}} & & & \\ \sqrt{z_{\pm k}} & y_{\pm k} & \sqrt{z_{\pm k}} & & \\ & \sqrt{z_{\pm k}} & y_{\pm k} & \ddots & \\ & & \ddots & \ddots & \sqrt{z_{\pm k}} \\ & & & \sqrt{z_{\pm k}} & y_{\pm k} \end{bmatrix},$$

by Theorem 1, we have for $n > 1$

$$\det M^{(3)}_{n,\pm k} = A^{(3)}_{p(n,\pm k,c)}. \tag{3.9}$$

Define the $(n \times n)$ tridiagonal matrix Q_3 as shown:

$$Q_3 = \begin{bmatrix} -1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

Thus the characteristic polynomial of the matrix Q_3 satisfies the following recurrence relation (for $n > 2$)

$$g_n(\mu) = -\mu g_{n-1}(\mu) - g_{n-2}(\mu),$$

where $g_1(\mu) = -\mu - 1$, $g_2(\mu) = \mu^2 + \mu - 1$.

If we take $\delta \equiv -2x$, then the family $\{g_n(\mu) : n = 1, 2, \dots\}$ is reduced to the family of Chebyshev polynomials of the third kind

$\{V_n(x) : n = 1, 2, \dots\}$. From [**7**–**9**], the zeros of Chebyshev polynomials of the third kind are known; then we obtain

$$\mu_k = -2\cos\frac{(k-(1/2))\,\pi}{n+(1/2)}, \quad k = 1, 2, \dots, n. \tag{3.10}$$

We can also write $M^{(3)}_{n,\pm k} = y_{\pm k} I_n + \sqrt{z_{\pm k}} Q_3$.

Theorem 4. *Let $A_{p(1,\pm k,c)} - y_{\pm k} = -1$ and $A_{p(0,\pm k,c)} = 1$. For $n > 1$, $k > 0$,*

$$A^{(3)}_{p(n,\pm k,c)} = \prod_{r=1}^{n} \left(y_{\pm k} - 2\sqrt{z_{\pm k}} \cos\frac{(2k-1)\,\pi}{2n+1} \right).$$

Proof. Let the eigenvalues of the matrix Q_3 be μ_r with respect to the eigenvectors w_r, and write

$$M^{(3)}_{n,\pm k} w_r = \left(y_{\pm k} I_n + \sqrt{z_{\pm k}} Q_3 \right) w_r = \left(y_{\pm k} + \sqrt{z_{\pm k}} \mu_r \right) w_r.$$

Thus the eigenvalues of $M^{(3)}_{n,\pm k}$ are $(y_{\pm k} + \sqrt{z_{\pm k}} \mu_r)$. By equations (3.9) and (3.10), we have

$$\det M^{(3)}_{n,\pm k} = A^{(3)}_{p(n,\pm k,c)} = \prod_{r=1}^{n} \left(y_{\pm k} - 2\sqrt{z_{\pm k}} \cos\frac{(2k-1)\,\pi}{2n+1} \right).$$

Thus the theorem is proven. $\square$

The following corollary follows directly from Theorem 4 since, for $1 \le k \le n/2$, we have $\cos(\frac{k\pi}{n}) = -\cos(((n-k)/\pi)/n)$.

Corollary 9. *For $n > 1$, $k > 0$,*

$$A^{(3)}_{p(n,\pm k,c)} = \begin{cases} y_{\pm k} \displaystyle\prod_{r=1}^{(n-1)/2} \left(y^2_{\pm k} - 4 z_{\pm k} \cos^2\left(\tfrac{(2k-1)\pi}{2n+1} \right) \right) & \text{if } n \text{ is odd}, \\ \displaystyle\prod_{r=1}^{n/2} \left(y^2_{\pm k} - 4 z_{\pm k} \cos^2\left(\tfrac{(2k-1)\pi}{2n+1} \right) \right) & \text{if } n \text{ is even}. \end{cases}$$

A particular case of the previous corollary can be found in the literature.

Corollary 10. *For $n > 1$,*

$$T_{4n+2} = 2^n T_2 \prod_{k=1}^{n} \left(T_4 - \cos \frac{(2k-1)\,\pi}{2n+1} \right). \tag{3.11}$$

Proof. For $k = 2$, $p = 2x$, $q = -1$, the first coefficient is $y_4 = 2T_4$ (Chebyshev polynomial of the first kind). Also under the conditions $a = 1$ and $b = 2x$, we obtain $A^{(3)}_{p(1,4,-2)} = y_4 = 2T_4$ and $A^{(3)}_{p(0,4,-2)} = 1$,

$$A^{(3)}_{p(1,4,-2)} = \frac{T_6}{T_2} = 2T_4 - 1 \quad \text{and} \quad A^{(3)}_{p(0,4,-2)} = \frac{T_2}{T_2} = 1,$$

and in general,

$$A^{(3)}_{p(n,4,-2)} = \frac{T_{4(n+1)-2}}{T_2}.$$

Then by Theorem 4, we get for $y_4 = 2T_4$ and $z_4 = 1$

$$A^{(3)}_{p(n,4,-2)} = \frac{T_{4(n+1)-1}}{T_2} = \prod_{r=1}^{n} \left(2T_4 - 2\cos \frac{(2k-1)\,\pi}{2n+1} \right), \tag{3.12}$$

and so equation (3.11) is obtained. $\square$

By Corollary 9 and the above result, we have

$$T_{4n+2} = \begin{cases} 2^n T_4 T_2 \prod\limits_{r=1}^{(n-1)/2} \left(T_4^2 - \cos^2 \left(\frac{(2k-1)\pi}{2n+1} \right) \right) & \text{if } n \text{ is odd,} \\ 2^n T_2 \prod\limits_{r=1}^{n/2} \left(T_4^2 - \cos^2 \left(\frac{(2k-1)\pi}{2n+1} \right) \right) & \text{if } n \text{ is even.} \end{cases}$$

Further, we consider our final case $A_{p(1,\pm k,c)} - y_{\pm k} = 1$ and $A_{p(0,\pm k,c)} = 1$. In this case, we label the matrix $M_{n,\pm k}$ and the sequence $A_{p(n,\pm k,c)}$ by $M^{(4)}_{n,\pm k}$ and $A^{(4)}_{p(n,\pm k,c)}$, respectively. By Theorem 1, we have

$$M^{(4)}_{n,\pm k} = \begin{bmatrix} y_{\pm k}+1 & \sqrt{z_{\pm k}} & & & \\ \sqrt{z_{\pm k}} & y_{\pm k} & \sqrt{z_{\pm k}} & & \\ & \sqrt{z_{\pm k}} & y_{\pm k} & \ddots & \\ & & \ddots & \ddots & \sqrt{z_{\pm k}} \\ & & & \sqrt{z_{\pm k}} & y_{\pm k} \end{bmatrix},$$

and for $n > 1$

$$\det M^{(4)}_{n,\pm k} = A^{(4)}_{p(n,\pm k,c)}. \tag{3.13}$$

We also define the $(n \times n)$ matrix Q_4 as follows:

$$Q_4 = \begin{bmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & 1 & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

Thus the characteristic polynomial of Q_4 satisfies the recurrence

$$h_{n+1}(\omega) = -\omega h_n(\omega) - h_{n-1}(\omega)$$

where $h_1(\omega) = 1 - \omega$ and $h_2(\omega) = \omega^2 + \omega + 1$.

Taking $\omega \equiv -2x$, the family $\{h_n(\omega)\}$ is converted into the family of Chebyshev polynomials of the fourth kind $\{W_n(x) : n = 1, 2, \dots\}$. By [**7**–**9**], the zeros of Chebyshev polynomials of the fourth kind are known, namely

$$\omega_k = -2\cos\frac{(k-1)\pi}{n+\frac{1}{2}}, \quad k = 1, 2, \dots, n. \tag{3.14}$$

The matrix $M^{(4)}_{n,\pm k}$ can be expressed as $y_{\pm k} I_n + \sqrt{z_{\pm k}} Q_4$.

Theorem 5. *Let $A_{p(1,\pm k,c)} - y_{\pm k} = 1$ and $A_{p(0,\pm k,c)} = 1$. For $n > 1$, $k > 0$,*

$$A^{(4)}_{p(n,\pm k,c)} = \prod_{r=1}^{n}\left(y_{\pm k} - 2\sqrt{z_{\pm k}}\cos\frac{2(r-1)\pi}{2n+1}\right).$$

Proof. If ω_r's are the eigenvalues of $M^{(4)}_{n,\pm k}$ according to the eigenvectors w_r, then

$$M^{(4)}_{n,\pm k} w_r = \left(y_{\pm k} I_n + \sqrt{z_{\pm k}} Q_4\right) w_r = y_{\pm k} + \sqrt{z_{\pm k}} \omega_r.$$

Thus the eigenvalues of $M^{(4)}_{n,\pm k}$ are $(y_{\pm k} + \sqrt{z_{\pm k}} \omega_r)$ and by (3.13), (3.14), we obtain the conclusion:

$$\det M^{(4)}_{n,\pm k} = \prod_{r=1}^{n} \left(y_{\pm k} - 2\sqrt{z_{\pm k}} \cos \frac{2\,(r-1)\,\pi}{2n+1} \right). \qquad \square$$

As a consequence of Theorem 5, we have the following corollary.

Corollary 11. *For $n > 1$, $k > 0$,*

$$A^{(4)}_{p(n,\pm k,c)} = \begin{cases} y_{\pm k} \displaystyle\prod_{r=1}^{(n-1)/2} \left(y^2_{\pm k} - 4 z_{\pm k} \cos^2 \left(\tfrac{2(r-1)\pi}{2n+1} \right) \right) & \textit{if } n \textit{ is odd,} \\ \displaystyle\prod_{r=1}^{n} \left(y^2_{\pm k} - 4 z_{\pm k} \cos^2 \left(\tfrac{2(r-1)\pi}{2n+1} \right) \right) & \textit{if } n \textit{ is even.} \end{cases}$$

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