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GLOBAL BIFURCATION OF NONNEGATIVE SOLUTIONS FOR A QUASILINEAR ELLIPTIC PROBLEM

PENG FENG

ABSTRACT. We study the structure of positive solutions for an m-Laplacian boundary value problem involving singular nonlinearities. We obtain the precise global bifurcation diagram of the solutions.

1. Introduction. In this paper we study the structure of positive solutions u of the following quasilinear problem

$$\begin{cases} (|u'|^{m-2}u')' + u^{-\alpha} - u^{-\beta} = 0 & \text{in } (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.1}$$

where $L > 0$ is a bifurcation parameter, and $m > 1$.

The equation in (1.1) arises in many interesting applications such as the steady states of thin films, chemical heterogeneous catalysts, non-Newtonian fluids. The thin film dynamics can be modeled by:

$$u_t = -\nabla \cdot (f(u)\nabla \Delta u) - \nabla \cdot (g(u)\nabla u). \tag{1.2}$$

Here, the air liquid interface is represented by the graph of the function $z = u(x,t)$. The coefficient $f(u)$ is used to model the surface tension. It is degenerate at 0, i.e., $f(0) = 0$ and $f(u) > 0$ if $u > 0$. The coefficient $g(u)$ models the additional forces such as gravity. If a thin film is hanging from a flat surface in the presence of gravity, it is typical to choose $f(u) = u^3$ and $g(u) = u^3$. We refer the readers to the review articles [**13, 14, 17, 18**] for more information on the modeling of thin liquid films.

If we consider the van der Waals interactions $g(u) = u^{l_1} - cu^{l_2}$ with $l_1 < 0$, $l_2 < 0$ and $c \geq 0$ and $|l_1| > |l_2|$ [**18**]. The steady-state of (1.2) satisfies:

$$f(u)\nabla\Delta u + g(u)\nabla u = \mathbf{C}$$

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where $\mathbf{C}$ is a constant vector. Assuming $\mathbf{C} = \mathbf{0}$, we have

$$\Delta u + \frac{1}{l_1 - 2} u^{l_1 - 2} - \frac{c}{l_2 - 2} u^{l_2 - 2} = C,$$

where C is a constant. If we assume $C = 0$ and take $v = (|l_1| + 2)^{1_2/(2-l_1)} u$, we have the following equation for v:

$$\Delta v = v^{l_1 - 2} - \frac{c(|l_1| + 2)^{(3-l_2)/(3-l_1)}}{|l_2| + 2} v^{l_2 - 2},$$

which is in the form of the equation

$$\Delta v = v^{-\beta} - \lambda v^{-\alpha}. \tag{1.3}$$

As usual, we may replace the parameter λ by the radius of the ball domain. Equation (1.3) is equivalent to

$$\Delta u + (u^{-\alpha} - u^{-\beta}) = 0, \quad u > 0 \text{ in } B_L, \quad u = 0 \text{ on } \partial B_L, \tag{1.4}$$

where B_L is a ball in $\mathbf{R}^N$ with radius L.

If we consider the related m-Laplacian equation in one dimensional space, we arrive at equation (1.1).

A similar problem

$$-\Delta u = \frac{\lambda}{(1-u)^2} \text{ in } \Omega, \quad 0 < u < 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

arises in the model for a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with supposed boundary at 0 below a rigid plate located at 1. λ corresponds to the voltage applied to the membrane. As the voltage increases, the membrane will deflect toward the plate. One of the primary device design goals is to achieve the maximum possible stable deflection, referred to as the pull-in stability, or rather instability. This problem has been studied by Feng and Zhou in [**7**] where they obtained the multiplicity result for radial solutions on annular domain. For more details on this model, we refer the interested readers to [**22**].

Problems of similar type $\Delta u + f(u) = 0$ with $f(u) = u^q - u^p$ have been studied by many authors. For example, in Serrin and Tang's work [**23**], they obtained uniqueness of radial solutions with Dirichlet-Newmann boundary condition under the condition $p < q$ and $N > m > 1$.

Several authors have considered the case $f(u) = \lambda u^p - u^{-\beta}$ where $p > 0$ and λ serves as a bifurcation parameter. The case $p = 1$ has been considered [**3**]. Chen showed that there exists λ^*, $\overline{\lambda}$ with $\lambda_1 \leq \lambda^* < \overline{\lambda}$, where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, such that there exists a unique positive radial solution if $\lambda^* < \lambda < \lambda$. Ouyang, Shi and Yao [**21**] studied the case $0 < p < 1$, Related results for nonlinearity $f(u) = \lambda u^{-\alpha} + u^p$ can be found in [**11**]. For nonlinearity with one singular term $f(u) = -u^{-\beta}$, the existence, uniqueness results can be found in [**8, 15**].

To the best of our knowledge, there are few results concerning the nonlinearity with double singularities. The fact that the nonlinearity is of concave-convex type and is singular at the origin makes the analysis more challenging. We shall mention that in some recent papers [**9, 10**] the authors studied the elliptic equations with $f(u) = \lambda(u^{-\alpha} - u^{-\beta})$ with $u = k$ on $\partial\Omega$. However, they considered the case where $k \in (0, 1]$. Thus the singularity at 0 is avoided. In an upcoming paper [**7**], we studied the structure of radial solutions in higher dimension. We are able to obtain some similar results using the shooting method.

We consider the following range of values for α and β:

Case 1. $0 < \alpha < \beta < 1$;

Case 2. $1 > \alpha > \beta > 0$.

We note the following properties of f:

Case 1. (H1) $f(u) < 0$ for $u < 1$, $f(u) \geq 0$ for $u \geq 1$;

(H2) $f'(u) > 0$ for $0 < u < (\beta/\alpha)^{1/(\beta-\alpha)}$, $f'(u) < 0$ for $u > (\beta/\alpha)^{1/(\beta-\alpha)}$;

(H3) $f''(u) < 0$ for $u < [\beta(\beta+1)/\alpha(\alpha+1)]^{1/(\beta-\alpha)}$, $f''(u) > 0$ for $u > [\beta(\beta+1)/\alpha(\alpha+1)]^{1/(\beta-\alpha)}$;

Case 2. (H1) $f(u) < 0$ for $u < 1$, $f(u) \leq 0$ for $u \geq 1$;

(H2) $f'(u) < 0$ for $0 < u < (\beta/\alpha)^{1/(\beta-\alpha)}$, $f'(u) > 0$ for $u > (\beta/\alpha)^{1/(\beta-\alpha)}$;

(H3) $f''(u) > 0$ for $u < [\beta(\beta+1)/\alpha(\alpha+1)]^{1/(\beta-\alpha)}$, $f''(u) < 0$ for $u > [\beta(\beta+1)/\alpha(\alpha+1)]^{1/(\beta-\alpha)}$;

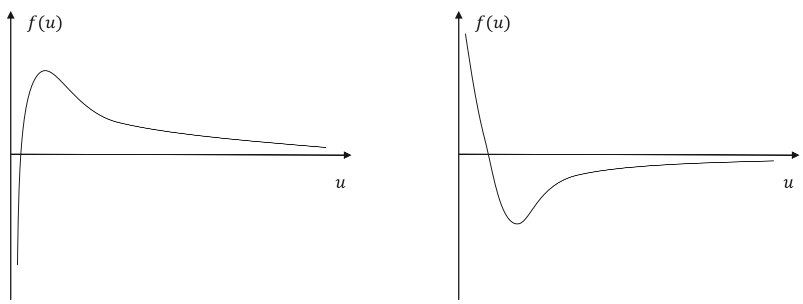

FIGURE 1. Illustration of $f(u)$.

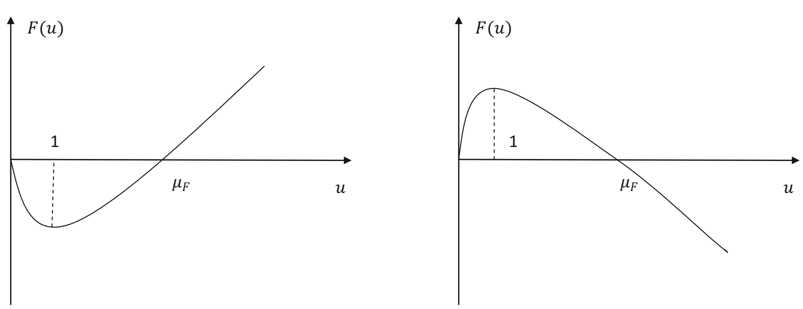

FIGURE 2. Illustration of $F(u)$.

The graphs of f of both cases are illustrated in Figure 1.

We introduce the notation $f(t) = t^{-\alpha} - t^{-\beta}$ and $F(t) := \int_0^t f(s)\,ds$. We note that $\mu_F = (1-\alpha/1-\beta)^{1/(\beta-\alpha)}$ is the unique positive zero of $F(t)$. Another zero of $F(t)$ is at $t = 0$. See Figure 2 for an illustration of $F(u)$.

Our main results are the following:

Theorem 1.1. *For* $0 < \alpha < \beta < 1$, $\beta \le 1/(m+1)$, *there exist* $L_1 > L_0 > 0$ *such that*:

(a) *if* $L < L_0$, *there are no positive solutions*;

(b) *if* $L = L_0$, *there is a unique solution*;

(c) *if* $L_0 < L < L_1$, *there are exactly two positive solutions*;

(d) *if* $L > L_1$, *there is one positive solution.*

Theorem 1.2. *For* $1 > \alpha > \beta > 0$ *and* $\alpha \le 1/(m+1)$, *there exists* L_2, *such that there is a unique solution for* $0 < L < L_2$ *and no solution for* $L > L_2$.

The proof of our results is based on an explicit formula for the time map as a function of the maximum value $\|u\|_\infty$ of the corresponding solution. More precisely, we study the behavior of the time map as $\|u\|_\infty$ varies. This approach has been applied by Laetsch [**12**]. It has also appeared in the work by Smoller [**24, 25**] and Wang [**26**].

2. Proof of Theorem 1.1. Since we are dealing only with positive solutions, $u(x)$ is symmetric with respect to $x = 0$ and $u' > 0$ for $-L < u < 0$. We consider

$$\begin{cases} -(|u'|^{m-2}u')' = f(u) & \text{in } (-L,0), \\ u(0) = p & u'(0) = 0. \end{cases} \tag{2.1}$$

Multiplying the equation by u', integrating by parts and using the initial conditions we find

$$u' = \left(\frac{m}{m-1}\right)^{1/m} [F(p) - F(u)]^{1/m}.$$

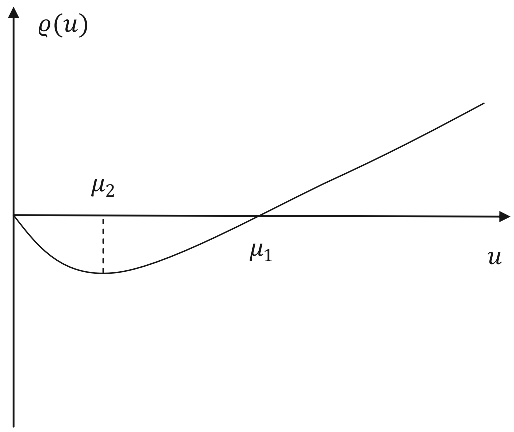

FIGURE 3. Illustration of $\varrho(u)$.

Now, integrating on $[-L, 0]$,

$$L = \left(\frac{m-1}{m}\right)^{1/m} \int_0^p \frac{du}{(F(p)-F(u))^{1/m}} := T(p).$$

Solutions u of (1.1) correspond to $T(p) = L$ and $\|u\|_\infty = p$. Thus to study the positive solutions of (1.1) is equivalent to studying the shape of the time map $T(p)$. It's easy to see that if $p < 1$, no solutions of (2.1) will satisfy (1.1). Moreover, if $1 < p < \mu_F$, no solutions of (2.1) will satisfy (1.1).

It is useful to introduce the following auxiliary function along with its properties.

Lemma 2.1. *Let $\varrho(t) = mF(t) - tf(t)$. Then there exists $\mu_1 \in (\mu_F, +\infty)$ such that $\varrho(t) < 0$ on $(0, \mu_1)$ and $\varrho(t) > 0$ on $(\mu_1, +\infty)$. There exists a μ_2 such that $\varrho'(t) < 0$ on $(0, \mu_2)$ and $\varrho'(t) > 0$ on $(\mu_2, +\infty)$. Moreover, $\mu_F < \mu_2$ if $\alpha + \beta < 2 - m$. $\varrho(u)$ is illustrated in Figure 3.*

The proof of this lemma is elementary based on the explicit formula for μ_1, μ_2 and μ_F. In fact,

$$\mu_1 = \left(\frac{(m/(1-\beta)) - 1}{(m/(1-\alpha)) - 1}\right)^{1/(\beta-\alpha)}, \qquad \mu_2 = \left(\frac{m+\beta-1}{m+\alpha-1}\right)^{1/(\beta-\alpha)}.$$

Lemma 2.2.

$$T(p) \sim \left(\frac{m-1}{m}\right)^{1/m} (1-\alpha)^{(1-m)/m} \mathbf{B}\left(\frac{1}{1-\alpha}, \frac{m-1}{m}\right) p^{(m-1+\alpha)/m}$$

as $p \to +\infty$. *Here* $\mathbf{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, $x, y > 0$ *is the beta function.*

The proof of this lemma is a modification of the proof of Theorem 2.7 in [**12**].

Proposition 2.3. (i) $T(p) \in C[\mu_F, +\infty) \cap C^1(\mu_F, +\infty)$;

(ii) $T(p) \to +\infty$ *as* $p \to +\infty$;

(iii) *If* $\beta \leq 1/(m+1)$, *then* $T'(p) \to -\infty$ *as* $p \to \mu_F$.

Proof. (i) A direct calculation shows

$$T'(p) = \left(\frac{m-1}{m}\right)^{1/m} \frac{1}{m} \int_0^p \frac{\varrho(p) - \varrho(u)}{(F(p) - F(u))^{(m+1)/m}} \frac{du}{p}$$

For $p \in (\mu_F, +\infty)$, $F'(p) > 0$, it is easy to verify that the above integral is convergent and $T'(p) \in C(\mu_F, +\infty)$. It follows that $T(p) \in C[\mu_F, +\infty) \cap C^1(\mu_F, +\infty)$.

(ii) This follows directly from Lemma 2.2.

(iii) Apply Fatou's lemma and the fact that $\int_0^\delta 1/(-F(u))^{(m+1)/m}\, du$ is not integrable for any $\delta > 0$ small if $\beta \leq 1/(m+1)$, we can prove this property. □

In view of Lemma 2.1, we have $T'(p) < 0$ on (μ_F, μ_2) and $T'(p) > 0$ on (μ_1, ∞). It is necessary that $T'(p)$ have at least one zero in the interval $[\mu_2, \mu_1]$. In our next proposition, we show that $T(p)$ can have only one zero in this interval.

Proposition 2.4. *The equation* $T'(p) = 0$ *has a unique root in* (μ_2, μ_1).

We directly compute $T''(p)$ from $T'(p)$, see also

(2.2)

$$T''(p) = \left(\frac{m-1}{m}\right)^{1/m} \frac{1}{m}\frac{1}{p^2}\int_0^p \frac{-[(m+1)/m]\Delta\varrho\Delta\widetilde{f} + \Delta F\Delta\widetilde{\varrho}'}{(\Delta F)^{(2m+1)/m}}\, du,$$

where

$$\begin{aligned}\Delta\varrho &= \varrho(p) - \varrho(u),\\ \Delta\widetilde{f} &= pf(p) - uf(u),\\ \Delta\widetilde{\varrho}' &= p\varrho'(p) - u\varrho'(u),\\ \Delta F &= F(p) - F(u).\end{aligned}$$

A careful calculation yields

$$\begin{aligned}(2.3)\quad & T''(p) + \frac{C}{p}T'(p)\\ &= \left(\frac{m-1}{m}\right)^{1/m}\frac{1}{m}\\ &\times \int_0^p \frac{C\Delta F(m\Delta F - \Delta\widetilde{f}) + [(m+1)/m](\Delta\widetilde{f})^2 - 2\Delta\widetilde{f}\Delta F - \Delta\widehat{f}'\Delta F}{p^2(\Delta F)^{(2m+1)/m}}\\ &= \left(\frac{m-1}{m}\right)^{1/m}\frac{1}{m}\\ &\times \int_0^p \frac{[(m+1)/m]\Delta\widetilde{f}^2 - (2 + (\Delta\widehat{f}'/\Delta\widetilde{f}) + C)\Delta\widetilde{f}\Delta F + mC(\Delta F)^2}{p^2(\Delta F)^{(2m+1)/m}},\end{aligned}$$

where $\Delta\widehat{f'} = p^2 f'(p) - u^2 f'(u)$.

For fixed $p > \mu_F$, it is easy to verify the following inequalities

(2.4)

$$\frac{(1-\beta)p^{-\beta} - (1-\alpha)p^{-\alpha}}{p^{-\beta} - p^{-\alpha}} \le \frac{\Delta\widetilde{f}}{\Delta F} \le \frac{p^{1-\alpha} - p^{1-\beta}}{[(p^{1-\alpha})/(1-\alpha)] - [(p^{1-\beta})/(1-\beta)]},$$

and

$$\frac{\alpha(1-\alpha)p^{-\alpha} - \beta(1-\beta)p^{-\beta}}{-(1-\alpha)p^{-\alpha} + (1+\beta)p^{-\beta}} \le \frac{\Delta\widehat{f}'}{\Delta\widetilde{f}} \le \frac{-\alpha p^{1-\alpha} + \beta p^{1+\beta}}{p^{1-\alpha} - p^{1-\beta}}.$$

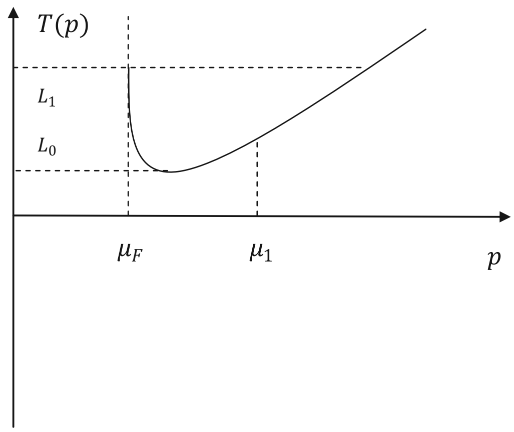

FIGURE 4. Bifurcation curve $T(p)$.

It is then straightforward to verify that

(2.5)

$$\begin{aligned} Q(s) &:= \frac{C\Delta F(m\Delta F - \Delta\widetilde{f}) + [(m+1)/m](\Delta\widetilde{f})^2 - 2\Delta\widetilde{f}\Delta F - \Delta\widehat{f}'\Delta F}{p^2(\Delta F)^{(2m+1)/m}} \\ &\geq (\Delta F)^2\left[\frac{m+1}{m}\lambda^2 - (2+M+C)\lambda + mC\right] \\ &\geq 0 \end{aligned}$$

for suitable C. Here M denotes the maximum value of $(\Delta\widehat{f}'/\Delta\widetilde{f})$ and λ denotes $\Delta\widehat{f}/\Delta F$.

The results of Theorem 1.1 follow directly from Propositions 2.3 and 2.4, see Figure 4 for the bifurcation diagram.

3. Proof of Theorem 1.2.

Lemma 3.1. *Suppose that* $0 \leq \lim_{u\to 0^+} f(u)/u^{m-1} := l_0 \leq \infty$. *Then*

$$\lim_{p\to 0^+} = \frac{m-1}{l_0}^{1/m}\frac{\pi}{m}\csc\frac{\pi}{m} \geq 0.$$

In particular, for $f(u) = u^{-\alpha} - u^{-\beta}$, $T(p) \to 0$ *as* $p \to 0$.

This lemma appeared in, for example, [**26**]. The proof of this lemma is very similar to the proofs of Theorems 2.5–2.7 and Theorem 2.9 in [**12**].

Lemma 3.2. *If* $\alpha \le 1/m+1$, *then* $\lim_{p\to\mu_F^-} T'(p) = +\infty$.

Lemma 3.3. *For* $p \in [0,\mu_F]$, $T'(p) \ge 0$.

Proof. For $p \in [0,\mu_2]$, since $\varrho'(t) > 0$, it's clear that $T'(p) > 0$. Moreover, if $\mu_F < \mu_2$, this result is then trivial.

To prove the general case, we suppose otherwise that there exists p_0 such that $T'(p_0) < 0$. In view of Lemma 3.2, there exists $\widetilde{p} \in [0,\mu_F]$ such that $T''(\widetilde{p}) < 0$ and $T'(\widetilde{p}) = 0$. However, similar to the proof of Proposition 2.4, we can show that $T''(p) + (C/p)T'(p) > 0$ for suitable constant $C > 0$. This is clearly a contradiction. □

Theorem 1.2 follows directly from Lemma 3.1, 3.2 and 3.3.

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Department of Mathematics, Florida Gulf Coast University, 10501 FGCU Blvd., Fort Myers, FL 33965
Email address: pfeng@fgcu.edu