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ON THE CONDITIONAL EXPECTATION OF THE FIRST EXIT TIME OF BROWNIAN MOTION

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ABSTRACT. Let U be a domain, convex in x and symmetric about the y-axis, which is contained in a centered and oriented rectangle R. If τ_A is the first exit time of a Brownian motion from A and $A^+ = A \cap \{(x,y) : x > 0\}$, it is proved that $E^z(\tau_{U^+} \mid \tau_{R^+} > t) \leq E^z(\tau_U \mid \tau_R > t)$ for every $t > 0$ and every $z \in U^+$.

1. Introduction. In this note we prove an inequality for the conditional expectation of the first exit time of a Brownian motion from a bounded domain. This inequality is in the spirit of the ratio inequalities proved in [**3, 8, 9, 11, 17**].

Davis [**8**] proved the first inequality of this kind for the heat kernel of Laplacian, in order to obtain a lower estimate for the gap between the first two eigenvalues of the Laplacian. Bañuelos and Méndez-Hernández [**3**] extended Davis's result to the heat kernel of Schrödinger operators and integrals of these kernels. You [**17**] proved an inequality of this type for the trace of Schrödinger operators. Davis and Hosseini [**9**] proved the extension to the heat content.

We call a set $A \subset \mathbf{R}^2$ convex in x if its intersection with every line parallel to the x-axis is a single interval or empty. We put $A^+ = A \cap \{(x,y) \mid x > 0\}$. Let $B_t = (B_{1,t}, B_{2,t})$, $t \geq 0$, be a standard two-dimensional Brownian motion and $\tau_A = \inf\{t > 0 : B_t \notin A\}$ (for a general reference on Brownian motion, random walks, and other topics in probability theory discussed in this note, see [**6**]). The literature on the estimates for τ_A for various A is quite extensive (see, for instance [**2, 4, 7, 10, 14, 15**]). We will prove the following.

Theorem 1. *Let U be an open, bounded and connected set in $\mathbf{R}^2$ which is symmetric about the y-axis and convex in x. Also, let R be an open rectangle containing U that is symmetric with respect to the*

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y-axis, and has sides parallel to the axes. If z is a point in U^+, and $t > 0$, then

$$E^z(\tau_{U^+} \mid \tau_{R^+} > t) \leq E^z(\tau_U \mid \tau_R > t). \tag{1}$$

For an open, bounded and connected set $V \subset \mathbf{R}^2$, the Brownian motion conditioned to stay forever in V is the diffusion process $\{Y_s^V\}_{s\geq 0}$ with generator

$$L_V = \frac{1}{2}\Delta + \frac{\nabla\varphi_0^V}{\varphi_0^V}\nabla,$$

where φ_0^V is the positive eigenfunction corresponding to λ_0, the simple eigenvalue at the bottom of the spectrum of $-(1/2)\Delta$, with the Dirichlet boundary condition on ∂V [**16**]. If $U \subset V$ is also open and connected, define the first exit time of this process out of U by $\tau_U^V = \inf\{s > 0 : Y_s^V \notin U\}$.

If we let $t \to \infty$ in (1), we get the following.

Proposition 2. *Let U, R and z be as in Theorem 1. Then*

$$E^z\left(\tau_{U^+}^{R^+}\right) \leq E^z\left(\tau_U^R\right).$$

Since

$$E^z(\tau_U \mid \tau_R > t) = \int_0^\infty P^z(\tau_U > s \mid \tau_R > t)\, ds,$$

and a similar equation holds for $E^z(\tau_{U^+} \mid \tau_{R^+} > t)$, Theorem 1 will be proved if we prove the following.

Proposition 3. *Let R, U and z be as in Theorem 1. Then for every $s, t > 0$,*

$$P^z(\tau_{U^+} > s \mid \tau_{R^+} > t) \leq P^z(\tau_U > s \mid \tau_R > t). \tag{2}$$

To prove Proposition 3, we first prove a discrete analog of inequality (2) and then use scaling. The discrete analog is stated and proved in the following section. The method of deriving inequality (2) from its discrete counterpart is a standard application of the invariance principle. We will omit this derivation for the sake of brevity. See [**9**] for a detailed description of an almost identical derivation.

Note that Proposition 2 follows from a weaker form of Proposition 3; we need only to prove inequality (2) for $t > s$. We will discuss later in this note that this case of Proposition 3 has been essentially proved in [**9**].

The following example shows that the conclusions of Theorem 1 and Proposition 3 are not valid if the convexity condition is removed. Let $0 < d < 1/2$, and take $U = (-1,1) \times (-1,1) \setminus \{(0,y) : |y| \geq 1/4\}$. Let $R = (-1,1) \times (-1,1)$ and $z = (d, 1/2)$. We will show at the end of Section 2 that, for this example, if $t > 0$ is fixed, the right sides of (1) and (2) converge to zero as $d \to 0$ while the left side of (1) is bounded away from zero for all $0 < d < 1/2$ and the left side of (2) equals 1 if $s \leq t$.

If all boundary points of U are regular, then the function $f_U(z) = E^z(\tau_U)$ is the unique solution of the Poisson equation

$$\begin{cases} \Delta f_U = -2 & \text{in } U; \\ f_U = 0 & \text{on } \partial U. \end{cases}$$

Theorem 1 and the ratio inequalities proved in [**3, 8, 9, 17**] lead to the following conjecture.

Conjecture 4. *Let U, R and z be as in Theorem* 1. *Then*

$$\frac{E^z(\tau_{U^+})}{E^z(\tau_{R^+})} \leq \frac{E^z(\tau_U)}{E^z(\tau_R)}.$$

2. Discrete-time inequalities. In this section we prove a discrete version of inequality (2) which, as we pointed out earlier, will imply inequality (2) by an application of invariance principle, see [**13**, Theorem 2.4.20]. To keep the notation simple, we now assume that U and R are subsets of $\mathbf{Z}^2$ and $z = (x, y) \in U^+$.

Let $\{X_i\}_{i\geq 0}$ and $\{Y_i\}_{i\geq 0}$ be independent sequences of random variables such that both sequences $\{X_{i+1} - X_i\}$ and $\{Y_{i+1} - Y_i\}$ are i.i.d.

sequences of random variables, each taking values 0, 1 and -1 with probability 1/3. Let $Z_i = (X_i, Y_i)$. The process $\{Z_i\}_{i \geq 0}$ is a random walk on $\mathbf{Z}^2$ started at Z_0. For any $A \subset \mathbf{Z}^2$, let $\tau_A = \inf\{i \geq 0 : Z_i \notin A\}$. We call $A \subset \mathbf{Z}^2$ connected if any two elements of A can be joined by a path that is entirely within A.

Proposition 5. *Let U be a bounded and connected subset of $\mathbf{Z}^2$ which is symmetric about the y-axis and convex in x. Let R be a rectangle containing U with sides parallel to the axes. Then, for all nonnegative integers m and n,*

$$P^z\left(\tau_{U^+} > m \mid \tau_{R^+} > n\right) \leq P^z\left(\tau_U > m \mid \tau_R > n\right).$$

We will prove the equivalent statement that, for all nonnegative integers m and n,

$$\frac{P^z\left(\tau_{U^+} > m, \tau_{R^+} > n\right)}{P^z\left(\tau_U > m, \tau_R > n\right)} \leq \frac{P^z\left(\tau_{R^+} > n\right)}{P^z\left(\tau_R > n\right)}. \tag{3}$$

We consider two cases: $m \leq n$ and $m > n$. The proof of the case $m \leq n$ is essentially the same as the proof of the case $m = n$ which was done in [**9**], cf. Lemma 5 and its proof in [**9**]. To avoid repetition, we omit this proof.

If $m > n$, then (3) becomes

$$\frac{P^z\left(\tau_{U^+} > m\right)}{P^z\left(\tau_U > m\right)} \leq \frac{P^z\left(\tau_{R^+} > n\right)}{P^z\left(\tau_R > n\right)}. \tag{4}$$

By Lemma 5 in [**9**],

$$\frac{P^z\left(\tau_{U^+} > m\right)}{P^z\left(\tau_U > m\right)} \leq \frac{P^z\left(\tau_{R^+} > m\right)}{P^z\left(\tau_R > m\right)}. \tag{5}$$

Thus, if we prove that the right side of inequality (4) is a decreasing function of n, then the case $m > n$ of (3) will be proved.

To prove that the right side of (4) is decreasing in n, we will show that, for all $n \geq 0$,

$$\frac{P^z\left(\tau_{R^+} > n+1\right)}{P^z\left(\tau_R > n+1\right)} \leq \frac{P^z\left(\tau_{R^+} > n\right)}{P^z\left(\tau_R > n\right)}. \tag{6}$$

Since R is a rectangle with sides parallel to the axes, and $\{X_i\}_{i\geq 0}$ and $\{Y_i\}_{i\geq 0}$ are independent, both sides of inequality (6) are independent of y, where $z=(x,y)$, and can be written in terms of $\{X_i\}_{i=0}^{n+1}$. Assume that $R=(-L,L)\times(-W,W)$ with $L,W\geq 2$ (so that $R^+\neq\varnothing$). Then inequality (6) can be rewritten as

$$\frac{P^x\left(0<X_i<L,0\leq i\leq n+1\right)}{P^x\left(|X_i|<L,0\leq i\leq n+1\right)}\leq\frac{P^x\left(0<X_i<L,0\leq i\leq n\right)}{P^x\left(|X_i|<L,0\leq i\leq n\right)},$$

or equivalently,

$$\frac{P^x\left(0<X_i<L,0\leq i\leq n+1\right)}{P^x\left(0<X_i<L,0\leq i\leq n\right)}\leq\frac{P^x\left(|X_i|<L,0\leq i\leq n+1\right)}{P^x\left(|X_i|<L,0\leq i\leq n\right)}. \tag{7}$$

Let $R_n^+=\{0<X_i<L,0\leq i\leq n\}$ and $R_n=\{|X_i|<L,0\leq i\leq n\}$. Note that $R_{n+1}^+\subset R_n^+$ and $R_{n+1}\subset R_n$. Thus, inequality (7) will be proved if we prove the following.

$$P^x\left(0<X_{n+1}<L\mid R_n^+\right)\leq P^x\left(|X_{n+1}|<L\mid R_n\right). \tag{8}$$

It is shown in [**9**] that (see the proof of Lemma 5 there), for any $t\in\mathbf{R}$,

$$P^x\left(X_n\geq t\mid R_n^+\right)\geq P^x\left(|X_n|\geq t\mid R_n\right). \tag{9}$$

Suppose that μ is the distribution of X_n (under P^x) given R_n^+, and ν is the distribution of $|X_n|$ given R_n. Thus, μ and ν are probability measures on integers with $\mu(\{1,\dots,L-1\}^c)=0$ and $\nu(\{0,\dots,L-1\}^c)=0$. Furthermore, by (9), for every $l\in\{0,\dots,L-1\}$, we have

$$\mu\{l,\dots,L-1\}\geq\nu\{l,\dots,L-1\}. \tag{10}$$

By the definition of R_n^+, for every $l\in\{1,\dots,L-1\}$, we have

$$\begin{aligned}P^x\left(X_{n+1}\geq L\mid X_n=l,R_n^+\right)&=P^x\left(X_{n+1}\geq L\mid X_n=l\right)\\&=P^l\left(X_1\geq L\right).\end{aligned}$$

Similarly, for every $l \in \{0, \ldots, L-1\}$, we have

$$P^x\left(|X_{n+1}| \geq L \mid |X_n| = l, R_n\right) = P^l\left(|X_1| \geq L\right).$$

Therefore, inequality (8) is equivalent to

$$P^\mu\left(0 < X_1 < L\right) \leq P^\nu\left(|X_1| < L\right). \tag{11}$$

We will prove that

$$P^\mu\left(X_1 \geq L\right) \geq P^\nu\left(|X_1| \geq L\right), \tag{12}$$

which implies (11).

To prove (12), note that

$$\begin{aligned}
&P^\mu\left(X_1 \geq L\right) - P^\nu\left(|X_1| \geq L\right) \\
&\qquad = \sum_{l=1}^{L-1} P^l\left(X_1 \geq L\right)\mu\{l\} - \sum_{l=0}^{L-1} P^l\left(|X_1| \geq L\right)\nu\{l\} \\
&\qquad = P^{L-1}\left(X_1 \geq L\right)\mu\{L-1\} - P^{L-1}\left(|X_1| \geq L\right)\nu\{L-1\} \\
&\qquad = \frac{1}{3}\left(\mu\{L-1\} - \nu\{L-1\}\right) \quad (\text{since } L \geq 2) \\
&\qquad \geq 0 \quad (\text{by } (10)).
\end{aligned}$$

This completes the proof of inequality (12). Thus, inequality (6) is also proved. As we pointed out earlier, inequality (6) implies Proposition 5. Finally, the invariance principle implies that Proposition 3 follows from Proposition 5. Therefore, the proof of Proposition 3 is complete.

Note that the proofs of inequality (9) and the case $m \leq n$ of inequality (3) are based on the idea of conditioning on the times that a random walk equals zero. Since the number of times a Brownian motion hits zero are uncountable, we cannot apply the method by which they are proved to Brownian motion directly.

Now we will show that, for the example in the introduction, inequalities (1) and (2) fail.

First we show that, for any $0 < s \leq t$, the left side of (2) is equal to 1, while the right side of (2) converges to zero as $d \to 0$ for all $s, t > 0$.

Recall that $B_r = (B_{1,r}, B_{2,r})$, $r \geq 0$, is a standard two-dimensional Brownian motion. Note that $R^+ = U^+$ and therefore the left side of (2) equals 1 if $s \leq t$.

Now assume that $s > 0$ is arbitrary. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$P^z(\tau_U > s) \leq P^{(0,1/2)}(\tau_U > s) + \varepsilon \tag{13}$$

for $0 < d < \delta$, see [**12**, page 55, inequality (20)]. On the other hand, by the law of the iterated logarithm for $B_{1,r}$, see [**13**, Theorem 2.9.23], we have $P^{(0,1/2)}(\tau_U = 0) = 1$ and, therefore, $P^{(0,1/2)}(\tau_U > s) = 0$. This, together with (13), implies that $\lim_{d\to 0} P^z(\tau_U > s) = 0$. Thus,

$$0 \leq \limsup_{d\to 0} P^z\left(\tau_U > s, \tau_R > t\right) \leq \lim_{d\to 0} P^z\left(\tau_U > s\right) = 0,$$

and therefore

$$\lim_{d\to 0} P^z\left(\tau_U > s, \tau_R > t\right) = 0 \quad \text{for all} \quad s, t > 0. \tag{14}$$

On the other hand,

$$\lim_{d\to 0} P^z(\tau_R > t) = P^{(0,1/2)}(\tau_R > t) > 0. \tag{15}$$

It follows from (14) and (15) that

$$\lim_{d\to 0} P^z(\tau_U > s \mid \tau_R > t) = 0 \quad \text{for all} \quad s, t > 0. \tag{16}$$

Now we will show that, for this choice of U and R, the right side of (1) converges to zero as $d \to 0$, while the left side of (1) is bounded away from zero.

First note that, since $U^+ = R^+$, for all $z \in U^+$,

$$E^z\left(\tau_{U^+} \mid \tau_{R^+} > t\right) > t.$$

Also, for any $t > 0$, the function $z \to P^z(\tau_R > t)$ is continuous on R (since $f(t,z) = P^z(\tau_R > t)$ is the solution of the heat equation on R, with initial value 1, and boundary value zero, see [**5**, Theorem II.4.14]) and, therefore, there is a $\delta_t > 0$ such that if $z \in [0, 1/2] \times \{1/2\}$ then

$$P^z(\tau_R > t) > \delta_t. \tag{17}$$

Similarly, for any $s > 0$, the function $z \to P^z(\tau_U > s)$ is continuous on U. Furthermore, (13) and the comments following it show that this function is also continuous at $(0, 1/2)$. Therefore, it is continuous on $[0, 1/2] \times \{1/2\}$. Hence, there exist constants c_1 and c_2, see [**1**, Theorem 3], such that, for all $z \in [0, 1/2] \times 1/2$,

$$P^z(\tau_U > s) \leq c_1 e^{-c_2 s}.$$

Therefore,

$$P^z(\tau_U > s \mid \tau_R > t) \leq \frac{P^z(\tau_U > s)}{P^z(\tau_R > t)} \leq c_1 \delta_t^{-1} e^{-c_2 s}. \tag{18}$$

Thus, by (16), (18), and the dominated convergence theorem, the right side of (1) converges to zero as $d \to 0$.

3. Higher dimensions. The analog of Proposition 3 holds for an arbitrary dimension k. We state it as Proposition 6, and we will show how the proof of Proposition 3 can be modified to prove it.

Represent a point in $\mathbf{R}^k$ by $z = (z_1, \dots, z_k)$. Also, for $A \subseteq \mathbf{R}^k$, put $A^+ = A \cap \{z \in \mathbf{R}^k \mid z_1 > 0\}$. Call A convex in z_1 if the intersection of A with every line parallel to z_1-axis is a connected interval or empty.

Proposition 6. *Let k be a positive integer, and let U be a bounded, connected and open subset of $\mathbf{R}^k$ which is symmetric about $\{z_1 = 0\}$ and convex in z_1. Also, let $R = (-L_1, L_1) \times \cdots \times (-L_k, L_k)$ be a k-dimensional rectangle, containing U. Then, for every $z \in U^+$ and every $s, t > 0$,*

$$P^z(\tau_{U^+} > s \mid \tau_{R^+} > t) \leq P^z(\tau_U > s \mid \tau_R > t). \tag{19}$$

The proof of Proposition 6 is very similar to the proof of Proposition 3. Let $\{X_i^1\}_{i \geq 0}, \dots, \{X_i^k\}_{i \geq 0}$ be k independent one-dimensional random walks, each constructed as $\{X_i\}_{i\geq 0}$ in Section 2. Then the analog of Proposition 5 holds by independence of $\{X_i^1\}_{i\geq 0}$ from $\{(X_i^2, \dots, X_i^k)\}_{i\geq 0}$.

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