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MATRIX SUMMABILITY METHODS AND WEAKLY UNCONDITIONALLY CAUCHY SERIES

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ABSTRACT. We study new sequence spaces determined by series in normed spaces and a matrix summability method, giving new characterizations of weakly unconditionally Cauchy series. We obtain characterizations for the completeness of a normed space, and a version of the Orlicz-Pettis theorem via matrix summability methods is also proved.

1. Introduction. Let X be a real normed space. A series $\sum_i x_i$ in X is said to be unconditionally convergent (uc) if $\sum_i x_{\pi(i)}$ converges for every permutation π of $\mathbf{N}$. We say that $\sum_i x_i$ is weakly unconditionally Cauchy (wuc) if, for every permutation π of $\mathbf{N}$, we have that the sequence $(\sum_{i=1}^n x_{\pi(i)})_n$ is weakly Cauchy. It is a well-known fact, see [**5**], that $\sum_i x_i$ is wuc if and only if $\sum_i |f(x_i)| < \infty$ for every $f \in X^*$, where X^* denotes the dual space of X. The following results are also well known, see [**3, 5, 6**]:

Let X be a Banach space, and let $\sum_i x_i$ be a series in X. Then:

1. $\sum_i x_i$ is uc if and only if $\sum_i a_i x_i$ is convergent for every $(a_i)_i \in l_\infty$.

2. $\sum_i x_i$ is wuc if and only if $\sum_i a_i x_i$ is convergent for every $(a_i)_i \in c_0$.

3. There exists a series $\sum_i x_i$ wuc and not uc in X if and only if X has a copy of c_0.

The following concepts and definitions can be found in [**4**].

A matrix method of limit is defined by a matrix $A = (\alpha_{ij})_{(i,j)\in\mathbf{N}\times\mathbf{N}}$ of real entries in the following way: If $(x_i)_i$ is a sequence in a normed space X, we say that $A\lim_i x_i = x_0$ if, for every $i \in \mathbf{N}$, the series $\sum_j \alpha_{ij} x_j$ is convergent and $\lim_i \sum_j \alpha_{ij} x_j = x_0$.

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A matrix method of summability is also defined by a matrix $A = (\alpha_{ij})_{(i,j)\in \mathbf{N}\times\mathbf{N}}$, and we will say that $A\sum_i x_i = x_0$ if $A\lim_n(\sum_{i=1}^n x_i) = x_0$, that is,

$$\lim_i \left(\sum_j \alpha_{ij}\left(\sum_{k=1}^j x_k\right)\right) = x_0.$$

We will also say that a matrix $A = (\alpha_{ij})_{(i,j)}$ is regular ([**4**]) if:

(i) $\sup_i \sum_j |\alpha_{ij}| < \infty$,

(ii) $\lim_i \alpha_{ij} = 0$ if $j \in \mathbf{N}$, and

(iii) $\lim_i \sum_j \alpha_{ij} = 1$.

In this paper, all regular matrices will be considered to have nonnegative entries.

We have that (i), (ii) and (iii) above are equivalent to saying that if $\lim_i x_i = x_0$, then $A\lim_i x_i = x_0$, and also equivalent to saying that if $\sum_i x_i = x_0$, then $A\sum_i x_i = x_0$. On the other hand, it can happen that $A\lim_i x_i = x_0$ and that $\lim_i x_i$ does not exist in X.

Let X be a normed space, $\sum_i x_i$ a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a matrix. In this work we will study the following subspaces of l_∞,

$$S_A\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty : A\sum_i a_i x_i \quad \text{exists}\right\},$$
$$S_{Aw}\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty : wA\sum_i a_i x_i \quad \text{exists}\right\},$$

and

$$S_{A*-w}\left(\sum_i f_i\right) = \left\{(a_i)_i \in l_\infty : * - wA\sum_i a_i x_i \quad \text{exists}\right\},$$

endowed with the sup norm, where $wA\sum_i a_i x_i$ and $* - wA\sum_i a_i x_i$ denote, respectively, the limit in the weak topology and in the weak beginning topology of the sequence

$$\left(\sum_j \alpha_{ij}\left(\sum_{k=1}^j a_k x_k\right)\right)_i.$$

In [**7**] the authors study these spaces in the case of the usual convergence and, in [**1**], these spaces are studied in the particular case of the Cesàro convergence.

We will give necessary and sufficient conditions for normed spaces to be complete by means of the previous spaces $S_A(\sum_i x_i)$ and $S_{Aw}(\sum_i x_i)$, and we will characterize the series $\sum_i x_i$ that make the previous spaces complete. A version of the Orlicz-Pettis theorem via wA-summability and matrix methods is also presented. As usual, $\mathcal{S}_{X^*}$ and $\mathcal{B}_{X^*}$ will denote, respectively, the unit sphere and the unit ball of X^* (the dual of X). Also, c_{00} denotes the space of sequences with finitely many nonzero terms.

2. The A-summability space. In the following result we give a sufficient condition for the space $S_A(\sum_i x_i)$ to be complete:

Theorem 2.1. *Let X be a normed space, $\sum_i x_i$ a series in X, and a matrix $A = (\alpha_{ij})_{(i,j)}$ of real entries. If*

1. *X is complete,*
2. *$\sum_i x_i$ is* wuc, *and*
3. *$\sup_i \sum_j |\alpha_{ij}| = M > 0$,*

then $S_A(\sum_i x_i)$ is complete.

Proof. Let us first observe that, if $(a_i)_i \in l_\infty$, then, since $\sum_i x_i$ is wuc, we have that the sequence $(\sum_{k=1}^j a_k x_k)_j$ is bounded. Therefore,

$$\sum_j \left\| \alpha_{ij} \left(\sum_{k=1}^{j} a_k x_k \right) \right\| < \infty$$

for every $i \in \mathbf{N}$.

Let $(a^n)_n$ be a sequence in $S_A(\sum_i x_i)$, $a^n = (a^n(i))_i$, and suppose that for some $a^0 = (a^0(i))_i$ we have that $\lim_n \|a^n - a^0\| = 0$ in l_∞. We will show that $a^0 \in S_A(\sum_i x_i)$, and we will do this by showing that the sequence

$$\left(\sum_j \alpha_{ij} \left(\sum_{k=1}^{j} a^0(k) x_k \right) \right)_i$$

is a Cauchy sequence.

Now, and since $\sum_i x_i$ is wuc, there is an $H \geq 0$ such that

$$\sup\left\{\left\|\sum_{i=1}^{n} a_i x_i\right\| : n \in \mathbf{N},\ |a_i| \leq 1 \text{ if } i \in \{1,\dots,n\}\right\} = H.$$

Let $\varepsilon > 0$, and fix $m \in \mathbf{N}$ such that $\|a^m - a^0\| < \varepsilon/(4MH)$. Now, and since the sequence $(\sum_j \alpha_{ij}(\sum_{k=1}^{j} a^m(k)x_k))_i$ is convergent, there exists $n_0 \in \mathbf{N}$ such that, if $p, q \geq n_0$, we have

$$\left\|\sum_j (\alpha_{pj} - \alpha_{qj})\left(\sum_{k=1}^{j} a^m(k)x_k\right)\right\| < \frac{\varepsilon}{2}.$$

Thus,

$$(\star)\quad \left\|\sum_j (\alpha_{pj} - \alpha_{qj})\left(\sum_{k=1}^{j} a^0(k)x_k\right)\right\| \leq \left\|\sum_j (\alpha_{pj} - \alpha_{qj})\left(\sum_{k=1}^{j} (a^0(k) - a^m(k))x_k\right)\right\| + \left\|\sum_j (\alpha_{pj} - \alpha_{qj})\left(\sum_{k=1}^{j} a^m(k)x_k\right)\right\|$$

On the other hand, if $r \in \mathbf{N}$, we have

$$\|(a^0(1) - a^m(1))x_1 + \cdots + (a^0(r) - a^m(r))x_r\| \leq H \cdot \|a^0 - a^m\|.$$

Therefore, if $h \in \mathbf{N}$, we obtain the following:

$$\begin{aligned}
&\left\|\sum_{j=1}^{h} (\alpha_{pj} - \alpha_{qj})\left(\sum_{k=1}^{j} (a^0(k) - a^m(k))x_k\right)\right\| \\
&\leq |\alpha_{p1} - \alpha_{q1}| \cdot \|(a^0(1) - a^m(1))x_1\| + \dots \\
&\quad + |\alpha_{ph} - \alpha_{qh}| \cdot \|(a^0(1) - a^m(1))x_1 + \cdots + (a^0(h) - a^m(h))x_h\| \\
&\leq \left(\sum_{j=1}^{h} |\alpha_{pj}| + \sum_{j=1}^{h} |\alpha_{qj}|\right) \cdot H \cdot \|(a^0 - a^m)\| \\
&\leq 2MH\|a^0 - a^m\| < \frac{\varepsilon}{2}.
\end{aligned}$$

Now, and from the convergence of the previous series and taking limits, we obtain that $(\star) \leq \varepsilon$ if $p, q \geq n_0$, and we are done. $\square$

The theorem that follows gives us a characterization of wuc series. In order to obtain this characterization, we will need to add an extra condition to our matrix A.

Theorem 2.2. *Let X be a normed space, $\sum_i x_i$ a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a matrix of real entries such that*:

(i) $\sup_i \sum_j |\alpha_{ij}| = M > 0$, *and*

(ii) *if $(\beta_j)_j$ is a nondecreasing sequence of real numbers with $\lim_j \beta_j = +\infty$, then $\lim_i(\sum_j \alpha_{ij}\beta_j)$ does not exist.*

If, in addition, X is complete, then we have that $\sum_i x_i$ is wuc *if and only if $S_A(\sum_i x_i)$ is complete.*

Proof. It suffices to prove that, if $S_A(\sum_i x_i)$ is complete, then $\sum_i x_i$ is wuc. Suppose that there exists an $f \in \mathcal{S}_{X^*}$ such that $\sum_i |f(x_i)| = +\infty$. Let us take $m_1 \in \mathbf{N}$ with $\sum_{i=1}^{m_1} |f(x_i)| > 2 \cdot 2$. If $i \in \{1, \dots, m_1\}$, we define

$$a_i = \begin{cases} 1/2 & \text{if } f(x_i) \geq 0, \text{ and} \\ -1/2 & \text{if } f(x_i) < 0. \end{cases}$$

Similarly, we can choose $m_2 > m_1$ such that $\sum_{i=m_1+1}^{m_2} |f(x_i)| > 2^2 \cdot 2^2$, and, if $i \in \{m_1 + 1, \dots, m_2\}$, we can also define

$$a_i = \begin{cases} 1/2^2 & \text{if} f(x_i) \geq 0, \text{ and} \\ -1/2^2 & \text{if } f(x_i) < 0. \end{cases}$$

Following this process, we can carry on and construct a sequence $(a_i)_i \in c_0$ such that

$$a_i f(x_i) \geq 0 \quad \text{for every} \quad i, \quad \text{and} \quad \sum_i a_i f(x_i) = +\infty.$$

Define now, for every $j \in \mathbf{N}$, $\beta_j = \sum_{i=1}^{j} a_i f(x_i)$. We have that $(\beta_j)_j$ is a nondecreasing sequence with $\lim_j \beta_j = +\infty$; therefore, $\lim_i(\sum_j \alpha_{ij}\beta_j)$ does not exist.

On the other hand, let us denote by e_i the ith canonical vector. Then $e_i \in S_A(\sum_i x_i)$ for every i, but, and since $S_A(\sum_i x_i)$ is complete, we deduce that $c_0 \subset S_A(\sum_i x_i)$ and, therefore, $(a_i)_i \in S_A(\sum_i x_i)$, reaching a contradiction. $\square$

Remark 2.3. 1. Now, we will see that condition (ii) of the matrix A from the previous theorem holds if A is regular (and of nonnegative entries, as considered in this paper). That is, A enjoys the following properties:

(a) $\alpha_{ij} \geq 0$ for each $i, j \in \mathbf{N}$ and $\sup_i \sum_j \alpha_{ij} < \infty$,

(b) $\lim_i \alpha_{ij} = 0$ if $j \in \mathbf{N}$, and

(c) $\lim_i \sum_j \alpha_{ij} = 1$.

Let us show the previous assertion. Take $(\beta_i)_i$ any motonotic nondecreasing sequence, and divergent to ∞. Without loss of generality, we can suppose that $\sum_j \alpha_{ij} > 1/2$ for every $i \in \mathbf{N}$ and that $\beta_i \geq 0$ for every $i \in \mathbf{N}$. Next, and since $\lim_i \alpha_{i1} = 0$, there exists $k_1 \in \mathbf{N}$ such that $\alpha_{k_1 1} < 1/4$, and we have that

$$\alpha_{k_1 1}\beta_1 + \alpha_{k_1 2}\beta_2 + \alpha_{k_1 3}\beta_3 + \cdots \geq (\alpha_{k_1 2} + \alpha_{k_1 3} + \cdots)\beta_2 \geq \frac{1}{4}\beta_2.$$

Now, since $\lim_i(\alpha_{i1} + \alpha_{i2}) = 0$, there exists $k_2 > k_1$, $k_2 \in \mathbf{N}$, such that $\alpha_{k_2 1} + \alpha_{k_2 2} < 1/4$, and we have

$$\alpha_{k_2 1}\beta_1 + \alpha_{k_2 2}\beta_2 + \alpha_{k_2 3}\beta_3 + \alpha_{k_2 4}\beta_4 + \cdots \geq (\alpha_{k_2 3} + \alpha_{k_2 4} + \cdots)\beta_3 \geq \frac{1}{4}\beta_3.$$

Proceeding in this way, we obtain that the sequence $(\sum_j \alpha_{ij}\beta_j)_i$ has a subsequence, which is divergent to ∞.

2. If the matrix A is regular and if

$$S = \left\{ (a_i)_i \in l_\infty : \sum_i a_i x_i \text{ converges} \right\},$$

then we have that $S \subset S_A(\sum_i x_i)$ but, in general, $S \neq S_A(\sum_i x_i)$.

3. If A is regular and X is not complete, then we can show that there exists a wuc series $\sum_i x_i$ in X so that $S_A(\sum_i x_i)$ is not complete.

Indeed, let $\sum_i x_i$ be a series in X so that $\|x_i\| < 1/i2^i$ for every $0 < i \in \mathbf{N}$ and $\sum_i x_i = x^{**} \in X^{**} \backslash X$. Clearly, it will be $A \sum_i x_i = x^{**}$. Now, let us consider the series $\sum_i z_i$ defined as $z_{2i-1} = ix_i$, and $z_{2i} = -ix_i$ for $i \in \mathbf{N}$. Then $\sum_i z_i$ is wuc. Let $(a_i)_i \in c_0$ be the sequence given by $a_{2i-1} = 1/(2i)$ and $a_{2i} = -1/(2i)$. We have that $\sum_i a_i z_i \in X^{**} \setminus X$; therefore, $(a_i)_i \notin S_A(\sum_i x_i)$ and, thus, $S_A(\sum_i x_i)$ is not complete.

4. Let X be a normed space, $\sum_i x_i$ a series in X, and $A = (\alpha_{ij})_{(i,j)}$ a regular matrix. Let us also define the following linear mapping:

$$\begin{aligned} T : S_A\left(\sum_i x_i\right) &\longrightarrow X \\ a \quad &\longmapsto T(a) = A \sum_i a_i x_i. \end{aligned}$$

We will show that T is continuous if and only if $\sum_i x_i$ is wuc. Indeed, let us suppose that T is continuous. We have that $c_{00} \subset S_A(\sum_i x_i)$ and, if $a \in c_{00}$ with $\|a\| \leq 1$ and $a_i = 0$ for all $i > n$, we have that

$$\|a_1 x_1 + \cdots + a_n x_n\| = \|Ta\| \leq \|T\|,$$

and we obtain that

$$\sup\left\{\left\| \sum_{i=1}^n a_i x_i \right\| : n \in \mathbf{N}, |a_i| \leq 1, i \in \{1, \ldots, n\}\right\} \leq \|T\|.$$

Conversely, let us suppose that $\sum_i x_i$ is wuc, and let us call

$$H = \sup\left\{\left\| \sum_{i=1}^n a_i x_i \right\| : n \in \mathbf{N}, |a_i| \leq 1, i \in \{1, \ldots, n\}\right\} < \infty.$$

Consider $a = (a_i)_i \in S_A(\sum_i x_i)$ with $\|a\| \leq 1$. If $i, p \in \mathbf{N}$, we obtain that

$$\left\| \sum_{j=1}^p \alpha_{ij} \left(\sum_{k=1}^j a_k x_k \right) \right\| \leq MH,$$

where $M = \sup_i(\sum_j |\alpha_{ij}|)$. From here, it follows that $\|T(a)\| \leq MH$ and that T is continuous, with $\|T\| \leq H$.

From the previous results and the above remarks, we can give the following corollary, which characterizes the wuc series by means of the completeness of $S_A(\sum_i x_i)$, amongst other consequences.

Corollary 2.4. (a) *Let X be a Banach space, $\sum_i x_i$ a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a regular matrix. The following are equivalent*:

(i) $\sum_i x_i$ *is* wuc.

(ii) $c_0 \subset S_A(\sum_i x_i)$.

(iii) $S_A(\sum_i x_i)$ *is complete.*

(iv) $A \sum_i |f(x_i)|$ *exists if* $f \in X^*$.

(b) *Let X be a normed space, $\sum_i x_i$ a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a regular matrix. The following are equivalent*:

(i) X *is complete.*

(ii) *If* $\sum_i x_i$ *is* wuc, *then* $S_A(\sum_i x_i)$ *is complete.*

3. The weak A-summability space. In this section we will see that some of the results seen in the previous section can also be extended for the weak topology. We start with the following result.

Theorem 3.1. *Let X be a Banach space, $\sum_i x_i$ a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a matrix with real entries and $\sup_i(\sum_j |\alpha_{ij}|) = M > 0$. We have that, if $\sum_i x_i$ is* wuc, *then $S_{Aw}(\sum_i x_i)$ is complete.*

Proof. First of all, let us call

$$H = \sup\left\{\left\|\sum_{i=1}^{n} a_i x_i\right\| : n \in \mathbf{N}, |a_i| \leq 1, i \in \{1, \ldots, n\}\right\} < \infty,$$

and suppose that $(a^n)_n \in S_{Aw}(\sum_i x_i)$, $a^n = (a^n(i))_i)$ for $n \in \mathbf{N}$. Let us also suppose that for some $a^0 = (a^0(i))_i) \in l_\infty$ we have that $\lim_n \|a^n - a^0\| = 0$. We will show that $a^0 \in S_{Aw}(\sum_i x_i)$.

For every $m \in \mathbf{N}$ there is a $z_m \in X$ such that $wA \sum_i a^m(i) x_i = z_m$. $(z_m)_m$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. There exists a $k_0 \in \mathbf{N}$ such that, if $p, q \geq k_0$, we have $\|a^p - a^q\| < \varepsilon/(HM)$. Consider $p, q \geq k_0$.

There is an $f \in \mathcal{S}_{X^*}$ such that

$$\|z_p - z_q\| = |f(z_p - z_q)| = \left| \lim_i \left(\sum_j \alpha_{ij} \left(\sum_{k=1}^{j} (a^p(k) - a^q(k)) f(x_k) \right) \right) \right|.$$

On the other hand, for every $h, i \in \mathbf{N}$ we have

$$\begin{aligned} &\left| f\left(\sum_{j=1}^{h} \alpha_{ij} \left(\sum_{k=1}^{j} (a^p(k) - a^q(k)) x_k \right) \right) \right| \\ &\qquad \le \left\| \sum_{j=1}^{h} \alpha_{ij} \left(\sum_{k=1}^{j} (a^p(k) - a^q(k)) x_k \right) \right\| \le MH \|a^p - a^q\| \le \varepsilon. \end{aligned}$$

Next, and since X is complete, there is a $z_0 \in X$ such that $\lim_m z_m = z_0$. We will now prove that $wA \sum_i a^0(i) x_i = z_0$.

Let $f \in \mathcal{B}_{X^*}$, and let $\varepsilon > 0$. Fix any $r \in \mathbf{N}$ such that $\|z_r - z_0\| < \varepsilon/3$, and $\|a^r - a^0\| < \varepsilon/(3MH)$. Now, let $i_1 \in \mathbf{N}$ be such that

$$\left| \left(\sum_j \alpha_{ij} \left(\sum_{k=1}^{j} a^r(k) f(x_k) \right) \right) - f(z_r) \right| < \frac{\varepsilon}{3} \quad \text{if} \quad i \ge i_1.$$

Thus, if $i \ge i_1$, we have that

$$\begin{aligned} &\left| \sum_j \alpha_{ij} \left(\sum_{k=1}^{j} a^0(k) f(x_k) \right) - f(z_0) \right| \\ &\qquad \le \left| \sum_j \alpha_{ij} \left(\sum_{k=1}^{j} (a^0(k) - a^r(k)) f(x_k) \right) \right| \\ &\qquad\quad + \left| \sum_j \alpha_{ij} \left(\sum_{k=1}^{j} a^r(k) f(x_k) \right) - f(z_r) \right| \\ &\qquad\quad + |f(z_r) - f(z_0)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \qquad \square \end{aligned}$$

Next, we present the weak-topology version of Theorem 2.2.

Theorem 3.2. *Let X be a normed space, $\sum_i x_i$ a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a matrix of real entries such that:*

(i) $\sup_i \sum_j |\alpha_{ij}| = M > 0$, *and*

(ii) *if $(\beta_j)_j$ is a nondecreasing sequence of real numbers with $\lim_j \beta_j = +\infty$, then $\lim_i(\sum_j \alpha_{ij}\beta_j)$ does not exist.*

If, in addition, X is complete, then we have that $\sum_i x_i$ is wuc *if and only if $S_{Aw}(\sum_i x_i)$ is complete.*

Proof. By Theorem 3.1, it suffices to prove that, if $S_{Aw}(\sum_i x_i)$ is complete, then $\sum_i x_i$ is wuc. Suppose that there exists an $f \in \mathcal{S}_{X^*}$ such that $\sum_i |f(x_i)| = +\infty$. Similarly as in the proof of Theorem 2.2, we can obtain a sequence $(a_i)_i \in c_0$ such that $a_i f(x_i) \geq 0$ for all i, and $\sum_i a_i f(x_i) = +\infty$. For every $j \in \mathbf{N}$, let $\beta_j = \sum_{i=1}^{j} a_i f(x_i)$. By hypothesis, we have that $\lim_i(\sum_j \alpha_{ij}\beta_j)$ does not exist. Now, and since we supposed $S_{Aw}(\sum_i x_i)$ to be complete, we deduce that $c_0 \subset S_{Aw}(\sum_i x_i)$ and, from here, it follows that $\lim_i(\sum_j \alpha_{ij}\beta_j)$ does exist, which is a contradiction. $\square$

Remark 3.3. 1. If the matrix A is regular and if $S_w = \{(a_i)_i \in l_\infty : w\sum_i a_i x_i \text{ exists}\}$, then we have that $S_w \subset S_{Aw}(\sum_i x_i)$ but, in general, $S_w \neq S_{Aw}(\sum_i x_i)$.

2. If A is regular and X is not complete, we can take the series from Remark 2.3 (3), and consider the same sequence $(a_i)_i \in c_0$. We obtain that $(a_i)_i \notin S_{Aw}(\sum_i x_i)$, since, if $f \in X^*$, then $wA\sum_i a_i f(x_i) = f(x^{**})$.

3. Let X be a normed space, $\sum_i x_i$ be a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a regular matrix. We can now define the following linear mapping:

$$\begin{aligned} T : S_{Aw}\Big(\sum_i x_i\Big) &\longrightarrow X \\ a &\longmapsto T(a) = wA\sum_i a_i x_i. \end{aligned}$$

As we did in Remark 2.3 (4), one can see that T is continuous if and only if the series $\sum_i x_i$ is wuc.

4. Let $\sum_i f_i$ be a series in X^*. It is a known fact, see, e.g. [**5**], that if X is complete, then $\sum_i f_i$ is wuc if and only if $\sum_i |f_i(x)| < \infty$ for every $x \in X$.

Now, if $\sum_i f_i$ is a series in X^*, and $A = (\alpha_{ij})_{(i,j)}$ a regular matrix, we can define the subspace of l_∞ given by

$$S_{A*-w}\left(\sum_i f_i\right) = \left\{(a_i)_i \in l_\infty : * - wA\sum_i a_i x_i \text{ exists}\right\}.$$

Consider the following assertions:

(a) $\sum_i f_i$ is wuc.

(b) $S_{A*-w}(\sum_i f_i) = l_\infty$.

(c) If $x \in X$, and $M \subset \mathbf{N}$, then $A\sum_{i\in M} f_i(x)$ exists.

We have that $(a) \Rightarrow (b) \Rightarrow (c)$. Indeed, in [**7**] it is shown that, if $\sum_i f_i$ is wuc, then the limit $* - wA\sum_i a_i x_i$ exists, for every $(a_i)_i \in l_\infty$. This implies that $S_{A*-w}(\sum_i f_i) = l_\infty$. Is is straightforward to prove that $(b) \Rightarrow (c)$.

If the normed space X is barreled, then we have that (a), (b), and (c) are equivalent. In order to do this, it suffices to notice that (c) implies (a), and for this we will show that

$$E = \left\{\sum_i a_i f_i : n \in \mathbf{N}, |a_i| \leq 1, i \in \{1, \dots, n\}\right\}$$

is pointwise bounded. Suppose not. Then we have that there would be an $x \in X$ such that $\sum_i |f_i(x)| = +\infty$. We can assume that the set $M = \{i \in \mathbf{N} : f_i(x)\}$ is infinite and that $\sum_{i\in M} f_i(x) = +\infty$. This is in contradiction with the existence of $A\sum_{i\in M} f_i(x)$.

To finish this section we can obtain, from Theorem 3.2 and the previous remarks, the following corollary.

Corollary 3.4. (a) *Let X be a Banach space, $\sum_i x_i$ a series in X, and let $A = (\alpha_{ij})_{(i,j)}$ be a regular matrix. The following are equivalent*:

(i) $\sum_i x_i$ *is* wuc.

(ii) $c_0 \subset S_{Aw}(\sum_i x_i)$.

(iii) $S_{Aw}(\sum_i x_i)$ *is complete.*

(b) *Let* X *be a normed space and* $A = (\alpha_{ij})_{(i,j)}$ *a regular matrix. Then we have that* X *is complete if and only if for every* wuc *series* $\sum_i x_i$ *we have that* $S_{Aw}(\sum_i x_i)$ *is complete.*

4. Some versions of the Orlicz-Pettis theorem for wA-summability. Let us remember that the Orlicz-Pettis theorem ([**5, 6**]) says that if X is a Banach space and $\sum_i x_i$ is a series in X, then $\sum_i x_i$ is uc if and only if $w\sum_i \chi_M(i)x_i$ exists for every $M \subset \mathbf{N}$, where χ_M is the sequence given by $\chi_M(i) = 1$ if $i \in M$ and $\chi_M(i) = 0$ if $i \notin M$.

In this section we study the Orlicz-Pettis theorem in the case of the weak A-summability.

Theorem 4.1. *Let* X *be a Banach space,* $\sum_i x_i$ *a series in* X*, and let* $A = (\alpha_{ij})_{(i,j)}$ *be a regular matrix. Then* $\sum_i x_i$ *is* uc *if and only if* $wA\sum_i \chi_M(i)x_i$ *exists for every* $M \subset \mathbf{N}$.

Proof. We have that, if $\sum_i x_i$ is uc then $w\sum_i \chi_M(i)x_i$ exists for every $M \subset \mathbf{N}$, and, since A is regular, the limit $wA\sum_i \chi_M(i)x_i$ also exists if $M \subset \mathbf{N}$.

Conversely, let $\sum_i x_i$ be a series in X so that the limit $wA\sum_i \chi_M(i)x_i$ also exists if $M \subset \mathbf{N}$. Let us first show that $\sum_i x_i$ is wuc. Suppose that there is an $f \in X^*$ such that $\sum_i |f(x_i)| = +\infty$, and consider the sequence $(\varepsilon_i)_i$ so that $\varepsilon_i = 1$ if $f(x_i) \geq 0$, and $\varepsilon_i = -1$ if $f(x_i) < 0$. It is clear that $\sum_i \varepsilon_i f(x_i) = +\infty$ and that $A\sum_i \varepsilon_i f(x_i) = +\infty$.

Let $M = \{i \in \mathbf{N} : \varepsilon_i = +1\}$ and $N = \mathbf{N} \setminus M$. We have that there exist y_1 and y_2 so that $wA\sum_i \chi_M(i)x_i = y_1$ and $wA\sum_i \chi_M(i)x_i = y_2$. Thus, we will have that $wA\sum_i \varepsilon_i x_i = y_1 - y_2$, and it will be $A\sum_i \varepsilon_i f(x_i) = f(y_1) - f(y_2)$, and this is a contradiction.

We will now show that, if $M \subset \mathbf{N}$, then $w\sum_i \chi_M(i)x_i$ exists and, from here, we will have that (by the classical Orlicz-Pettis theorem) $\sum_i x_i$ is uc. Indeed, let $M \subset \mathbf{N}$, and $f \in X^*$. We have that there exists a $y_0 \in X$ such that $wA\sum_i \chi_M(i)x_i = y_0$. On the other hand, since $\sum_i x_i$ is wuc, the series $\sum_i \chi_M(i)f(x_i)$ is convergent, and we will have that

$$\sum_i \chi_M(i) f(x_i) = A \sum_i \chi_M(i) f(x_i) = f(y_0).$$

Thus, $w \sum_i \chi_M(i) x_i = y_0$. $\square$

As a consequence of the previous theorem, we have the following result:

Corollary 4.2. *Let X be a Banach space, $\sum_i x_i$ a series in X, and $A = (\alpha_{ij})_{(i,j)}$ a regular matrix. The following are equivalent*:

1. $\sum_i x_i$ *is uc.*
2. $S_A(\sum_i x_i) = l_\infty$.
3. $S_{Aw}(\sum_i x_i) = l_\infty$.

To finalize, let us make a last remark.

Remark 4.3. Let X be a Banach space, $A = (\alpha_{ij})_{(i,j)}$ a regular matrix, and let $\sum_i x_i$ be a wuc series in X. It is clear that, if $(a_i)_i \in l_\infty$, then $\sum_i a_i x_i$ is wuc. From here, it follows that, if $(a_i)_i \in S_A(\sum_i x_i)$ then $\sum_i a_i x_i$ is wuc and $A \sum_i a_i x_i$ exists. So, therefore, $wA \sum_i a_i x_i$ also exists. From this last fact, together with the fact that $\sum_i a_i x_i$ is wuc, we obtain that $w \sum_i a_i x_i$ exists. Thus, we have that $S_A(\sum_i x_i) \subset S_w(\sum_i x_i)$, where

$$S_w\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty : w \sum_i a_i x_i \text{ exists}\right\}.$$

On the other hand, we do not know a necessary and sufficient condition for the equality $S_A(\sum_i x_i) = S_w(\sum_i x_i)$ to be true.

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