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LINEAR ALGEBRA IN THE CATEGORY OF C(M)-LOCALLY CONVEX MODULES

J.W. KITCHEN AND D.A. ROBBINS

ABSTRACT. This paper is concerned with the "linear algebra" of Banach bundles (otherwise known as bundles of Banach spaces). Although it is not entirely self-contained, it furnishes a unified, detailed treatment of the category Bun_X consisting of all Banach bundles $\pi : E \to X$(X a compact Hausdorff space). For the most part, we accomplish our study through a formulation of results for the equivalent (and, for our purposes more convenient) category LCMod_X whose objects are the $C(X)$−locally convex $C(X)$-modules.

Many results are new (such as the study of projective and injective objects in the category): some topics, like tensor products of Banach bundles, have been studied previously, but are developed further here. In other cases, "known" facts (that Bun_X is a cocomplete category, for instance), which have appeared in the literature in scattered and sometimes sketchy form, receive a systematic exposition.

Introduction. Throughout this paper, the space X mentioned in the title will be both compact and Hausdorff. We will direct our attention toward two naturally equivalent categories of Banach structures over X: the category LCMod_X, whose objects are the $C(X)$-locally convex modules over $C(X)$, and the category Bun_X of all Banach bundles (i.e., bundles of Banach spaces) over X.

Our chief interest will be in the construction of new objects from old in these categories - limits, colimits, tensor products, hom sets, injective envelopes, etc. These constructions involve not only the objects of these categories, but also their morphisms. Not surprisingly, therefore, several important functors become involved.

This paper continues the study, begun by the authors in earlier papers (see Kitchen and Robbins [**10**], [**11**], and [**12**]), of the relationship

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between Banach modules and Banach bundles. We direct the reader to these papers, their bibliographies, and other works cited in the References, for definitions, terminology, etc. (Among these papers in the References, we cite especially those of Hofmann [**7**], [**8**] and Hofmann and Keimel [**9**], which provide some of the earliest development of the theory of Banach bundles. The books of Dupré and Gillette [**2**] and Gierz [**4**] are also invaluable sources of information about related aspects of Banach bundles and Banach modules.)

Many of the results of this paper are new (such as the fact that every object in Bun_X is a colimit of trivial bundles with finite-dimensional fibers, and the work of §3 on injective objects). We have also taken the opportunity to draw together, systematize, and amplify previous work on the construction of Banach bundles and their fibers which has heretofore appeared in the literature only in sketchy and scattered form.

1. Colimits and tensor products in $\mathbf{LCMod}_X$. The key to all constructions presented in this paper is the use of the *Gelfand functor*

$$\mathcal{CB} : \mathrm{Mod}_X \to \mathrm{Bun}_X$$

with the *section space functor*

$$\Gamma : \mathrm{Bun}_X \to \mathrm{LCMod}_X,$$

whose descriptions we shall briefly review.

The functor CB assigns to each module M in Mod_X its *canonical bundle* $\pi : E \to X$. The fibers of this bundle are quotient spaces of M. Specifically, if $p \in X$, we denote by I_p the maximal ideal in $C(X)$ consisting of the functions which vanish at p, and we denote by I_pM the closed submodule of M generated by all possible products $f \cdot X$, where $f \in I_p$ and $X \in M$. (Since, as a Banach algebra, I_p possesses approximate identities, and since I_pM is an essential I_p-modules, it follows from the Cohen factorization that I_pM is simply $\{f \cdot X : f \in I_p, X \in M\}$.) The corresponding stalk $E_p = \pi^{-1}(p)$ is the canonical bundle is (isomorphic to) or the quotient space M/I_pM. (More precisely, the fiber space is defined to be the disjoint union of the family $\{M/I_pM : p \in X\}$.) For each element $X \in M$, one can define a selection $\hat{X} : X \to E$ in the following natural way: for each $p \in X, \hat{X}(p)$ is the coset of X modulo I_pM, i.e., $\hat{X}(p) = X + I_pM$. One can then show that there is a unique topology on the fiber space E which makes the triple $\pi : E \to X$ a bundle of Banach spaces and makes each of the selections $\hat{X}$ continuous. In this way, we get not only our canonical bundle $\pi : E \to X$ but also a map $\wedge : M \to \Gamma(\pi)$, called the *Gelfand representation* of M. This map is a morphism of the category Mod_X and has an important *universal property*: if $\rho : F \to X$ is a Banach bundle over X, and if $\phi : M \to \Gamma(\rho)$ is a bounded $C(X)$-linear homomorphism (i.e., ϕ is a "sectional representation of Gelfand type", to use earlier terminology), then ϕ factors through $\wedge$; more precisely, there is a unique $C(X)$-linear map $\tilde{\phi} : \Gamma(\pi) \to \Gamma(\rho)$ such that $\phi(X) = \tilde{\phi}(\hat{X})$ for all $X \in M$, and, moreover, $||\tilde{\phi}|| = ||\phi||$. If, in addition, ϕ is a morphism of the category (i.e., we have additionally $||\phi|| \leq 1$), then it follows that $\tilde{\phi}$ is also a morphism of the category. (For details and generalizations of these results, the reader can consult Kitchen and Robbins [**10** and **11**].

The functor Γ assigns to each object $\pi : E \to X$ in Bun_X its section space $\Gamma(\pi)$. These section spaces are not only $C(X)$-modules, but have the added property of being $C(X)$-*locally convex*, which is to say that the unit balls in these spaces are closed under forming "convex combinations" of the form $f \cdot X + (1 - f) \cdot y$, where f is a nonnegative function in $C(X)$ with values between 0 and 1. (For equivalent formulations of this property, the reader can consult Gierz [**4**].) More importantly, the property of $C(X)$-local convexity precisely characterizes section spaces as $C(X)$-modules: specifically, if M is in Mod_X, then the Gelfand representation $\wedge : M \to \Gamma(\pi)$ is an isomorphism (of the category Mod_X) if and only if M is $C(X)$-locally convex. This important result was first proved by Varela in [**16**]; a more streamlined proof can be found in [**4**]. Thus, the range of the functor Γ is the full subcategory LCMod_X of Mod_X consisting of the $C(X)$-locally convex modules. In this way, we get a functor

$$\Gamma : \mathrm{Bun}_X \to \mathrm{LCMod}_X$$

which is an equivalence of categories.

By composing the functors CB and Γ we get the Gelfand functor

$$\mathcal{G} = \Gamma \cdot \mathcal{CB} : \mathrm{Mod}_X \to \mathrm{LCMod}_X.$$

Thus, $\mathcal{G}$ assigns to each module M in Mod_X the section space $\Gamma(\pi)$ of its canonical bundle $\pi : E \to X$. What is the assignment of morphisms? If N is a second object in Mod_X and if $\phi : M \to N$ is a morphism, we denote by $\rho : F \to X$ the canonical bundle of N and observe that $\wedge \circ \phi : M \to \Gamma(\rho)$ is a morphism. Hence, by the universal property of Gelfand representations, there is a unique morphism $\mathcal{G}(\phi) : \Gamma(\pi) \to \Gamma(\rho)$ which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \wedge\big\downarrow & & \big\uparrow\wedge \\ \Gamma(\pi) = \mathcal{G}(M) & \xrightarrow{\mathcal{G}(\phi)} & \mathcal{G}(N) = \Gamma(\rho) \end{array}$$

commute. (We have not described the morphism assignments for the functors $\mathcal{CB}$ and Γ. The reader can either consult the literature or figure out for himself how these should go, and, having done so, he may then satisfy himself that $\Gamma(\mathcal{CB}(\phi))$ is the same as the map $\mathcal{G}(\phi)$ characterized by the commuting rectangle above.)

THEOREM 1.1. *The Gelfand functor $\mathcal{G}$: $\mathrm{Mod}_X \to \mathrm{LCMod}_X$ is a reflector.*

PROOF. It suffices to show that, for each M in Mod_X, the Gelfand representation $\wedge : M \to \mathcal{G}(M)$ is LCMod_X-universal. If we are given any morphism $\phi : M \to N$, where N is in LCMod_X, then the Gelfand representation $\wedge : N \to \mathcal{G}(N)$ is an isormorphism; consequently, $\mathcal{G}(\phi)$ followed by the inverse of $\wedge : N \to \mathcal{G}(N)$ gives us a morphism $\overline{\phi} : \mathcal{G}(M) \to N$ which makes the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\wedge"'] & N \arrow[d, "\wedge"'] \\ \mathcal{G}(M) \arrow[r, "\mathcal{G}(\phi)"] \arrow[ur, "\overline{\phi}"] & \mathcal{G}(M) \end{tikzcd}$$

commute. Thus $\wedge : M \to \mathcal{G}(M)$ is LCMod_X-universal. □

COROLLARY 1.2. *The categories* LCMod_X *and* Bun_X *are cocomplete and the Gelfand functor preserves colimits.*

PROOF. This result follows from the facts that Mod_X is cocomplete and that $\mathcal{G}$ is a reflector. □

Thus, if we have (in the terminology of Cigler et al. [**1**]) a spectral family $\{M_j : j \in J\}$ in Mod_X, then $\{\mathcal{G}(M_j) : j \in J\}$ is a spectral family in LCMod_X whose colimit in the latter category is $\mathcal{G}(\mathrm{colim}_j\, M_j)$: in particular, if we have a spectral family $\{\Gamma(\pi_j) : j \in J\}$ in LCMod_X, then its colimit in LCMod_X is the section space of the canonical bundle $\pi : E \to X$ of

$$\mathrm{colim}_j \Gamma(\pi_j) \quad (\text{in } \mathrm{Mod}_X).$$

This raises the following question: how are the fibers of the colimit bundle $\pi : E \to X$ related to the fibers of the given bundles $\{\Gamma(\pi_j) : j \in J\}$? The answer can be arrived at through the construction of a family of reflectors Q_I.

Let I be a fixed closed ideal in $C(X)$. We define a functor $Q_I : \mathrm{Mod}_X \to \mathrm{Mod}_X$ as follows: Q_I assigns to each module M in Mod_X the quotient module $Q_I(M) = M/IM$; Q_I assigns to any morphism $\phi : M \to N$ the unique morphism $Q_I(\phi) : Q_I(M) \to Q_I(N)$ which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M/IM & \xrightarrow{Q_I(\phi)} & N/IN \end{array}$$

commute, where the π's are the canonical surjections. (Since ϕ takes IM into IN, the existence and uniqueness of $Q_I(\phi)$ is assured.)

The range of the covariant functor Q_I is a subcategory of Mod_X which we shall temporarily denote by $\mathcal{C}_I$. The objects in $\mathcal{C}_I$ may be described as those modules in Mod_X which are annihilated by functions in I. Alternatively, $\mathcal{C}_I$ consists of those modules M in Mod_X for which the canonical surjection $\pi_M : M \to Q_I(M)$ is an isomorphism.

THEOREM 1.3. *For each closed ideal I in $C(X)$, the functor Q_I :* $\mathrm{Mod}_X \to \mathcal{C}_I$ *is a reflector.*

PROOF. It suffices to show that, for each M in Mod_X, the map $\pi_M : M \to Q_I(M)$ is $\mathcal{C}_I$-universal.

If we are given any morphisms $\phi : M \to N$ where N is in $\mathcal{C}_I$, then $\pi_N : N \to Q_I(N)$ is an isomorphism, so that the map $\overline{\phi} = \pi_N^{-1} \circ Q_I(\phi)$ causes the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\pi_M"'] & N \arrow[d, "\pi_N"] \\ Q_I(M) \arrow[r, "Q_I(\phi)"'] \arrow[ur, "\overline{\phi}"] & Q_I(N) \end{tikzcd}$$

to commute. Thus, $\pi_M : M \to Q_I(M)$ is $\mathcal{C}_I$-universal. □

Since the category Mod_X is cocomplete, we get:

COROLLARY 1.4. *The reflectors Q_I preserve colimits.*

By taking I to be a maximal ideal in $C(X)$ we can answer the question posed in the paragraph following Corollary 1.2.

COROLLARY 1.5. *Let $\{M_j : j \in J\}$ be a spectral family in* Mod_X *and let*

$$M = \mathrm{colim}_j\ M_j.$$

Let $\pi_j : E_j \to X$ be the canonical bundle for $M_j (j \in J)$, and let $\pi : E \to X$ be the canonical bundle for M. Then, for each $p \in X$,

$$E_p = \mathrm{colim}_j (E_j)_p.$$

PROOF. Again we denote by I_p the maximal ideal $\{f \in C(X) : f(p) = 0\}$. Then, since Q_{I_p} preserves colimits, we have

$$E_p = M/I_pM = Q_{I_p}(M) = \mathrm{colim}_j Q_{I_p}(M_j) = \mathrm{colim}_J (E_j)_p.$$

□

COROLLARY 1.6. *Let* $\{\Gamma(\pi_j) : j \in J\}$ *be a spectral family in* LCMod_X, *and let*

$$\Gamma(\pi) = \mathrm{colim}_j \Gamma(\pi_j) \quad (\textit{in } LCMod_X).$$

If we denote by E_j *and* E *the fiber spaces of the bundles* π_j *and* π *respectively, then for every* $p \in X$,

$$E_p = \mathrm{colim}_j (E_j)_p.$$

PROOF. The bundle $\pi : E \to X$ is the canonical bundle for $M = \mathrm{colim}_j \Gamma(\pi_j)$ (in Mod_X). Moreover, the canonical bundle for $\Gamma(\pi_j)$ can be naturally identified with the given bundles $\pi_j : E_j \to X$. Hence, the stated result is an immediate consequence of the preceding corollary. □

Because of the equivalence of the categories Bun_X and LCMod_X we can reinterpret the preceding results in terms of bundles rather than $C(X)$-modules. Suppose that $\{\pi_j : E_j \to X\}, j$ ranging over an index category J, is a spectral family in Bun_X. Then $\{\Gamma(\pi_j) : j \in J\}$ is a spectral family in LCMod_X. The family $\{\pi_j : E \to X\}$ has a colimit $\pi : E \to X$ in the category Bun_X, namely, $\pi : E \to X$ is the canonical bundle for

$$\mathrm{colim}_j \Gamma(\pi_j) \quad (\text{in } \mathrm{Mod}_X).$$

The preceding corollary says that the fibers of this colimit bundle $\pi : E \to X$ are colimits of the fibers of the given bundles $\pi_j : E_j \to X$.

Before turning to tensor products of modules in LCMod_X, we shall prove two additional theorems concerning colimits. We will show that every Banach bundle over X is a colimit of trivial bundles with finite-dimensional fibers, which is a rather surprising result, perhaps, in the light of Corollary 1.6 and the fact that the fibers of a Banach bundle can be prescribed arbitrarily (see Theorem 2.13 of Kitchen and Robbins [**11**]). We shall also show that every quotient module $M/IM = Q_I(M)$ has a natural interpretation as a colimit of other quotient modules. We begin with the latter.

Every closed ideal in $C(X)$ is of the form $I_C = \{f \in C(X) : f = 0 \text{ on } C\}$, where C is a closed subset of X. If M is in Mod_X, then $I_C M$ can be defined initially as the smallest closed submodule of M which contains all products $f \cdot X$ where $f \in I_C$ and $x \in M$. As a Banach algebra, however, I_C has approximate identities, and $I_C M$ can be viewed as an I_C-module. It then follows from the Cohen factorization theorem that every element of $I_C M$ is simply a product $f \cdot X$ where $f \in I_C$ and $X \in M$.

THEOREM 1.7. *Let M be in* Mod_X *and let C be a closed subset of X. Then*

$$M/IM = \operatorname{colim}\{M/I_S M : S \in U\},$$

where $\mathcal{U}$ is any fundamental system of closed neighborhoods of C.

PROOF. The theorem, as stated, is not completely formulated since we have not yet explained exactly how the family $\{M/I_S M : S \in U\}$ is to be regarded as a spectral family. Specifically, we take as our index category the family $\mathcal{U}$ ordered by inclusion. If S and T are in $\mathcal{U}$ and $S \subset T$, then $I_T \subset I_S$, so $I_T M \subset I_S M$; consequently, there is a unique morphism $\phi_{ST} : M/I_T M \to M/I_S M$ which makes the diagram

$$\begin{tikzcd} M \arrow[r, "\pi_S"] \arrow[dr, "\pi_T"'] & M/I_S M \arrow[d, "\phi_{ST}"] \\ & M/I_T M \end{tikzcd}$$

commute, the π's being the canonical surjections. One can then verify that the family $\{M/I_S M : S \in \mathcal{U}\}$, together with the morphism $\{\phi_{ST} : S, T \in \mathcal{U}, S \subset T\}$, is a spectral family in Mod_X.

If $S \in U$, then, since $C \subset S$, there is (by the above argument) a unique morphism $\phi_S : M/I_S M \to M/I_C M$ which makes the diagram

$$\begin{tikzcd} M \arrow[r, "\pi_S"] \arrow[dr, "\pi_C"'] & M/I_S M \arrow[d, "\phi_S"] \\ & M/I_c M \end{tikzcd}$$

commute. We will now show that M/I_CM together with the family of morphisms $\{\phi_S : S \in U\}$ is a colimit of the family $\{M/I_S : S \in U\}$.

We observe, first of all, that if $S, T \in U$ and $S \subset T$, then

$$\phi_S \circ \phi_{ST} \circ \phi_T = \phi_S \circ \pi_S = \pi_C = \phi_T \circ \pi_T,$$

which implies that $\phi_S \circ \phi_{ST} = \phi_T$, since π_T is surjective. Thus, the triangle

$$\begin{tikzcd} M/I_SM \arrow[r, "\phi_S"] & M/I_CM \\ M/I_TM \arrow[u, "\phi_{ST}"] \arrow[ur, "\phi_T"'] & \end{tikzcd}$$

commutes.

We next suppose that we have a family of morphisms $\psi_S : M/I_SM \to N, S \in U$, such that the triangle

$$\begin{tikzcd} M/I_SM \arrow[r, "\psi_S"] & N \\ M/I_TM \arrow[u, "\phi_{ST}"] \arrow[ur, "\psi_T"'] & \end{tikzcd}$$

commutes whenever $S, T \in U$ and $S \subset T$. We must prove that there is a unique morphism $\psi : M/I_CM \to N$ such that the triangle

$$\begin{tikzcd} M/I_SM \arrow[r, "\psi_S"] \arrow[d, "\phi_S"'] & N \\ M/I_CM \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes for each $S \in U$.

We show first that, for any two elements S and T in U, the composed maps $\psi_S \circ \pi_S$ and $\psi_T \circ \pi_T$ are the same. It suffices to consider the case in which $S \subset T$. The general case then follows, since by selecting $U \in \mathcal{U}$ with $U \subset S \cap T$, we then get $\psi_S \circ \pi_S = \psi_U \circ \pi_U = \psi_T \circ \pi_T$. When $S \subset T$ we get the diagram We show first that, for any two elements S and T in $\mathcal{U}$, the composed maps $\psi_S \circ \pi_S$ and $\psi_T \circ \pi_T$ are the same. It suffices to consider the case in which $S \subset T$. The general case then follows, since by selecting $U \in \mathcal{U}$ with $U \subset S \cap T$, we then get $\psi_S \circ \pi_S = \psi_U \circ \pi_U = \psi_T \circ \pi_T$. When $S \subset T$ we get the diagram

$$\begin{tikzcd}
M \arrow[r, "\pi_S"] \arrow[d, "\pi_T"'] & M/I_S M \arrow[d, "\psi_S"] \\
M/I_T M \arrow[ur, "\phi_{ST}"] \arrow[r, "\psi_T"] & N
\end{tikzcd}$$

in which the two triangles are known to commute. It follows that $\psi_S \circ \pi_S = \psi_S \circ \phi_{ST} \circ \pi_T = \psi_T \circ \pi_T$.

Let us denote by $\alpha : M \to N$ the common value of the maps $\psi_S \circ \pi_S$. We will prove that $I_C M$ is contained in the kernel of α. Suppose not. Then $||\alpha(f \cdot X)|| = ||f \cdot \alpha(X)|| = 1$ for some $f \in I_C$ and some $X \in M$. Since f vanishes on C, we can approximate f as closely as we please by a function g which vanishes throughout a neighborhood of C. Consequently, we can choose a function $g \in I_S$, for some $S \in \mathcal{U}$, such that $||g \cdot \alpha(X)|| > 1/2$. But $g \cdot \alpha(X) = \alpha(g \cdot X) = \psi_G(\pi_G(g \cdot X)) = 0$ since $g \cdot X \in I_S M = \ker \pi_S$, a contradiction.

Since $I_C M \subset \ker \alpha$, there is a unique morphism $\psi : M/I_C M \to N$ which makes the diagram

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\pi_C"'] & N \\
M/I_C M \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commute. Thus, for each $S \in \mathcal{U}$,

$$\psi \circ \phi_S \circ \pi_S = \psi \circ \pi_C = \alpha = \psi_S \circ \pi_S,$$

which implies that $\psi \circ \phi_S = \psi_S$. Hence, the triangle

$$\begin{tikzcd} M/I_SM \arrow[r, "\psi_S"] \arrow[d, "\phi_S"'] & N \\ M/I_CM \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes. □

COROLLARY 1.8. *Let M be a module in Mod_X, let $\pi : E \to X$ be its canonical bundle, and let p be a point in X. Then*

$$E_p = \pi^{-1}(p) = \operatorname{colim}\{M/I_SM : S \in U\},$$

where U is any fundamental system of closed neighborhoods of p.

We will next show that every Banach bundle is a colimit of trivial bundles with finite-dimensional fibers. More precisely, these fibers will be the form $\mathbf{C}^n$, where in *all* instances $\mathbf{C}^n$ will be endowed with the ℓ^1-norm:

$$||(\alpha_1, \alpha_2, \ldots, \alpha_n)|| = |\alpha_1| + \cdots + |\alpha_n|.$$

In other words, $\mathbf{C}^n$ can be regarded as the space ℓ_n^1 of all summable families of complex numbers indexed by the set $\{1, \ldots, n\}$. The space of sections of the trivial bundle $\tau_n : X \times \ell_n^1 \to X$ can be identified with the space $C(X, \ell_n^1) = C(X, \mathbf{C}^n)$ of all continuous functions from X to $\mathbf{C}^n$ or with the inductive tensor product $C(X)\hat{\odot}\ell_n^1$, i.e.,

$$\Gamma(\tau_n) \cong C(X, \ell_n^1) \cong C(X)\hat{\odot}\ell_n^1.$$

We shall also make use of the fact that the trivial bundle $\tau_n : X \times \ell_n^1 \to X$ is the canonical bundle for the following $C(X)$-modules:

$$\begin{aligned} & C(X)^n \text{ (with the) } \ell^1 - \text{norm)} \\ & \cong \ell_n^1(C(X)) \text{ (the space of summable families in } C(X) \\ & \qquad \text{indexed by } \{1, \ldots, n\}) \\ & \cong \ell_n^1 \hat{\odot} C(X). \end{aligned}$$

Generalizations of the results just quoted will be proved in the next section of this paper (Theorem 2.4, in particular).

THEOREM 1.9. *Every module in* Mod_X *is a colimit of spaces of the form* $\ell^1_n \hat{\otimes} C(X) \cong C(X)^n$ *(with the* ℓ^1*-norm), i.e., every unital* $C(X)$*-module is a colimit of free, finitely-generated* $C(X)$*-modules.*

PROOF. One can follow Cigler, Losert, and Michor almost verbatim [**1**, p.21]. We shall merely begin the proof in order to indicate the sort of modifications that are needed.

Let M be a fixed module in Mod_X. We take for our index category the collection of all pairs $(C(X)^n, \phi)$ such that $\phi : C(X)^n \to M$ is a morphism of the category Mod_X. Such morphisms are easily seen to be of the form

$$\phi((f_1, \ldots, f_n)) = \sum_{j=1}^{n} f_j \cdot X_j$$

for $X_j \in M$ and $||X_j|| \leq 1$.

By a morphism of the index category we mean any morphism $\alpha : C(X)^n \to C(X)^m$ for which the diagram

$$\begin{tikzcd} C(X)^n \arrow[r, "\phi"] \arrow[d, "\alpha"'] & M \\ C(X)^m \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes, where $(C(X)^n, \phi)$ and $(C(X)^m, \psi)$ are objects of the index category, etc. □

COROLLARY 1.10. *Every Banach bundle* $\pi : E \to X$ *is a colimit of trivial bundles of the form* τ_n.

PROOF. In the category Mod_X the section space $\Gamma(\pi)$ is a colimit of spaces of the form $\ell^1_n \hat{\otimes} C(X)$. Hence, in the category $\mathrm{LCMod}_X, \mathcal{G}(\Gamma(\pi))$

$=\Gamma(\pi)$ is a colimit of spaces of the form $\mathcal{G}(\ell^1_n \hat{\otimes} C(X)) = \Gamma(\tau_n)$. Equivalently, in the category Bun_X the bundle $\pi : E \to X$ is a colimit of trivial bundles of the form $\tau_n : X \times \ell^1_n \to X$. □

We turn next to tensor products. Suppose we are given two objects in LCMod_X; we write them as $\Gamma(\pi)$ and $\Gamma(\rho)$, where $\pi : E \to X$ and $\rho : F \to X$ are Banach bundles. In the category Mod_X, they have a projective tensor product, namely

$$\Gamma(\pi) \hat{\otimes}_X \Gamma(\rho) := \Gamma(\pi) \hat{\otimes}_{C(X)} \Gamma(\rho).$$

(Throughout the paper, we shall usually replace $C(X)$ by X when it appears as a subscript.) The $C(X)$-module $\Gamma(\pi) \hat{\otimes}_X \Gamma(\rho)$ is the quotient space of the projective tensor product $\Gamma(\pi) \hat{\otimes} \Gamma(\rho)$ modulo the closed subspace which is generated by elements of the form

$$(f\sigma) \otimes \tau - \sigma \otimes (f\tau),$$

where $f \in C(X), \sigma \in \Gamma(\pi)$, and $\tau \in \Gamma(p)$. See, for example, [**6**], [**10**], or [**14**]. We then define the projective tensor product of $\Gamma(\pi)$ and $\Gamma(\rho)$ in the category LCMod_X to be

$$\Gamma(\pi) \hat{\otimes}_{LC} \Gamma(\rho) := \mathcal{G}(\Gamma(\pi) \hat{\otimes}_X \Gamma(\rho)).$$

Thus, $\Gamma(\pi) \hat{\otimes}_{LC} \Gamma(\rho)$ is the section space of the canonical bundle of the $C(X)$-module $\Gamma(\pi) \hat{\otimes}_X \Gamma(\rho)$. Let us denote the latter bundle by $\pi \hat{\otimes}_X \rho : E \hat{\otimes}_X F \to X$. In [**10**] it is observed that, for each $p \in X$, the fiber $(\pi \hat{\otimes}_X \rho)^{-1}(p)$ can be identified with $E_p \hat{\otimes} F_p$ (where $E_p = \pi^{-1}(p)$ and $F_p = \rho^{-1}(p)$) in such a way that the Gelfand representation

$$\wedge : \Gamma(\pi) \hat{\otimes}_X \Gamma(\rho) \to \Gamma(\pi \hat{\otimes}_X \rho) = \Gamma(\pi) \hat{\otimes}_{LC} \Gamma(\rho)$$

has the property

$$(\sigma \hat{\otimes}_X \tau)^\wedge(p) = \sigma(p) \hat{\otimes} \tau(p) := (\sigma \hat{\odot} \tau)(p)$$

for all $\sigma \in \Gamma(\pi)$ and $\tau \in \Gamma(\rho)$. (As in previous papers, the authors choose to use $\hat{\odot}$ for pointwise tensor products of sections. The notation is not standard, but it helps to relieve the over-burdened tensor sign.)

In addition to the properties already mentioned, the tensor product $\Gamma(\pi) \odot_{LC} \Gamma(\rho)$ has an important universal property.

PROPOSITION 1.11. *Let $\pi : E \to X$ and $\rho : F \to X$ be Banach bundles. The map*

$$\odot : \Gamma(\pi) \times \Gamma(\rho) \to \Gamma(\pi) \otimes_{LC} \Gamma(\rho)$$

defined by $\odot(\sigma, \tau) = \sigma \odot \tau$ is a $C(X)$-bilinear map of norm one or less. If $\xi : \mathcal{G} \to X$ is a Banach bundle, and if

$$\phi : \Gamma(\pi) \times \Gamma(\rho) \to \Gamma(\xi)$$

is any bounded $C(X)$-bilinear map, then there exists a unique bounded $C(X)$-bilinear map

$$\tilde{\phi} : \Gamma(\pi) \otimes_{LC} \Gamma(\rho) \to \Gamma(\xi)$$

such that $\tilde{\phi}(\sigma \odot \tau) = \phi(\sigma, \tau)$ for all $\sigma \in \Gamma(\pi)$ and $\tau \in \Gamma(\rho)$. Moreover, $||\tilde{\phi}|| \leq ||\phi||$.

PROOF. The map $\odot$ is clearly $C(X)$-bilinear, i.e.,

$$(f\sigma) \odot \tau = f(\sigma \odot \tau) = \sigma \odot (f\tau)$$

for all $f \in C(X), \sigma \in \Gamma(\pi)$, and $\tau \in \Gamma(\rho)$. Also,

$$||\sigma \odot \tau|| = ||(\sigma \otimes_X \tau)^\wedge|| \leq ||\sigma \otimes_X \tau|| \leq ||\sigma|| \cdot ||\tau||,$$

which shows that $\odot$ has norm one or less.

Since $\phi : \Gamma(\pi) \times \Gamma(\rho) \to \Gamma(\xi)$ is assumed to be bounded and $C(X)$-bilinear, there is a unique linear map

$$\overline{\phi} : \Gamma(\pi) \otimes_X \Gamma(\rho) \to \Gamma(\xi)$$

such that

$$\overline{\phi}(\sigma \otimes_X \tau) = \phi(\sigma, \tau)$$

for all $\sigma \in \Gamma(\pi)$ and $\tau \in \Gamma(\rho)$. (Such a $\overline{\phi}$ exists if we merely assume that ϕ is $C(X)$-balanced.) Moreover, $\overline{\phi}$ is bounded with $||\overline{\phi}|| \leq ||\phi||$.

Since ϕ is $C(X)$-bilinear, $\overline{\phi}$ is $C(X)$-linear. By the universal property of the Gelfand representation, there exists a unique bounded $C(X)$-linear map

$$\tilde{\phi} : \Gamma(\pi) \odot_{LC} \Gamma(\rho) \to \Gamma(\xi)$$

which makes the diagram

$$\begin{tikzcd} \Gamma(\pi) \odot_X \Gamma(\rho) \arrow[r, "\overline{\phi}"] \arrow[d, "\wedge"'] & \Gamma(\xi) \\ \Gamma(\pi) \odot_{LC} \Gamma(\rho) \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

commute. Thus

$$\begin{aligned} \phi(\sigma, \tau) &= \overline{\phi}(\sigma \odot_X \tau) = (\tilde{\phi} \circ \wedge)(\sigma \odot_X \tau) \\ &= \tilde{\phi}((\sigma \odot_X \tau)^\wedge) = \tilde{\phi}(\sigma \odot \tau). \end{aligned}$$

From the universal property just proved, one can easily show the existence of tensor products of morphisms in LCMod_X.

COROLLARY 1.12. *Let $\pi_i : E_i \to X$ and $\rho_i : F_i \to X$ be Banach bundles $(i = 1, 2)$ and let $\phi_i : \Gamma(\pi_i) \to \Gamma(p_i)$ be morphisms $(i = 1, 2)$. Then there is a unique morphism*

$$\phi_1 \odot_{LC} \phi_2 : \Gamma(\pi_1) \odot_{LC} \Gamma(\pi_2) \to \Gamma(\rho_1) \odot_{LC} \Gamma(\rho_2)$$

such that

$$(\phi_1 \odot_{LC} \phi_2)(\sigma_1 \odot \sigma_2) = \phi_1(\sigma_1) \odot \phi_2(\sigma_2)$$

for $\sigma_i \in \Gamma(\pi_i)(i = 1, 2)$. Moreover,

$$||\phi_1 \odot_{LC} \phi_2|| \leq ||\phi_1|| \cdot ||\phi_2||.$$

PROOF. One observes that, as a function of σ_1 and σ_2, $\phi_1(\sigma_1) \odot \phi_2(\sigma_2)$ is $C(X)$-bilinear, etc. □

One can also show that $\phi_1 \odot_{LC} \phi_2$, when viewed as a function of ϕ_1 and ϕ_2, is $C(X)$-linear. Moreover, if in addition to the data of Corollary 1.12 we are given morphisms $\psi_i : \Gamma(\rho_i) \to \Gamma(\xi_i), i = 1, 2$, then

$$(\psi_1 \odot_{LC} \psi_2) \circ (\phi_1 \odot_{LC} \phi_2) = (\psi_1 \circ \phi_1) \odot_{LC} (\psi_2 \circ \phi_2),$$

etc.

As might be expected, our tensor product in LCMod_X commutes with colimits. Let $\rho : F \to X$ be a given Banach bundle and let $\{\Gamma(\pi_j) : j \in J\}$ be a spectral family in LCMod_X. It can be verified that $\{\Gamma(\pi_j) \odot_{LC} \Gamma(\rho) : j \in J\}$ can be made into a spectral family in the following way: to each morphism $\delta : j \to j'$ of the index category we associate the morphism

$$\psi_\delta \odot_{LC} \mathrm{id} : \Gamma(\pi_j) \odot_{LC} \Gamma(\rho) \to \Gamma(\pi_{j'}) \odot_{LC} \Gamma(\rho),$$

where $\psi_\delta : \Gamma(\pi_j) \to \Gamma(\pi_{j'})$ is the associated morphism of the spectral family $\{\Gamma(\pi_j) : j \in J\}$.

PROPOSITION 1.13. *With the data of the preceding paragraph, we have*

$$\mathrm{colim}_j \Gamma(\pi_j) \odot_{LC} \Gamma(\rho) = (\mathrm{colim}_j \Gamma(\pi_j)) \odot_{LC} \Gamma(\rho).$$

PROOF. (Outline). It is a matter of interpretation whether the equality sign should be replaced by $\cong$ in the statement of the theorem. The statement is incomplete, in any case, since it appears to concern only objects, and not morphisms, of the category LCMod_X. What *is* true is the following. If $\Gamma(\pi)$ is *a* colimit of the family $\{\Gamma(\pi_j) : j \in J\}$ and if

$$\overline{\psi}_j : \Gamma(\pi_j) \to \Gamma(\pi) \quad (j \in J)$$

are the associated morphisms (which make $\Gamma(\pi)$ a colimit), then the space $\Gamma(\pi) \odot_{LC} \Gamma(\rho)$, together with the morphisms

$$\overline{\psi}_j \odot \mathrm{id} : \Gamma(\pi_j) \odot_{LC} \Gamma(\rho) \to \Gamma(\pi) \odot_{LC} \Gamma(\rho),$$

is *a* colimit of the family $\{\Gamma(\pi_j) \odot_{LC} \Gamma(\rho) : j \in J\}$

It is also the case that colimits in LCMod_X are unique up to isomorphism. In other words, if $\Gamma(\pi')$ is *another* colimit of the family $\{\Gamma(\pi_j) : j \in J\}$ and if

$$\overline{\psi}_j' : \Gamma(\pi_j) \to \Gamma(\pi')$$

are the associated morphisms, then there is a unique isomorphism $\phi : \Gamma(\pi) \to \Gamma(\pi')$ which makes the diagram

$$\begin{tikzcd} \Gamma(\pi_j) \arrow[r, "\overline{\psi}_j"] \arrow[d, "\overline{\psi}_j'"'] & \Gamma(\pi) \arrow[dl, "\phi"] \\ \Gamma(\pi') & \end{tikzcd}$$

commute for all $j \in J$. □

A full verification of all the assertions in the Proof Outline would require several pages full of induced maps and commuting diagrams. There is, however, a deeper explanation as to why the theorem is true. This explanation involves another functor.

If $\rho : F \to X$ is a fixed bundle of Banach spaces, we define a functor

$$\otimes_\rho : \mathrm{LCMod}_X \to \mathrm{LCMod}_X$$

as follows: for each object $\Gamma(\pi)$ in LCMod_X,

$$\otimes_\rho(\Gamma(\pi)) = \Gamma(\pi) \otimes_{LC} \Gamma(\rho),$$

and, for each morphism $\phi : \Gamma(\pi) \to \Gamma(\xi)$,

$$\otimes_\rho(\phi) = \phi \otimes_{LC} \mathrm{id}.$$

Then it is easily verified that $\otimes_\rho$ is a covariant functor. Moreover, the previous theorem simply states that the functor $\otimes_\rho$ preserves colimits. The real reason that $\otimes_\rho$ preserves colimits is that it is a left adjoint to another functor, which we will denote by Hom_ρ. The functor Hom_ρ assigns to each object $\Gamma(\pi)$ in LCMod_X the set

$$\mathrm{Hom}_X(\Gamma(\rho), \Gamma(\pi)) := \mathrm{Hom}_{C(X)}(\Gamma(\rho), \Gamma(\pi))$$

of all bounded $C(X)$-linear homomorphisms from $\Gamma(\rho)$ to $\Gamma(\pi)$. (Thus, the unit ball of $\mathrm{Hom}_X(\Gamma(\rho), \Gamma(\pi))$ is the set of morphisms from $\Gamma(\rho)$ to $\Gamma(\pi)$ in the category LCMod_X.)

We first observe

PROPOSITION 1.14. *For any two Banach bundles $\pi : E \to X$ and $\rho : F \to X$, the space $\mathrm{Hom}_X(\Gamma(\rho), \Gamma(\pi))$ is a $C(X)$-module which is $C(X)$-locally convex.*

PROOF. This is a special case of a result appearing in [**3**]. □

We now define the functor

$$\mathrm{Hom}_\rho : \mathrm{LCMod}_X \to \mathrm{LCMod}_X$$

as follows: for each object $\Gamma(\pi)$ in LCMod_X,

$$\mathrm{Hom}_\rho(\Gamma(\pi)) = \mathrm{Hom}_X(\Gamma(\rho), \Gamma(\pi));$$

for each morphism $\phi : \Gamma(\pi) \to \Gamma(\xi)$ in LCMod_X, the corresponding morphism

$$\mathrm{Hom}_\rho(\phi) : \mathrm{Hom}_X(\Gamma(\rho), \Gamma(\pi)) \to \mathrm{Hom}_X(\Gamma(\rho), \Gamma(\xi))$$

is defined by the equation

$$[\mathrm{Hom}_\rho(\phi)](\psi) = \phi \circ \psi$$

for all $\psi \in \mathrm{Hom}_X(\Gamma(\rho), \Gamma(\pi))$. Then it is easy to show that Hom_ρ is a covariant functor.

The fact that $\otimes_\rho$ is a left adjoint of Hom_ρ is largely the result of the following "exponential law".

THEOREM 1.15. *Let* $\pi : E \to X, \rho : F \to X$, *and* $\xi : \mathcal{G} \to X$ *be Banach bundles. Then the spaces*

$$\mathrm{Hom}_X(\Gamma(\pi) \otimes_{LC} \Gamma(\rho), \Gamma(\xi)) \textit{ and } \mathrm{Hom}_X(\Gamma(\pi), \mathrm{Hom}_X(\Gamma(\rho), \Gamma(\xi)))$$

are isomorphic in the category LCMod_X. *Specifically, there is a unique isomorphism*

$$\theta : \mathrm{Hom}_X(\Gamma(\pi) \otimes_{LC} \Gamma(\rho)) \to \mathrm{Hom}_X(\Gamma(\pi), \mathrm{Hom}_X(\Gamma(\rho), \Gamma(\xi)))$$

such that

$$\{[\theta(\phi)](\sigma)\}(\tau) = \phi(\sigma \odot \tau)$$

for all $\phi \in \mathrm{Hom}_X(\Gamma(\pi) \otimes_{LC} \Gamma(\rho), \Gamma(\xi)), \sigma \in \Gamma(\pi)$, *and* $\tau \in \Gamma(\rho)$.

PROOF. It is well-known that there is a $C(X)$-linear isometric isomorphism from

$$\mathrm{Hom}_X(\Gamma(\pi) \otimes_X \Gamma(\rho), \Gamma(\xi)) \text{ to } \mathrm{Hom}_X(\Gamma(\pi), \mathrm{Hom}_X(\Gamma(\rho), \Gamma(\xi)))$$

given by $\phi \longmapsto \tilde{\phi}$, where

$$[\tilde{\phi}(\sigma)](\tau) = \phi(\sigma \otimes_X \tau)$$

for all $\sigma \in \Gamma(\pi)$ and $\tau \in \Gamma(\rho)$. (See **14**.)

On the other hand, there is also a natural $C(X)$-linear isometric isomorphism from

$$\mathrm{Hom}_X(\Gamma(\pi) \otimes_X \Gamma(\rho), \Gamma(\xi)) \text{ to } \mathrm{Hom}_X(\Gamma(\pi) \otimes_{LC} \Gamma(\rho), \Gamma(\xi)).$$

For, if $\phi \in \mathrm{Hom}_X(\Gamma(\pi) \otimes_X \Gamma(\rho), \Gamma(\xi))$, then by the universal property of Gelfand representations there is a unique $\overline{\phi} \in \mathrm{Hom}_X(\Gamma(\pi) \otimes_{LC} \Gamma(\rho), \Gamma(\xi))$ such that the diagram

$$\begin{tikzcd} \Gamma(\pi) \otimes_X \Gamma(\rho) \arrow[r, "\phi"] \arrow[d, "\wedge"'] & \Gamma(\xi) \\ \Gamma(\pi) \otimes_{LC} \Gamma(\rho) \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

commutes; moreover, $||\overline{\phi}|| \leq ||\phi||$ and

$$\overline{\phi}(\sigma \odot \tau) = (\overline{\phi}((\sigma \otimes_X \tau)^\wedge) = (\overline{\phi} \circ \wedge)(\sigma \otimes_X \tau) = \phi(\sigma \otimes_X \tau).$$

Thus, $\phi \longmapsto \overline{\phi}$ provides an isometric map from

$$\mathrm{Hom}_X(\Gamma(\pi) \otimes_X \Gamma(\rho), \Gamma(\xi))$$

to

$$\mathrm{Hom}_X(\Gamma(\pi) \otimes_{LC} \Gamma(\rho), \Gamma(\xi)).$$

The map is obviously surjective - if ψ belongs to the space displayed immediately above, then its preimage is $\psi \cdot \wedge$ - and it is easily seen to be $C(X)$-linear, etc. □

THEOREM 1.16. *For each bundle of Banach spaces $\rho : F \to X$ the functor $\otimes_\rho$ is a left adjoint of the functor Hom_ρ.*

PROOF. Let us denote by $\langle \Gamma(\pi), \Gamma(\xi) \rangle$ the morphisms from $\Gamma(\pi)$ to $\Gamma(\xi)$ in LCMod$_X$. Thus, $\langle \Gamma(\pi), \Gamma(\xi) \rangle$ is the unit ball in $\mathrm{Hom}_X(\Gamma(\pi),$

$\Gamma(\xi))$. For any two bundles π and ξ over X, we have, by the preceding theorem, a bijection

$$\phi_{\pi\xi} : \langle \otimes_\rho(\Gamma(\pi)), \Gamma(\xi) \rangle \to \langle < \Gamma(\pi),\ \mathrm{Hom}\ _\rho(\Gamma(\xi)) \rangle$$

which is characterized by the equation

$$\{[\phi_{\pi\xi}(\phi)](\sigma)\}(\tau) = \phi(\sigma \odot \tau)$$

which holds for all morphisms $\phi : \Gamma(\pi) \otimes_{LC} \Gamma(p) \to \Gamma(\xi)$ and for all $\sigma \in \Gamma(\pi)$ and $\tau \in \Gamma(\rho)$.

We must show that the family of maps $\{\phi_{\pi\xi}\}$ is a natural transformation from the bifunctor

$$F = \langle \otimes_\rho(\Gamma(?)), \Gamma(?') \rangle$$

to the bifunctor

$$\mathcal{G} = \langle \Gamma(?), \mathrm{Hom}_\rho(\Gamma(?')) \rangle.$$

What this comes to is this: given morphisms $\alpha : \Gamma(\pi') \to \Gamma(\pi)$ and $\beta : \Gamma(\xi) \to \Gamma(\xi')$ we must check that the diagram

$$\begin{tikzcd}
\langle \odot_\rho(\Gamma(\pi)), \Gamma(\xi)\rangle \arrow[r, "\varphi_{\pi\xi}"] \arrow[d, "F_{\alpha\beta}"'] & \langle \Gamma(\pi), \mathrm{Hom}_\rho(\Gamma(\xi))\rangle \\
\langle \odot_\rho(\Gamma(\pi')), \Gamma(\xi')\rangle \arrow[r, "\varphi_{\pi'\xi'}"] & \langle \Gamma(\pi'), \mathrm{Hom}_\rho(\Gamma(\xi'))\rangle \arrow[u, "\mathcal{G}_{\alpha\beta}"']
\end{tikzcd}$$

commutes, where $F_{\alpha,\beta}$ and $\mathcal{G}_{\alpha,\beta}$ are the morphisms assigned to the pair (α, β) by the bifunctors F and $\mathcal{G}$, respectively.

One can show first that

$$F_{\alpha,\beta}(o) = \beta \circ \phi \circ (\alpha \odot_{LC} \mathrm{id})$$

and that

$$\mathcal{G}_{\alpha\beta}(\psi) = (\mathrm{Hom}_\rho(\beta)) \cdot \psi \cdot \alpha$$

for all $\phi \in \langle < \odot_\rho(\Gamma(\pi)), \Gamma(\xi)\rangle$ and all $\psi \in \langle \Gamma(\pi), \mathrm{Hom}_\rho(\Gamma(\xi))\rangle$.

Let $\phi \in \langle \odot_\rho(\Gamma(\pi)), \Gamma(\xi)\rangle$. We must show that

$$(*) \qquad (\phi_{\eta'\xi'} \circ F_{\alpha,\beta}(\phi) = (\mathcal{G}_{\alpha,\beta} \circ \phi_{\pi\xi})(\phi)$$

or equivalently that

$$(**) \qquad [L(\sigma)](\tau) = [R(\sigma)](\tau)$$

holds for all $\sigma \in \Gamma(\pi)$ and $\tau \in \Gamma(\rho)$, where L and R denote the left - and right-hand sides of (*), respectively. Here are the unpleasant details:

$$\begin{aligned}
[L(\sigma)](\tau) &= [\{\phi_{\pi'\xi'}(F_{\alpha\beta}(\phi))\}(\sigma)](\tau) \\
&= F_{\alpha,\beta}(\phi)(\sigma \odot \tau) = \{\beta \circ \phi \circ (\alpha \odot_{LC} \mathrm{id})\}(\alpha \odot \tau) \\
&= (\beta \cdot \phi)(\alpha(\sigma) \odot \tau),
\end{aligned}$$

while

$$\begin{aligned}
[R(\sigma)](\tau) &= [\mathrm{Hom}_\rho(\beta) \circ \varphi_{\pi\xi}(\phi) \circ \alpha)(\sigma)](\tau) \\
&= [\mathrm{Hom}_\rho(\beta)\{\phi_{\pi\xi}(\phi)(\alpha(\sigma))\}](\tau) \\
&= [\beta \circ \{\phi_{\pi\xi}(\phi)(\alpha(\sigma))\}](\tau) \\
&= \beta\{[\phi_{\pi\xi}(\phi)(\alpha(\sigma))](\tau)\} = \beta\{\phi(\alpha(\sigma) \odot \tau)\} \\
&= (\beta \cdot \phi)(\alpha(\sigma) \odot \tau).
\end{aligned}$$

Thus, (**) holds. □

COROLLARY 1.17. *The functor $\otimes_\rho$ preserves colimits in the category* LCMod_X. *The functor* Hom_ρ *preserves limits in* LCMod_X.

2. Free and projective modules. In this section we shall consider free and projective modules in LCMod_X. The main result is that the projective objects in this category are the retracts of free objects. (The corresponding result for the category Mod_X was proved by Graven in [**6**]).

DEFINITION 2.1. Let $\mathcal{C}$ be either of the categories Mod_X or LCMod_X, and let S be any set. By a *free module* in $\mathcal{C}$ *generated* by S we mean an object F_S in $\mathcal{C}$ together with a map $i : S \to F_S$ having the following universal property: whenever M is an object in $\mathcal{C}$ and $g : S \to M$ is a bounded map (i.e., $||g||_\infty = \sup\{||g(s)|| : s \in S\} < +\infty$), there exists a unique $\tilde{g} \in \mathrm{Hom}_X(F_S, M)$ such that the diagram

$$\begin{tikzcd} F_S \arrow[r, "\tilde{g}"] & M \\ S \arrow[u, "i"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes and $||\tilde{g}|| = ||g||_\infty$.

Before considering the two cases, $\mathcal{C} = \mathrm{Mod}_X$ and $\mathcal{C} = \mathrm{LCMod}_X$, several remarks are in order. First, the usual argument shows that F_S is unique up to isomorphism. Second, the map $i : S \to F_S$ must be injective and, for all $s \in S, ||i(s)|| = 1$. Third, if $g : S \to M$ maps S into the unit ball of M, then the induced map $\tilde{g}$ is a morphism of $\mathcal{C}$.

Let us now consider the case in which $\mathcal{C} = \mathrm{Mod}_X$. In this case, our definition of a free $C(X)$-module is in agreement with that of Graven [**6**]. That is to say, we can take F_S to be the space $\ell^1(S, C(X))$ of all summable families $f = (f_s)_{s\in S}$ of functions in $C(X)$ with norm

$$||f|| = \sum_{s\in S} ||f_s||$$

and the obvious pointwise operations. The natural injection $i : S \to \ell^1(S, C(X))$ assigns to each $s \in S$ the family $(\delta_{st})_{t\in S}$, where δ_{st} is the constant function 0 if $t \neq s$ and δ_{ss} is the constant function 1.

PROPOSITION 2.2. *Let S be any set. Then the $C(X)$-module $\ell^1(S, C(X))$, together with the natural injection $i : S \to \ell^1(S, C(X))$ is the (up to isomorphism) free module in* Mod_X *generated by S.*

PROOF. Let M be any $C(X)$-module, and let $g : S \to M$ be a bounded map. We define a map $\tilde{g} : \ell^1(S, C(X)) \to M$ as follows:

$$\tilde{g}(f) = \sum_{s \in S} f_s g(s),$$

where $f = (f_s)_{s \in S}$. It is easy to verify that $\tilde{g}$ has the properties required of it in our definition of a free module. □

We consider next the case in which $\mathcal{C} = \mathrm{LCMod}_X$. Again, we let S be any set. If we apply the Gelfand functor $\mathcal{G}$ to $\ell^1(S, C(X))$ we get, as one might suspect, the free module in LCMod_X generated by S. For, suppose we are given a Banach bundle $\pi : E \to X$ and a bounded map $g : S \to \Gamma(\pi)$. Then there exists a map $\tilde{g} \in \mathrm{Hom}_X(\ell^1(S, C(X)), \Gamma(\pi))$ such that $g = \tilde{g} \cdot i$ and $||\tilde{g}|| = ||g||$. By the universal property of the Gelfand representation, there is a unique $\bar{g} \in \mathrm{Hom}_X(\mathcal{G}(\ell^1(S, C(X))), \Gamma(\pi))$ such that $\tilde{g} = \bar{g} \cdot \wedge$ and $||\tilde{g}|| = ||\bar{g}||$. Thus, we have the commuting diagram

$$\begin{tikzcd}
\mathcal{G}(\ell^1(S, C(X))) \arrow[dr, "\bar{g}"] & \\
\ell^1(S, C(X)) \arrow[u, "\wedge"] \arrow[r, "\tilde{g}"] & \Gamma(\pi) \\
S \arrow[u, "i"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Consequently, if we let $\bar{i} : S \to \mathcal{G}(\ell^1(S, C(X)))$ be the composed map $\wedge \cdot i$, it follows that the triangle

$$\begin{tikzcd}
\mathcal{G}(\ell^1(S, C(X))) \arrow[r, "\bar{g}"] & \Gamma(\pi) \\
S \arrow[u, "\bar{i}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. We have therefore proven:

PROPOSITION 2.3. *For any set $\mathcal{G}$, the space $\mathcal{G}(\ell^1(S, C(X)))$, together with the map $\bar{i} : S \to \mathcal{G}(\ell^1(S, C(X)))$, is the free module in* LCMod$_X$ *generated by S.*

The space $\ell^1(S, C(X))$ is isomorphic to the projective tensor product $\ell^1(S)\hat{\odot}C(X)$, where $\ell^1(S) = \ell^1(S, \mathbf{C})$ is the space of all summable familes of complex numbers indexed by S. The content of the next result is that the space $\mathcal{G}(\ell^1(S, C(X)))$ is isomorphic to the inductive tensor product $\ell^1(S)\hat{\hat{\odot}}C(X)$ which, in turn, is isomorphic to the space $C(X, \ell^1(S))$ of all continuous maps from X to $\ell^1(S)$.

PROPOSITION 2.4. *The canonical bundle of the $C(X)$-module $\ell^1(S, C(X))$ is (isomorphic to) the trivial bundle over S with constant fiber $\ell^1(S)$. Thus $\mathcal{G}(\ell^1(S, C(X)))$, the section space of this trivial bundle, can be identified with $C(X, \ell^1(S))$. Under these identifications,*

$$\hat{f}(X) = (f_{\mathcal{G}}(X))_{s\in S},$$

for all $f = (f_s)_{s\in S} \in \ell^1(S, C(X))$ and all $X \in X$.

PROOF. Suppose that $f = (f_s)_{s\in S} \in \ell^1(S, C(X))$. For each $X \in X$,

$$\sum_{s\in S} |f_{\mathcal{G}}(X)| \le \sum_{s\in S} ||f_s|| = ||f||,$$

which shows that the family $\tilde{f}(X) := (f_{\mathcal{G}}(X))_{s\in S}$ is in $\ell^1(S)$. Moreover,

$$||\tilde{f}(X)|| = \sum_{s\in S} |f_{\mathcal{G}}(X)| \le ||f||.$$

In this way, we get a map $\tilde{f} : X \to \ell^1(S)$. We will show that $\tilde{f}$ is continuous. Let $X_0 \in X$ and $\varepsilon > 0$ be given. Choose a finite subset F of S such that

$$\sum_{s\in S\setminus F} ||f_s|| < \varepsilon/4.$$

Because the functions $f_s, s \in S$ are continuous at X_0, we can find a neighborhood V of X_0 such that

$$|f_{\mathcal{G}}(X) - f_{\mathcal{G}}(X_0)| < \varepsilon/2n$$

for all $s \in F$ and $X \in V$, n being the number of elements in F. Consequently, if $X \in V$, then

$$\begin{aligned} ||\tilde{f}(X) - \tilde{f}(X_0)|| &= \sum_{s\in S} |f_{\mathcal{G}}(X) - f_{\mathcal{G}}(X_0)| \\ &\leq \sum_{s\in F} |f_{\mathcal{G}}(X) - f_{\mathcal{G}}(X_0)| + \sum_{s\in S\setminus F} 2 \cdot ||f_s|| \\ &< n \cdot \varepsilon/2n + 2 \cdot \varepsilon/4 = \varepsilon. \end{aligned}$$

Hence $\tilde{f}$ is continuous at X_0. Since X_0 was arbitrary, it follows that $\tilde{f} \in C(X, \ell^1(S))$. Moreover,

$$||\tilde{f}|| = \sup\{||\tilde{f}(X)|| : X \in X\} \leq ||f||.$$

The assignment $f \to \tilde{f}$ defines a norm-decreasing map $\theta : \ell^1(S, C(X))$

$\to C(X, \ell^1(S))$. It is easily checked that θ is linear and $C(X)$-linear. Since $C(X, \ell^1(S))$ can be viewed as the section space of the trivial bundle $\mathrm{pr}_1 : X \times \ell^1(S) \to X$, we can think of θ as being a sectional representation of $\ell^1(S, C(X))$ of Gelfand type. We want to show that $\mathcal{G}(\ell^1(S, C(X)))$ can be identified with $C(X, \ell^1(S))$ in such a way that $\tilde{f}$ and $\hat{f}$ become indistinguishable.

Since the section space $\mathcal{G}(\ell^1(S, C(X)))$ is determined up to isomorphism by the universal property of the Gelfand representation of $\ell^1(S, C(X))$, it suffices to show that our map

$$\theta : \ell^1(S, C(X)) \to C(X, \ell^1(S))$$

has this same universal property.

Let us suppose, therefore, that we are given any sectional representation $\phi : \ell^1(S, C(X)) \to \Gamma(\pi)$ of Gelfand type, where $\pi : E \to X$ is a Banach bundle. (In other words, $\phi \in \mathrm{Hom}_X(\ell^1(S, C(X)), \Gamma(\pi))$.) We must show that there is a unique map $\tilde{f} \in \mathrm{Hom}_X(C(X, \ell^1(S)), \Gamma(\pi))$ such that $||\tilde{\phi}|| = ||\phi||$ and $\phi = \tilde{\phi} \circ \theta$.

Let $\sigma_s = (\phi \circ i)(s)$, where $i : S \to \ell^1(S, C(X))$ is the natural injection. Then $\sigma_s \in \Gamma(\pi)$ and

$$||\phi|| = \sup\{\sigma_s : s \in S\}.$$

If $g \in C(X, \ell^1(S))$, then $\tilde{\phi}(g)$ can be defined by the series

$$\tilde{\phi}(g) = \sum_{s \in S} (p_s \circ g)\sigma_s,$$

where $p_s : \ell^1(S) \to \mathbf{C}$ is the s^{th} coordinate projection. The fact that the series $\sum_{s \in S}(p_s \circ g)\sigma_s$ converges uniformly on X to a section in $\Gamma(\pi)$ follows from the observation that

$$\begin{aligned}\sum_{s \in S} ||(p_s \circ g)(X)\sigma_{\mathcal{G}}(X)|| &\leq \sum_{s \in S} |p_s \circ g)(X)| \cdot ||\phi|| \\ &= ||g(X)|| \cdot ||\phi|| \leq ||g|| \cdot ||\phi||\end{aligned}$$

for all $X \in X$. Moreover,

$$||[\tilde{\phi}(g)](X)|| \leq \sum_{s \in S} ||(p_s \circ g)(X)\sigma_{\mathcal{G}}(X)|| \leq |g|| \cdot ||\phi||$$

for all $X \in X$, so

$$||\tilde{\phi}(g)|| = \sup\{||[\tilde{\phi}(g)](X)|| : X \in X\} \leq ||g|| \cdot ||\phi||.$$

It is easily checked that our map

$$\tilde{\phi} : C(X, \ell^1(S)) \to \Gamma(\pi)$$

belongs to $\mathrm{Hom}_X(C(X, \ell^1(S)), \Gamma(\pi))$; moreover, $||\tilde{\phi}|| \leq ||\phi||$. To prove that $\phi = \tilde{\phi} \circ \theta$, it suffices (by the universal property of $\ell^1(S, C(X))$) to show that $\phi \circ i = \tilde{\phi} \circ \theta \circ i$. However, for each $s \in S$ one can easily see that $(\tilde{\phi} \circ \theta \circ i)(s)$ is simply $\sigma_s = (\phi \circ i)(s)$. Thus, $\phi = \tilde{\phi} \circ \theta$, from which it follows that $||\phi|| = ||\tilde{\phi} \cdot \theta|| \leq ||\tilde{\phi}|| \cdot ||\theta|| = ||\tilde{\phi}||$. Hence, $||\tilde{\phi}|| = ||\phi||$.

The uniqueness of $\tilde{\phi}$ follows from the fact that the range of θ is dense in $C(X, \ell^1(S))$, the latter being a consequence of the bundle version of the Stone-Weierstrass Theorem (see [**4**]). (The constant functions in $C(X, \ell^1(S))$ belong to the range of θ and they completely fill the stalks of the trivial bundle $\mathrm{pr}_1 : X \times \ell^1(S) \to X$.) Alternatively, one can observe that the natural embeddings of the algebraic tensor product $\ell^1(S) \odot C(X)$ into $\ell^1(S, C(X))$ and $C(X, \ell^1(S))$ result in the commuting diagram

$$\begin{tikzcd}
\ell^1(S) \odot C(X) \arrow[r] \arrow[dr] & \ell^1(S, C(X)) \cong \ell^1(S) \hat{\odot} C(X) \arrow[d, "\theta"] \\
& C(X, \ell^1(S)) \cong \ell^1(S) \check{\odot} C(X)
\end{tikzcd}$$

Since the algebraic tensor product $\ell^1(S) \odot C(X)$ is dense in both the projective and inductive product spaces, it follows that the range of θ is dense. $\square$

Having described in some detail the free objects in LCMod_X, we will now show that every object in the category is a quotient of a free object. The actual theorem is somewhat sharper, and involves the notion of a *supermorphism*.

DEFINITION 2.5. Let $\mathcal{C}$ be either of the categories Mod_X or LCMod_X. By a *supermorphism* of $\mathcal{C}$, we mean a morphism $\phi : V \to W$ between objects of the category (so $||\phi|| \leq 1$, in particular) with the following additional property: for each $w \in W$ there is a $v \in V$ such that

$$\phi(v) = w \text{ and } ||v|| = ||w||.$$

A supermorphism $\phi : V \to W$ is, in particular, a quotient map, so it induces an isometry $V/(\ker \phi) \cong W$. Also, except in the trivial case where $W = 0$, $||\phi|| = 1$.

PROPOSITION 2.6. *If $\Gamma(\pi)$ is any object in* LCMod_X, *then there is a free object $\Gamma(\rho)$ in* LCMod_X *and a supermorphism $\phi : \Gamma(\rho) \to \Gamma(\pi)$. In particular, $\Gamma(\pi)$ is the quotient of a free object in* LCMod_X.

PROOF. By Graven [**6**], there exists a free object in Mod_X, $\ell^1(S, C(X))$, and a supermorphism

$$\alpha : \ell^1(S, C(X)) \to \Gamma(\pi).$$

(We may define S to be $\{\sigma \in \Gamma(\pi) : ||\sigma|| = 1\}$ and define α by

$$\alpha(f) = \sum_{\sigma \in S} f_\sigma \cdot \sigma$$

where $f = (f_\sigma)_{\sigma \in S}$.) By the universal property of the Gelfand representation, there is a unique morphism $\phi : \mathcal{G}(\ell^1(S, C(X))) \to \Gamma(\pi)$ which makes the diagram

$$\begin{tikzcd} \ell^1(S, C(X)) \arrow[r, "\alpha"] \arrow[d, "\wedge"'] & \Gamma(\pi) \\ \mathcal{G}(\ell^1(S, C(X))) \arrow[ur, "\phi"'] & \end{tikzcd}$$

commute. Given $\sigma \in \Gamma(\pi)$, we can choose $f \in \ell^1(S, C(X))$ so that $\alpha(f) = \sigma$ and $||f|| = ||\sigma||$. Then $\phi(\hat{f}) = \sigma$ and $||\hat{f}|| \leq ||f|| = ||\sigma||$. On the other hand, $||\sigma|| = ||\phi(\hat{f})|| \leq ||\phi|| \cdot ||\hat{f}|| \leq ||\hat{f}||$. Thus, $||\hat{f}|| = ||\sigma||$. This proves that ϕ is a supermorphism. □

We will show next that free objects are projective.

DEFINITION 2.7. Let $\mathcal{C}$ be either of the categories Mod_X or LCMod_X. An object V in $\mathcal{C}$ is *projective* if and only if, for any supermorphism $\phi : U \to W$ of the category, and for any $\psi \in \mathrm{Hom}_X(V, W)$, there exists a map $\alpha \in \mathrm{Hom}_X(V, U)$ such that $||\alpha|| = ||\psi||$ and $\psi = \phi \circ \alpha$, i.e., the diagram

$$\begin{tikzcd}
U \arrow[d, "\phi"'] \arrow[dr, "\alpha"] & \\
W & V \arrow[l, "\psi"']
\end{tikzcd}$$

commutes. (Since $||\psi|| = ||\phi \circ \alpha|| \leq ||\phi|| \cdot ||\alpha|| \leq ||\alpha||$, the requirement that $||\alpha|| = ||\psi||$ can be weakened to $||\alpha|| \leq ||\psi||$.)

PROPOSITION 2.8. *The free objects in either* Mod_X *or* LCMod_X *are projective.*

PROOF. Let $\mathcal{C}$ be either Mod_X or LCMod_X. Let S be a set, and let F_S, together with the map $i : S \to F_S$, be the free object in $\mathcal{C}$ generated by S. Suppose that we have a supermorphism $\phi : U \to V$ of the category, and suppose $\psi \in \mathrm{Hom}_X(F_s, V)$. For each $s \in S$, set $v_s = (\psi \circ i)(s)$. Since ϕ is a supermorphism, we can select $u_s \in U$ such that $\phi(u_s) = v_s$ and $||u_s|| = ||v_s||$. We define a map $g : S \to U$ by setting $g(s) = u_s$ for all $s \in S$. By the universal property of F_S, there is a unique $\alpha \in \mathrm{Hom}_X(F_S, U)$ such that $g = \alpha \cdot i$ and

$$||\alpha|| = ||g|| = \sup\{||u_s|| : s \in S\} = \sup\{||v_s|| : s \in S\} = ||v||.$$

Finally, for all $s \in S$,

$$\begin{aligned}
[(\phi \circ \alpha) \circ i](s) &= [\phi \circ (\alpha \circ i)](s) = \phi(g(s)) \\
&= \phi(u_s) = v_s = (\psi \circ i)(s),
\end{aligned}$$

so $(\phi \circ \alpha) \circ i = \psi \circ i$, from which we get $\phi \circ \alpha = \psi$. This proves that F_S is projective.

$$\begin{tikzcd}
U & & \\
V \arrow[u, "\phi"] \arrow[r, "\psi"'] & F_S \arrow[ul, "\alpha"'] \arrow[r, "i"'] & S \arrow[ull, "g"']
\end{tikzcd}$$

PROPOSITION 2.9. *If M is a projective object in* Mod_X, *then* $\mathcal{G}(M)$ *is projective in* LCMod_X.

PROOF. Suppose we have a supermorphsm $\phi : \Gamma(\pi) \to \Gamma(\rho)$ of the category LCMod_X, and suppose $\psi \in \mathrm{Hom}_X(\mathcal{G}(M), \Gamma(\rho))$. Because M is projective, there is a unique $\beta \in \mathrm{Hom}_X(M, \Gamma(\pi))$ such that $||\beta|| = ||\psi \circ \wedge|| = ||\psi||$ and $\phi \circ \beta = \psi \circ \wedge$, where $\wedge : M \to \mathcal{G}(M)$ is the Gelfand representation of M. By the universal property of the Gelfand representation, there is a unqiue $\alpha \in \mathrm{Hom}_X(\mathcal{G}(M), \Gamma(\pi))$ such that $\beta = \alpha \cdot \wedge$ and $||\alpha|| = ||\beta||$. Hence, $||\alpha|| = ||\psi||$ and $(\phi \circ \alpha) \circ \wedge = \phi \circ (\alpha \circ \wedge) = \phi \circ \beta = \psi \circ \wedge$, from which we get $\phi \circ \alpha = \psi$. Thus, the diagram commutes.

$$
\begin{tikzcd}
\Gamma(\pi) \arrow[d, "\phi"'] & & \\
\Gamma(\rho) & \mathcal{G}(M) \arrow[l, "\psi"'] \arrow[ul, "\alpha"'] & M \arrow[l, "\wedge"'] \arrow[ull, "\beta"']
\end{tikzcd}
$$

This proves that $\mathcal{G}(M)$ is projective. □

We have shown that free objects are projective. We will show next that retracts of projective objects are projective.

DEFINITION 2.10. Let $\mathcal{C}$ be either of the categories Mod_X or LCMod_X. If V and W are objects in $\mathcal{C}$, we say that W is a *retract* of V if and only if there are morphisms $\phi : V \to W$ and $\psi : W \to V$ such that ψ is an isometry and $\phi \cdot \psi = \mathrm{id}_W$.

With regard to this definition, we offer the following observations:

1) The map $\phi : V \to W$ is necessarily a supermorphism. (Let W be any element in W. Set $v = \psi(w)$. Then $||V|| = ||w||$ and $\phi(v) = (\phi \cdot \psi)(w) = \mathrm{id}_W(w) = w$.)

2) Up to isomorphism, W is both a subspace of V (since $\phi : W \to V$ is an isometry) and a quotient space of V (since $\phi : V \to W$ is a quotient

map). The situation is, however, even more specialized. If we let V_1 be the range of ψ and let V_2 be the kernel of ψ, then V_1 and V_2 are closed submodules of V, and V is their direct sum. We have (isometric) isomorphisms $W \cong V_1 \cong V/V_2$, and, additionally,

$$||v + V_2|| = ||v||$$

for all $v \in V_1$.

PROPOSITION 2.11. Let $\mathcal{C}$ be either of the categories Mod_X or LCMod_X. Suppose that V and W are objects in $\mathcal{C}$, that V is projective, and that W is a retract of V. Then W is also projective.

PROOF. By assumption, there are morphisms $\phi : V \to W$ and $\psi : W \to V$ such that ϕ is a supermorphism, ψ is an isometry, and $\phi \cdot \psi = \mathrm{id}_W$. Suppose we are given a supermorphism $\alpha : U_1 \to U_2$ and a map $\beta \in \mathrm{Hom}_X(W, U_2)$ as indicated in the diagram below.

$$\begin{tikzcd} U_1 \arrow[d, "\alpha"'] & V \arrow[l, "\gamma"'] \arrow[d, shift right, "\phi"'] \\ U_2 & W \arrow[l, "\beta"] \arrow[ul, "\delta"] \arrow[u, shift right, "\psi"'] \end{tikzcd}$$

Because V is projective, there is a unique $\Gamma \in \mathrm{Hom}_X(V, U_1)$ such that $\alpha \cdot \Gamma = \beta \cdot \phi$ and $||\Gamma|| = ||\beta \cdot \phi|| \leq ||\beta||$. Set $\delta = \Gamma \cdot \psi$. Then $||\delta|| \leq ||\Gamma|| \leq ||\beta||$ and $\alpha \cdot \delta = \alpha \cdot \Gamma \cdot \psi = \beta \cdot \phi \cdot \psi = \beta \cdot \mathrm{id}_W = \beta$. This proves that W is projective. □

It follows that the retract of a free object is projective. The converse of this was proved by Graven in [**6**] for the category Mod_X. We shall now prove it for the category LCMod_X.

PROPOSITION 2.12. *An object in* LCMod_X *is projective if and only if it is the retract of a free object.*

PROOF. Let us suppose that $\Gamma(\pi)$ is a projective object in the category LCMod_X. As in the proof of Proposition 2.6, we know that there is a free object $\ell^1(S, C(X))$ in Mod_X and a supermorphism ϕ : $\ell^1(S, C(X)) \to \Gamma(\pi)$. Furthermore, the induced map $\tilde{\phi} : C(X, \ell^1(S)) \to \Gamma(\pi)$ which causes the rectangle shown below to commute, is also a supermorphism (as shown in the proof of Proposition 2.6).

$$\begin{tikzcd}
\ell^1(S, C(X)) \arrow[r, "\wedge"] \arrow[d, "\psi"'] & C(X, \ell^1(S)) \cong \mathcal{G}(\ell^1(S, C(X))) \arrow[d, "\tilde{\phi}"] \\
\Gamma(\pi) \arrow[r, "\mathrm{id}"'] \arrow[ur, "\theta"] & \Gamma(\pi)
\end{tikzcd}$$

Since $\Gamma(\pi)$ is projective, there is a map $\theta : \Gamma(\pi) \to C(X, \ell^1(S))$ such that $\tilde{\phi} \cdot \theta = \mathrm{id}_{\Gamma(\pi)}$ and $||\theta|| = ||\mathrm{id}_{\Gamma(\pi)}|| = 1$. If $\sigma \in \Gamma(\pi)$, then $||\sigma|| = ||\tilde{\phi}(\theta(\sigma))|| \leq ||\theta(\sigma)||$, but, since $||\theta|| = 1$, we also have $||\theta(\sigma)|| \leq ||\sigma||$. Thus, $||\theta(\sigma)|| = ||\sigma||$. This proves that θ is an isometry. Consequently, $\Gamma(\pi)$ is a retract of the free object $C(X, \ell^1(S))$. $\square$

3. Injective objects. We turn our attention now to the injective objects in LCMod_X.

PROPOSITION 3.1. *Let $\mathcal{C}$ be either of the categories* Mod_X *or* LCMod_X, *and let M be an object in $\mathcal{C}$. The following are equivalent:*

1) *Whenever $\phi : N \to M$ and $i : N \to N'$ are morphisms of $\mathcal{C}$ and i is an isometry, there is a morphism $\tilde{\phi} : N' \to M$ such that $\phi = \tilde{\phi} \circ i$; i.e., the triangle*

$$\begin{tikzcd}
N \arrow[r, "\phi"] \arrow[d, "i"'] & M \\
N' \arrow[ur, "\tilde{\phi}"'] &
\end{tikzcd}$$

commutes.

1') *Whenever $i : N \to N'$ is an isometric morphism of $\mathcal{C}$ and $\phi \in \mathrm{Hom}_X(N, M)$, then there exists a $\tilde{\phi}$ in $\mathrm{Hom}_X(N', M)$ such that $\phi = \tilde{\phi} \circ i$ and $\|\tilde{\phi}\| = \|\phi\|$.*

2) *Whenever $\psi : M \to N$ and $j : M \to M'$ are morphisms of $\mathcal{C}$ and j is an isometry, there exists a morphism $\tilde{\psi} : M' \to N$ such that $\psi = \tilde{\psi} \circ j$:*

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "j"'] & N \\ M' \arrow[ur, "\tilde{\psi}"'] & \end{tikzcd}$$

2') *Whenever $j : M \to M'$ is an isometric morphism of $\mathcal{C}$ and whenever $\psi \in \mathrm{Hom}_X(M, N)$, there exists a $\tilde{\psi} \in \mathrm{Hom}_X(M', N)$ such that $\psi = \tilde{\psi} \circ j$ and $\|\tilde{\psi}\| = \|\psi\|$.*

3) *Whenever $j : M \to M'$ is an isometric morphism of $\mathcal{C}$, there exists a morphism $p : M' \to M$ (so, in particular, $\|p\| \leq 1$) such that $p \cdot j = \mathrm{id}_M$ (which means, in particular, that M is a retract of M').*

The above can be easily proved by arguments to be found in Graven [**6**]. In any case, an object M which satisfies the conditions above will be called *$\mathcal{C}$-injective*. For the category Mod_X the notion of an injective module is nothing new: an object M in Mod_X is Mod_X-injective if and only if it is $C(X)$-injective in the sense of Graven.

An immediate consequence of 3) above is that an object in LCMod_X which is Mod_X-injective is automatically LCMod_X-injective.

COROLLARY 3.2 *If $\Gamma(\pi)$ is an object in LCMod_X which is Mod_X-injective, then $\Gamma(\pi)$ is LCMod_X-injective.*

PROOF. Suppose $j : \Gamma(\pi) \to \Gamma(\rho)$ is an isometric morphism between objects in LCMod_X. Since $\Gamma(\pi)$ is LCMod_X-injective, there is a morphism $p : \Gamma(\rho) \to \Gamma(\pi)$ such that $P \circ j = \mathrm{id}_{\Gamma(\pi)}$. Since P is a morphism of the category LCMod_X it follows that $\Gamma(\pi)$ is LCMod_X-injective. □

Injective objects have also been studied in the category Ban_1 (of all Banach spaces with contractive linear maps as morphisms); see Semadeni [**15**], for instance. In this category, the injective objects prove to be quite exceptional: they are those Banach spaces which are isometrically isomorphic to $C(p)$, where p is compact and extremally disconnected. We will show first that, by taking products $\pi\{E_X : X \in X\}$ of Ban_1-injective Banach spaces,we get injective objects in the category LCMod_X. Later we will see that every injective object of LCMod_X is the retract of such a product.

PROPOSITION 3.3. *Let C be either of the categories* Mod_X *or* LCMod_X.

1) *If $\{M_s : s \in S\}$ is a family of modules belonging to C, then their product $\pi\{M_s : s \in S\}$ is C-injective if and only if M_s is C-injective for each $s \in S$.*

2) *If M is C-injective, and if N is a retract of M, then N is C-injective.*

Thus, the injective objects of C are closed under forming products and taking retracts.

The proof, which is quite straightforward, is omitted.

If $\{E_X : X \in X\}$ is any family of Banach spaces indexed by X, then we can apply 1) in the above proposition to

$$M := \pi\{E_X : X \in X\}$$

in the category Ban_1. If we view the elements of M as bounded functions (selections) with X as domain, then,under pointwise multiplication by functions in $C(X), M$ is a $C(X)$-locally convex module. Furthermore, we may regard M as a product in either of the categories Mod_X of LCMod_X. This comes about if, for each $X \in X$, we regard E_X as a $C(X)$-module. There multiplication of vectors in E_X by a function $f \in C(X)$ is defined to be multiplication of vectors by the scalar $f(X)$.

COROLLARY 3.4. *Let $\{E_X : X \in X\}$ be a family of Banach spaces, and let their* Ban_1*-product be regarded as a $C(X)$-locally convex module* *as described above. Then* $\pi\{E_X : X \in X\}$ *is* LCMod$_X$*-injective if and only if* E_X *is* Ban$_1$*-injective for each* $X \in X$.

PROOF. It suffices to show that, for each $X \in X, E_X$ is LCMod$_X$-injective if and only if E_X is Ban$_1$-injective.

Let us suppose first that E_X is Ban$_1$-injective. To show that E_X is LCMod$_X$-injective, we use 1) of Proposition 3.1. Suppose we have morphisms $\phi : \Gamma(\rho) \to E_X$ and $i : \Gamma(\rho) \to \Gamma(\rho')$, where $\rho : F \to X$ and $\rho' : F' \to X$ are Banach bundles and i is an isometry. We must show that there is a morphism $\tilde{\phi} : \Gamma(\rho') \to E_X$ such that $\phi = \tilde{\phi} \circ i$. To this end, we first observe that

$$\begin{tikzcd} \Gamma(\rho) \arrow[r, "\phi"] \arrow[d, "i"'] & E_X \cong \Gamma(\tau) \\ \Gamma(\rho') \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

where E_X may be regarded as the section space of the bundle $\tau : G \to X$ for which

$$G_y = \begin{cases} E_X, & \text{if } y = X, \\ 0, & \text{if } y \neq X. \end{cases}$$

If, for each $y \in X$, we let $\phi_y : F_y \to G_y$ be the restricted fiber map of the bundle map induced by ϕ, then clearly $\phi_y = \emptyset$ if $y \neq X$, and $||\phi_X|| = ||\phi|| \leq 1$. If, similarly, we let $i_y : F_y \to (F')_y$ be the fiber map induced by i, then i_y is an isometry (for each $y \in X$). Since E_X is Ban$_1$-injective, there is a contractive linear map $\tilde{\phi}_X : (F')_X \to G_X = E_X$ such that $\phi_X = \tilde{\phi}_X \circ i_x$:

$$\begin{tikzcd} F_X \arrow[r, "\phi_X"] \arrow[d, "i_X"'] & E_X \\ (F')_X \arrow[ur, "\tilde{\phi}_x"'] & \end{tikzcd}$$

We can then define a map $\tilde{\phi} : \Gamma(\rho') \to \Gamma(\tau) \cong E_X$ as follows:

$$[\tilde{\phi}(\sigma)](y) = \begin{cases} \tilde{\phi}_X(\sigma(X)), & \text{if } y = X \\ 0, & \text{if } y \neq X, \end{cases}$$

for $\sigma \in \Gamma(\rho')$ and $y \in X$. Then it is easily checked that $\tilde{\phi}$ is a morphism and that $\phi = \tilde{\phi} \circ i$. Thus, E_x is LCMod_X-injective.

Suppose, conversely, that E_X is LCMod_X-injective. To show that E_X is Ban_1-injective, we suppose that V and V' are Banach spaces, that $\phi : V \to E_x$ is a contractive linear map, and that $i : V \to V'$ is a linear isometry. We must show that there exists a contractive linear map $\tilde{\phi} : V \to E_x$ such that $\phi = \tilde{\phi} \circ i$:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "i"'] & E_X \\ V' \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

Let $\rho : F \to X$ and $\rho' : F' \to X$ be Banach bundles for which

$$F_y = \begin{cases} V, & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases}$$

and

$$(F')_y = \begin{cases} (V'), & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases}$$

and define maps $i' : \Gamma(\rho) \to \Gamma(\rho')$ and $\phi' : \Gamma(\rho) \to \Gamma(\tau) \cong E_x$ by

$$[i'(\sigma)](y) = \begin{cases} i(\sigma(X)), & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases}$$

and

$$[\phi'(\sigma)](y) = \begin{cases} \phi(\sigma(X)), & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases}$$

for $\sigma \in \Gamma(\rho)$ and $y \in X$. Then it is easily checked that i' and ϕ' are morphisms of LCMod_X and that i' is an isometry. Since E_x is assumed to be LCMod_X-injective, there is a morphism $\tilde{\phi} : \Gamma(\rho') \to \Gamma(\tau)$ such that $\phi' = \tilde{\phi}' \circ i'$. If we let $\tilde{\phi} : V' \to E_x$ be the restricted fiber map $(\tilde{\phi}')_x : (F')_x \to G_x$ induced by $\tilde{\phi}'$, then it is easily checked that $\tilde{\phi}$ has the required properties. □

We can now describe (sort of) the most general injective objects in LCMod_X.

THEOREM 3.5. *An object* $\Gamma(\pi)$ *in* LCMod_X *is* LCMod_X*-injective if and only if there is some set* Y *such that* $\Gamma(\pi)$ *is a retract of the product* $\pi\{\ell^\infty(Y) : x \in X\}$.

PROOF. The "if" part follows from Corollary 3.4 and the fact that injectivity is preserved by taking retracts.

To prove the "only if" part, let us assume that $\Gamma(\pi)$ is LCMod_X-injective, where $\pi : E \to X$ is a Banach bundle. It is well known that, for each $x \in X$, there is a set Y_x such that E_x is a retract (in Ban_1) of $\ell^\infty(Y_X)$. For $x \in X$, let $\alpha_x : E_x \to \ell^\infty(Y_X)$ be the isometric injection. Let Y be the disjoint union of $\{Y_x : x \in X\}$; it is then easy to check that $\ell^\infty(Y_x)$ is a retract of $\ell^\infty(Y)$ for each $x \in X$. Let $\beta_x : \ell^\infty(Y_x) \to \ell^\infty(Y)$ be the isometric injection. Let $N := \pi\{\ell^\infty(Y) : x \in X\}$. Then the map $\Gamma : \Gamma(\pi) \to N$ defined by

$$[\Gamma(\sigma)](x) = (\beta_x \circ \alpha_x)(\sigma(x)) \quad (\sigma \in \Gamma(\pi))$$

is easily seen to be an isometric morphism of the category LCMod_X. Since $\Gamma(\pi)$ was injective in LCMod_X, it follows (from Proposition 3.1) that $\Gamma(\pi)$ is a retract (in LCMod_X) of N. □

We shall next prove the converse of Corollary 3.2; in other words we shall prove that, for objects in LCMod_X, our two types of injectivity - Mod_X-injectivity and LCMod_X - injectivity - coincide. The crucial fact is:

LEMMA 3.6. *Let* $\Gamma(\pi)$ *be an object in* LCMod_X, *let* N *be an object in* Mod_X *with canonical bundle* $\rho : F \to X$ *and suppose that* $i : \Gamma(p) \to N$ *is an isometric morhpism. Then* $\wedge \circ i : \Gamma(\pi) \to \Gamma(\rho)$ *is also an*

isometry, where $\wedge : N \to \Gamma(\rho)$ *is the Gelfand map for* N.

PROOF. Let $\sigma \in \Gamma(\pi)$. It suffices to show that, for each $x \in X$,

$$||\sigma(X)|| = ||i(\sigma)^{\wedge}(X)||,$$

for then it follows that

$$\begin{aligned} ||\sigma|| = \sup\{||\sigma(X)|| : x \in X\} &= \sup\{||i(\sigma)^{\wedge}(X)|| : x \in X\} \\ &= ||i(\sigma)^{\wedge}|| = ||(\wedge \circ i)(\sigma)||. \end{aligned}$$

For each open neighborhood V of X, choose a closed neighborhood $U \subset V$ and a continuous function $f_V : X \to [0, 1]$ such that $f_V = 1$ on U and $f_V = 0$ outside of V. Then

$$\begin{aligned} ||\sigma(X)|| = \mathrm{inf}_V\{||f_V\sigma||\} &= \mathrm{inf}_V\{||i(f_V\sigma)||\} \\ \mathrm{inv}_V\{||f_V i(\omega)||\} &= ||i(\sigma)^{\wedge}(X)|| \end{aligned}$$

by Varela [**16**]. □

THEOREM 3.7. *An object* $\Gamma(\pi)$ *in* LCMod_X *is* LCMod_X*-injective if and only if it is* Mod_X*-injective.*

PROOF. Because of Corollary 3.2, it suffices to prove the "only if" part. Suppose, then, that $\Gamma(\pi)$ is LCMod_X-injective. Suppose also that M is an object in Mod_X and that $i : \Gamma(\pi) \to M$ is an isomorphic morphism (of Mod_X). By the lemma, the composed morphism $\wedge \cdot i : \Gamma(\pi) \to G(M)$ is also an isometry:

$$\begin{tikzcd} \Gamma(\pi) \arrow[r, "i"] & M \arrow[dl, dash, "\wedge"] \\ G(M) \arrow[u, "Q"] & \end{tikzcd}$$

Since $\Gamma(\pi)$ is assumed to be LCMod_X-injective, there is a morphism (of LCMod_X)$Q : G(M) \to \Gamma(\pi)$ such that $Q \circ (\wedge \circ i) = \mathrm{id}_{\Gamma(\pi)}$. If we set $p = Q \circ \wedge$, then we have a morphism $p : M \to \Gamma(\pi)$ such that $p \circ i = \mathrm{id}_{\Gamma(\pi)}$, which proves that $\Gamma(\pi)$ is Mod_X-injective. □

The next result may be viewed as a refinement of the previous one. If M is any object in Mod_X, then it is shown in Graven [**6**] that M has a Mod_X-*injective envelope* $\overline{M}$. That is, $\overline{M}$ is a smallest injective module which contains M isomorphically. More precisely, there is an isometric morphism $i : M \to \overline{M}$, $\overline{M}$ is injective, and in $\overline{M}$ there is no proper Mod_X-injective submodule which contains $i(M)$. Such an injective envelope, while not unique, is unique up to isomorphism (in the category Mod_X). Furthermore, $i : M \to \overline{M}$ is an *essential extension* of M, which means this: whenever $\phi : \overline{M} \to N$ is a morphism in Mod_X for which the composed map $\phi \cdot i$ is an isometry, then ϕ itself is an isometry.

THEOREM 3.8. *Let* $\Gamma(\pi)$ *be an object in* LCMod_X, *and let* N *be its* Mod_X-*injective envelope. Then* N *is* $C(X)$-*locally convex; moreover,* N *is the* LCMod_X-*injective envelope of* $\Gamma(\pi)$.

PROOF. Let $\rho : F \to X$ be the canonical bundle of N. To show that N is $C(X)$-locally convex, we will prove that the Gelfand map $\wedge : N \to \Gamma(\rho)$ is an isometric isomorphism. If $i : \Gamma(\pi) \to N$ is the isometric embedding of $\Gamma(\pi)$ into its injective envelope, then by Lemma 3.6, the composed morphism $\wedge \circ i : \Gamma(\pi) \to \Gamma(\rho)$ is an isometry. However, since $i : \Gamma(\pi) \to N$ is an essential extension, it follows that $\wedge : N \to \Gamma(\rho)$ is also an isometry. By the bundle version of the Stone-Weierstrass theorem, the range of $\wedge$ is dense in $\Gamma(\rho)$. Since, however, $\wedge$ is an isometry, it then follows that $\wedge : N \to \Gamma(\rho)$ is surjective. Therefore, the map $\wedge$ is an isometric isomorphism from N to $\Gamma(\rho)$.

Knowing that N is an object in LCMod_X, it is then easily argued that N is the injective envelope of $\Gamma(\pi)$ in the category LCMod_X. We have, first of all, our isometric morphism $i : \Gamma(\pi) \to N$. Suppose M is a submodule of N which is in LCMod_X, is LCMod_X-injective, and contains $i(\Gamma(\pi))$. Then M is Mod_X-injective, by the previous theorem. Hence, by the definition of an injective envelope, M must be N. Thus, N is the LCMod_X-injective envelope of $\Gamma(\pi)$. □

PROPOSITION 3.9. *Let $\pi : E \to X$ be a Banach bundle. If $\Gamma(\pi)$ is* LCMod_X*-injective, then $\Gamma(\pi)$ is a* Ban_1*-injective.* □

It turns out, at least in some simple instances, that Ban_1 - and LCMod_X - injective objects coincide.

PROPOSITION 3.10. *Let X be extremally disconnected. Then $C(X)$ is* LCMod_X*-injective.*

PROOF. Let βX_d denote the Stone-Cech compactification of X with its discrete topology. It is well-known (see, e.g., Lacy [**13**]) that X is a topological retract of βX_d. Let

$$X \xrightarrow{s} \beta X_d \xrightarrow{r} X$$

denote the mappings involved; we have $r \circ s = \mathrm{id}_X$.

It then follows, of course, that $C(X)$ is a Ban_1-retract of $C(\beta X_d)$:

$$C(X) \xleftarrow{s^*} C(\beta X_d) \xleftarrow{r^*} C(X)$$

where, for example,

$$s^*(f)(X) = (f \circ s)(X)$$

for $f \in C(\beta X_d)$ and $X \in X$. Moreover, $C(\beta X_d)$ is a $C(X)$-locally convex module under pullback by r: we define $a \cdot f = r^*(a) \cdot f$, for $a \in C(X)$ and $f \in C(\beta X_d)$.

We will be done if we can show that r^* and s^* are actually morphisms of LCMod_X and that the canonical algebra isometric isomorphism of $\ell^\infty(X)$ and $C(\beta X_d)$ is also a morphism of LCMod_X.

1) Let $a, b \in C(X)$. For $z \in \beta X_d$, we have

$$\begin{aligned} r^*(ab)(z) &= (ab)(r(z)) = a(r(z))b(r(z)) \\ &= r^*(a)(z)r^*(b)(z), \end{aligned}$$

so that

$$r^*(ab) = r^*(a)r^*(b) = a \cdot r^*(b).$$

2) Similarly, for $f \in C(\beta X_d), a \in C(X)$, and $X \in X$, we have

$$\begin{aligned}[s^*(a \cdot f)](X) &= (a \cdot f)(s(X)) = r^*(a(s(X)))f(s(X)) \\ &= a(r(s(X)))f(s(X)) = a(X)f(s(X)),\end{aligned}$$

or

$$s^*(a \cdot f) = a \cdot s^*(f).$$

Thus, $C(X)$ is an LCMod_X-retract of $C(\beta X_d)$.

3) Finally, let $\phi : \ell^\infty(X) \to C(\beta X_d)$ be the isometric algebra isomorphism which, to each bounded function on X, assigns its unique continuous extension to βX_d, and let $i : C(X) \to \ell^\infty(X)$ be the isometric embedding. We necessarily have $\phi \circ i = r^*$. Let $f \in \ell^\infty(X), a \in C(X)$, and $z \in \beta X_d$. Then

$$\begin{aligned}\phi(a \cdot f)(z) &= [\phi(i(a))f](z) = [\phi(i(a))](z)[\phi(f)](z) \\ &= [r^*(a)](z)[\phi(f)](z),\end{aligned}$$

so that

$$\phi(a \cdot f) = r^*(a)\phi(f) = a \cdot \phi(f).$$

By the same token, ϕ^{-1} is a morphism of LCMod_X. Thus, $C(X)$ is an LCMod_X retract of $\ell^\infty(X)$, and is LCMod_X-injective. □

COROLLARY 3.11. *$C(X)$ is LCMod_X-injective if and only if X is extremally disconnected.*

PROOF. Proposition 3.10 shows the "if" part; the other half is a consequence of Proposition 3.9 and the fact that if $C(X)$ is Ban_1-injective, then X is extremally disconnected. □

If X is any compact space, then it is well-known (see again Lacey [**13**]) that there is an extremally disconnected compact space p and a continuous surjection $p : P \to X$ such that $C(P)$ is the Ban_1-injective envelope of $C(X)$. From the known properties of extremally disconnected spaces, it follows that p is a topological retract of βX_d.

We thus have a diagram

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, shift left, "u"] & X \\ \beta X_d \arrow[u, shift left, "t"] \arrow[ur, "s"'] & \end{tikzcd}$$

where each of the maps p, s, and t is a surjection.

As usual, we may regard $C(P)$ and $C(\beta X_d)$ as $C(X)$-locally convex modules (under pullback by p and s, respectively).

COROLLARY 3.12. *Let X be compact, and let p be the extremally disconnected space such that $C(P)$ is the* Ban_1*-injective envelope of $C(X)$. If $C(P)$ is regarded as a $C(X)$-module via pullback, then $C(P)$ is also the* LCMod_X*-injective envelope of $C(X)$.*

PROOF. In the same fashion as in the proof of the Proposition 3.10, we may show that each of the maps in the above diagram induces a morphism of LCMod_X. Thus, $C(P)$ is LCMod_X-injective, since it is a retract in LCMod_X of the LCMod_X-injective object $C(\beta X_d)$.

Finally, let $\pi : E \to X$ be a Banach bundle, and suppose that $\delta : C(P) \to \Gamma(\pi)$ is a morphism of LCMod_X such that

$$\delta \cdot P^* : C(X) \to \Gamma(\pi)$$

is an isometry (where $p^* : C(X) \to C(P)$ is the isometry induced by p). Since $C(P)$ is the Ban_1-injective envelope of $C(X)$, it follows that δ is an isometry, and thus that $C(P)$ is an essential extention of $C(X)$, and finally that $C(P)$ is the LCMod_X-injective envelope of $C(X)$. □

COROLLARY 3.13. *Let p be any extremally disconnected space such that there exists a continuous surjection $p : P \to X$. Then $C(P)$, as a $C(X)$-module via pullback by p, is* LCMod_X*-injective.*

PROPOSITION 3.14. *Let V be a Banach space. Then V is a* Ban_1*-retract of $C(X, V)$; if $V \neq 0$, then $C(X)$ is a* Mod_X*-retract of $C(X, V)$.*

Thus, if $C(X,V)$ is LCMod_X*-injective, then V is* Ban_1*-injective, and $C(X)$ is* LCMod_X*-injective.*

PROOF. Let $i : V \to C(X,V)$ assign to each $v \in V$ the constant function on X whose value is v. Then i is an isometry. For each $x \in X, \mathrm{ev}_x : C(X,V) \to V(\ \mathrm{ev}_x(f) = f(x))$ is a contractive linear map, and $\mathrm{ev}_x \circ i = \mathrm{id}_V$. Thus, V is a Ban_1-retract of $C(X,V)$.

Assume $V \neq 0$. Choose a unit vector $v \in V$ and an $F \in V^*$ such that $F(v) = 1$ and $||F|| = 1$. Define $i : C(X) \to C(X,V)$ by

$$[i(f)](x) = f(x)v \quad (f \in C(X), x \in X),$$

and define $R : C(X,V) \to C(X)$ by

$$R(\xi) = F \cdot \xi.$$

Then i and R are morphisms of LCMod_X, i is an isometry, and $R \circ i = \mathrm{id}_{C(X)}$. □

COROLLARY 3.15. *If $C(X,\mathbf{C}^n)$ is* LCMod_X*-injective, then X is extremally disconnected and the norm on $\mathbf{C}^n$ is the sup-norm.*

COROLLARY 3.16. *Let p be extremally disconnected, and suppose that there is a continuous surjection $p : P \to X$. Regard $C(P)$ has a $C(X)$-module by pullback, and endow $\mathbf{C}^n$ with the sup-norm. Then the section space $C(p,\mathbf{C}^n)$ of the trivial bundle* $\mathrm{pr}_1 : P \times \mathbf{C}^n \to X$ *is* LCMod_X*-injective.*

PROOF. Let $S = \{1,\ldots,n\}$, and consider the "flip-flop" mapping

$$\phi : \ell^\infty(S, C(P)) \to C(P,\mathbf{C}^n)$$

given by

$$[\phi(f)(y)](i) = [f(i)](y) \quad (y \in p, i \in S).$$

It is evident that ϕ is both an isometry and surjective. Moreover, for $a \in C(X)$, we have

$$\begin{aligned}[\phi(a \cdot f)(y)](i) &= [(a \cdot f)(i)](y) \\ &= a(p(y))[f(i)](y) = p^*(a)(y)[\phi(f)(y)](i),\end{aligned}$$

so that $\phi(a \cdot f) = p^*(a)\phi(f) = a \cdot \phi(f)$, where $p^* : C(X) \to C(P)$ is the isometric embedding dual to p.

Thus, $C(P, \mathbf{C}^n)$ is $C(X)$-isometrically isomorphic to a product of $C(P)$, and the conclusion follows since $C(P)$ is LCMod_X-injective. □

Conversely, however, even if $C(P, \mathbf{C}^n)$ is LCMod_X-injective for an extremally disconnected P, then such a surjection need not exist. For example, if $P = \{x\}$ for some $x \in X$, then $M := C(P, \mathbf{C}^n) \cong \mathbf{C}^n$ as a Banach space. If we make M into a $C(X)$-module by $a \cdot f = a(x)f$ for $a \in C(X)$ and $f \in \mathbf{C}^n$, then M is evidently LCMod_X-injective, but if X is at all interesting, then there is no surjection $p : P \to X$.

If we are willing to make some strong assumptions about the fibers E_X of a bundle $\pi : E \to X$ such that $\Gamma(\pi)$ is LCMod_X-injective, then we may obtain some more information.

PROPOSITION 3.17. *Let X be extremally disconnected, and let $\pi : E \to X$ be a Hausdorff bundle such that each fiber E_x is n-dimensional. Then $\Gamma(\pi)$ is $C(X)$-isormorphic to the section space $C(X, \mathbf{C}^n)$ of the trivial bundle $\mathrm{pr}_1 : X \times \mathbf{C}^n \to X$. Thus, the section space of any Hausdorff bundle with constant, finite-dimensional fibers over X is "almost" LCMod_X-injective.*

PROOF. By Gierz [**4**, Theorem 18.5], $\pi : E \to X$ is locally trivial.

Since X has a basis for its topology consisting of open-and-closed sets, we may find a disjoint open-and-closed cover $U_1, \ldots, U_m$ of X such that $\Gamma(\pi|U_k)$ is $C(X)$-isomorphic to $C(U_k, \mathbf{C}^n)$ for each $k = 1, \ldots, m$. Let $\phi_k : \Gamma(\pi|U_k) \to C(U_k, \mathbf{C}^n)$ denote these isomorphisms $(k = 1, \ldots, m)$.

Since U_k is open-and-closed for each k it follows from the Tietze Extension Theorem for bundles [**10**] and the fact that the characteristic functions of the U_k are continuous on X, that $\Gamma(\pi)$ is the direct sum (in the sup-norm sense) of the $\Gamma(\pi|U_k)$; in particular, if $\sigma \in \Gamma(\pi)$, we may think of σ as $\sigma_1 + \cdots + \sigma_m$, where $\sigma_k \in \Gamma(\pi|U_k)$ for $k = 1, \ldots, m$. Similarly, $C(X, \mathbf{C}^n)$ is the direct sum (sup-norm) of $C(U_k|\mathbf{C}^n)$, and $f \in C(X, \mathbf{C}^n)$ corresponds to $f_1 + \cdots + f_m$ in that direct sum.

Define $\phi : \Gamma(\pi) \to C(X, \mathbf{C}^n)$ by

$$\phi(\sigma) = \sum_{i=1}^{n} \phi_i(\sigma_i).$$

Evidently ϕ is a $C(X)$-homomorphism, ϕ is bijective, and $||\phi|| = \sup\{||\phi_i|| i = 1, \ldots, m\} > 0$, so that ϕ is an isomorphism. □

To this point, we have tacitly assumed that all of our results referred to complex Banach spaces which were objects in LCMod_X, i.e., all fibers of bundles were complex, and so was $C(X)$. It is evident, however, that nothing we have done would be invalid if we had restricted ourselves to a category $\mathrm{LCMod}_X^{\mathbf{R}}$ of real Banach modules of functions over X with values in real Banach spaces, viewed as modules over $C_{\mathbf{R}}(X)$, the space of real-valued continuous functions on X. If we now restrict ourselves to that "real" subcategory (which we will continue to denote LCMod_X, for convenience), we may use some results of Gierz to obtain yet more information about injective objects in LCMod_X.

Suppose that $\pi : E \to X$ is a bundle such that each fiber E_X is $\mathbf{R}^n$ endowed with the sup-norm, for some $n \in \mathbf{N}$. If we put the pointwise ordering on the fibers $\mathbf{R}^n$, then $\Gamma(\pi)$ becomes a Banach lattice (and a bundle of Banach lattices) in the obvious fashion.

PROPOSITION 3.18. *Let $\pi : E \to X$ be a Hausdorff bundle such that:*

1) *the function $X \longmapsto ||\sigma(x)|| : X \to \mathbf{R}$ is continuous for each $\sigma \in \Gamma(\pi)$;*

2) *if $U \subset X$ is an open set, and if $\sigma : U \to E$ is a bounded continuous section, then σ may be extended to a global section $\sigma^* \in \Gamma(\pi)$ of the same norm; and*

3) *for each $x \in X$, the fiber E_x is $\mathbf{R}^n$, endowed with the sup-norm.*

Then $\Gamma(\pi)$ is $C(X)$-isometrically isomorphic to the section space $C(X, \mathbf{R}^n)$ of the trivial bundle $\mathrm{pr}_1 : X \times \mathbf{R}^n \to X$, *where $\mathbf{R}^n$ is given its sup-norm.*

PROOF. The stated conditions mean that $\pi : E \to X$ satisfies the hypotheses of Propositions 2.1-2.3 of [**5**]. In particular, then, $\Gamma(\pi)$ and $C(X, \mathbf{R}^n)$ are isomorphic. We claim that if $\mathbf{R}^n$ (regarded as a fiber of $\mathrm{pr}_1 : X \times \mathbf{R}^n \to X$) is given its sup-norm, then the isomorphism is actually an isometry.

From Propositions 2.1 and 2.2 of [**5**], we may find sections $\sigma_1, \ldots, \sigma_n$

≥ 0 such that the σ_i are pairwise orthogonal and such that $||\sigma_i(X)|| = 1$ for all $i = 1, \ldots, n$ and $x \in X$. (In this context, "pairwise orthogonal" means that $[\sigma_i(x)](k) \cdot [\sigma_j(x)](k) = \emptyset$ for each $1 \leq i < j \leq n$, each $k = 1, \ldots, n$, and $x \in X$. From this it follows that

$$||a\sigma_i(x) + b\sigma_j(x)|| = \max\{||a\sigma_i(x)||, ||b\sigma_j(x)||\}$$

when $i \neq j$ and $a, b \in \mathbf{R}$.)

As in the proof of Proposition 2.3 of [**5**], we define a map

$$\phi : C(X, \mathbf{R}^n) \to \Gamma(\pi)$$

by

$$[\phi(f)](x) = \sum_{k=1}^{n} (f_k \sigma_k)(x),$$

where $f_k \in C(X)$ is defined by $f_k(x) = [f(x)](k)$ for $k = 1, \ldots, n$ and $x \in X$.

Then ϕ is $C(X)$-linear. Moreover, since $\sigma_1(x), \ldots, \sigma_n(x)$ forms a basis for E_x, the Stone-Weierstrass theorem for bundles shows that the range of ϕ is dense in $\Gamma(\pi)$.

We claim that ϕ is the isometry we are seeking. For $f \in C(X, \mathbf{R}^n)$, we have

$$\begin{aligned} ||\phi(f)|| &= \sup \{||\phi(f)(x)|| : X \in X\} \\ &= \sup \left\{ \left|\left| \sum_{k=1}^{n} (f_k \sigma_k)(x) \right|\right| : x \in X \right\} \\ &= \sup \{||f_k(x)\sigma_k(x)|| : x \in X, k = 1, \ldots, n\} \\ &= \sup \{|f_k(x)|, x \in X, k = 1, \ldots, n\} \\ &= ||f|| \end{aligned}$$

□

COROLLARY 3.19. *Let $\pi : E \to X$ be a bundle satisfying* 1) *and* 2) *of Proposition* 3.19. *Suppose that each fiber E_x is $\mathbf{R}^{n(x)}$, endowed with the sup-norm, for some $n(x) \in \mathbf{N}$. Then there exists a pairwise disjoint cover $U_1, \ldots, U_n$ of (possibly empty) open-and-closed sets in X such that*

a) $\dim E_x = k$ *if and only if* $X \in U_k$ $(k = 1, \ldots, n)$; *and*

b) $\Gamma(\pi)$ *is $C(X)$-isometrically isomorphic to the product* $\prod_{k=1}^{n} C(U_k, \mathbf{R}^k)$, *where $\mathbf{R}^k$ is given its sup-norm. (In particular, then, $\Gamma(\pi)$ is the product of the section spaces of trivial bundles.)*

PROOF. The stated conditions also imply that the function

$$\dim : X \to \mathbf{N}, \quad x \to \dim E_x$$

is continuous. (See Proposition 2.3 of [**5**].) Thus, $U_k = \dim^{-1}(\{k\})$ is both open and closed in X for each $k \in \mathbf{N}$; since X is compact, there exists n such that $U_1, \ldots, U_n$ covers X. The Tietze Extension Theorem and the continuity of the characteristic functions of the U_k mean that $\Gamma(\pi)$ is $C(X)$-isometrically isomorphic to the product $\prod_{k=1}^{n} \Gamma(\pi|U_k)$. Now, apply the preceding proposition to each factor $\Gamma(\pi|U_k)$, regarded as a $C(U_k)$-module. □

COROLLARY 3.2. *Let $\pi : E \to X$ be a bundle satisfying the conditions of the preceding corollary. Assume additionally that X is extremally disconnected. Then $\Gamma(\pi)$ is* LCMod_X*-injective.*

PROOF. For each $k = 1, \ldots, n$, $\Gamma(\pi|U_k)$ is isometrically isomorphic to $C(U_k, \mathbf{R}^k)$, which is an LCMod_X-retract of the LCMod_X-injective modules $C(X, \mathbf{R}^k)$. □

COROLLARY 3.21. *Let $\pi : E \to X$ be a bundle satisfying the conditions of Corollary* 3.20. *If $\Gamma(\pi)$ is* LCMod_X*-injective, then X is extremally disconnected.*

PROOF. If each E_X is n-dimensional, then $\Gamma(\pi) \cong C(X, \mathbf{R}^n)$, and the result follows from Proposition 3.14.

The more general case follows immediately, since then each U_k in the open-and-closed cover of X is itself extremally disconnected, and X

itself, as the union of finitely many disjoint open-and-closed subsets, is then also extremally disconnected. □

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DUKE UNIVERSITY, DURHAM, NC 27706
TRINITY COLLEGE, HARTFORD, CT 06106