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ON THE RELATIVE GROWTH OF AREA FOR SUBORDINATE FUNCTIONS

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Introduction. Let f be analytic in the open unit disk Δ and let $A(r,f)$ denote the area of the region on the Riemann surface onto which the disk $|z| < r$ is mapped by f. Then

$$A(r,f) = \int_{|z|<r}\int |f(z)|^2 dxdy$$
$$= \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}.$$

If F is also analytic in Δ, we say f is *subordinate* to $F (f \prec F)$ if there exists a bounded analytic function ω, $\omega(0) = 0$, such that $f(z) = F(\omega(z)), z \in \Delta$. Golusin [**5**] has shown that if $f \prec F$, then

$$A(r,f) \leq A(r,F), \quad r \leq 1/\sqrt{2}.$$

Reich [**6**] has extended this result by showing that, for $0 < r < 1$,

$$A(r,f) \leq T(r)A(r,F), \tag{1}$$

where

$$T(r) = mr^{2m-2}$$

in the range

$$\frac{m-1}{m} \leq r^2 \leq \frac{m}{m+1} \quad (m = 1,2,\ldots).$$

He also finds, for each r, all pairs $(f;F)$ for which equality holds in (1). Waniurski and this author [**3**] have extended Reich's results to quasi-subordinate pairs. It is the purpose, however, of this paper to examine

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the asymptotic behavior of the ratio $A(r, f)/A(r, F)$ for subordinate pairs (f, F). The definition of $T(r)$ immediately yields the existence of positive constants A and B such that

$$\frac{A}{1-r} \leq T(r) \leq \frac{B}{1-r}, \quad \leq r < 1.$$

It then follows from (1) that $f \prec F$ implies

$$A(r, f)/A(r, F) = O\Big(\frac{1}{1-r}\Big), \quad \text{as } r \to 1. \tag{2}$$

We intend to examine the relation (2) for various choices of schlicht mappings F: a bounded mapping, a mapping onto an infinite strip, and a mapping onto a sector with central angle $\pi\alpha$. We find that the growth of $A(r, F)/A(r, F)$ becomes smaller as the range of F becomes more expansive. In particular, the relation (2) is almost best possible when F is bounded, while $A(r, f) \leq A(r, F)$ when F maps Δ onto a sector with central angle $\geq \pi$. We first establish these two extreme cases, and then we give some results which interpolate between them.

Main Results. Fix $\rho > 1$. We first exhibit a function f, analytic in Δ, continuous in $\overline{\Delta}$, for which

$$A(r, f) \geq \frac{K}{(1-r)(\log\frac{1}{1-r})^{2\rho}}$$

$$\frac{1}{2} < r < 1, \quad K \text{a constant}.$$

We simply define $f(z) = \sum_{n=1}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} \frac{1}{k^\rho} & \text{if } n = 2^k, \\ 0 & \text{otherwise.} \end{cases}$$

The justification that f has the desired properties can be found in [**4**].

Actually, one cannot hope to find a bounded f such that $A(r, f) \geq K(1-r)^{-1}$, as the following theorem states.

THEOREM 1. *If* $f \in H^2$ *then* $\lim_{r\to 1}(1-r)A(r, f) = 0$.

PROOF. Since for $r \leq r_n = 1 - \frac{1}{n}$ we have $A(r, f) \leq A(r_n, f)$, it suffices to show that $(1 - r_n)A(r_n, f) \to 0$ as $n \to \infty$. But

$$(3) \qquad \begin{aligned}(1 - r_n)A(r_n, F) \leq & \frac{\pi}{n}\sum_{k=1}^{n} k|a_k|^2 \\ & + \frac{\pi}{n}\sum_{k=n+1}^{\infty} k|a_k|^2\left(1 - \frac{1}{n}\right)^{2k}.\end{aligned}$$

Let $\varepsilon > 0$. Choose N such that $\sum_{k=N+1}^{\infty} |a_k|^2 < \varepsilon$. Then $\frac{1}{n}\sum_{k=N+1}^{n} k|a_k|^2 < \varepsilon$. Consequently, the first term of the right side of (3) satisfies

$$\begin{aligned}\frac{\pi}{n}\sum_{k=1}^{n} k|a_k|^2 &= \frac{\pi}{n}\sum_{k=1}^{N} k|a_k|^2 + \frac{\pi}{n}\sum_{k=N+1}^{n} k|a_k|^2 \\ &\leq \frac{\pi}{n}(\text{constant}) + \pi\varepsilon \\ &\leq 2\pi\varepsilon, \text{ if } n \text{ is sufficently large.}\end{aligned}$$

For the second term of the right side of (3), a differentiation shows $k(1 - \frac{1}{n})^{2k}$ is a decreasing function of k, for $k \geq (\log(\frac{n}{n-1}))^{-1}$. Since $\log(1 + x) > x - x^2/2$ for $0 < x < 1$, the choice $x = 1/(n-1)$ shows that $n > (\log(\frac{n}{n-1}))^{-1}$ for $n > 2$. Hence, $k(1 - \frac{1}{n})^{2k}$ is a decreasing function of k, for $k \geq n$, and so

$$\begin{aligned}\frac{1}{n}\sum_{k=n+1}^{\infty} k|a_k|^2\left(1 - \frac{1}{n}\right)^{2k} &\leq \frac{1}{n}\sum_{k=n+1}^{\infty} |a_k|^2 n\left(1 - \frac{1}{n}\right)^{2n} \\ &\leq \sum_{k=n+1}^{\infty} |a_k|^2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

The proof is complete. □

We now take up the case where F maps Δ onto a sector with central angle $\geq \pi$. Brannan, Clunie, and Kirwan [1] have shown that if

$$F(z) = \left(\frac{1 + cz}{1 - z}\right)^{\alpha}, \quad \alpha \geq 1, \ |c| \leq 1,$$

then every $f \prec F$ can be expressed as

$$f(z) = \int_{-\pi}^{\pi} F(ze^{-it})d\mu(t)$$

for some probability measure μ on $|z| = 1$.

THEOREM 2. *Let F be analytic in Δ, and let μ be a probability measure on $|z| = 1$. If F is defined by*

$$f(z) = \int_{-\pi}^{\pi} F(ze^{-it})d\mu(t),$$

then

$$A(r, f) \leq A(r, F).$$

PROOF. Letting $z = \rho e^{i\theta}$, we have

$$\begin{aligned}
A(r, f) &= \int_0^r \int_0^{2\pi} |f'(z)|^2 \rho d\theta d\rho \\
&= \int_o^r \int_0^{2\pi} \left| \int_{-\pi}^{\pi} F'(ze^{-it})e^{-it} d\mu(t) \right|^2 \rho d\theta d\rho \\
&\leq \int_o^r \int_0^{2\pi} \left(\int_{-\pi}^{\pi} |F'(ze^{-it})| d\mu(t) \right)^2 \rho d\theta d\rho \\
&\leq \int_0^r \int_0^{2\pi} \int_{-\pi}^{\pi} |F'(ze^{-it})|^2 d\mu(t) \rho d\theta d\rho
\end{aligned}$$

(by Jensen's inequality)

$$\begin{aligned}
&= \int_0^r \int_{-\pi}^{\pi} \left(\int_0^{2\pi} |F'(\rho e^{i(\theta - t)})|^2 d\theta \right) d\mu(t) \rho d\rho \\
&= \int_0^r \int_0^{2\pi} |F'(\rho e^{i\phi})|^2 d\phi \rho d\rho \\
&= A(r, F).
\end{aligned}$$

The proof is complete. □

In preparation for our final result, we need to establish some notation. First, K will denote a constant, not necessarily the same in each instance. Also, if $p(x)$ and $q(x)$ are positive functions on the same domain X, then $p(x) \sim q(x)$ will mean that the ratio $p(x)/q(x)$ is bounded away from 0 and ∞ on X. That is, there exist positive constants m and M such that

$$m < p(x)/q(x) < M, \quad x \in X.$$

LEMMA 1. [**2**, p. 84] *If* $z = re^{i\phi}, 1/2 < r < 1$, *then*

$$\int_{-\pi}^{\pi} \frac{d\phi}{|1-z|^p} \sim \begin{cases} \frac{1}{(1-r)^{p-1}} & \text{if } p > 1, \\ \log\frac{1}{1-r} & \text{if } p = 1. \end{cases}$$

In fact, a more careful analysis would show that the limits of integration can be replaced by $-\pi/2$ and $\pi/2$. That is, all of the growth is attained in the right half plane. This remark will be used in the proof of the next result.

LEMMA 2. *If* $F(z) = (\frac{1+z}{1-z})^\alpha, \alpha > 0$, *then*

$$A(r, F) \sim (1-r)^{-2\alpha}, \quad \frac{1}{2} < r < 1.$$

If $F(z) = \log(\frac{1+z}{1-z})$, *then*

$$A(r, F) \sim \log\frac{1}{1-r}, \quad \frac{1}{2} < r < 1,$$

PROOF. In the case $\alpha > 0$ we have

$$|F'(z)| \sim \frac{1}{|1-z|^{1+\alpha}}, \quad z \in \Delta\ , \mathrm{Re}\ z \geq 0,$$

and

$$|F'(z)| \sim \frac{1}{|1+z|^{1-\alpha}}, \quad z \in \Delta,\ \mathrm{Re}\ z < 0.$$

Since $1-\alpha \le 1+\alpha$, it follows that, for $z=\rho e^{i\theta}$, $1/2<r<1$,

$$\begin{aligned} A(r,F) &\sim \int_0^r \int_{-\pi/2}^{\pi/2} \frac{d\theta}{|1-z|^{2(1+\alpha)}} \\ &\sim (1-r)^{-2\alpha}, \text{ by Lemma 1.} \end{aligned}$$

The logarithm case follows by this same reasoning, but with $\alpha = 0$. The proof is complete. □

We now state our main result giving the growth of $A(r,f)/A(r,F)$ for various domains $F(\Delta)$.

THEOREM 3. *If $f \prec F$, where $F(z) = K(\frac{1+z}{1-z})^\alpha$, then*

$$(4) \qquad A(r,f)/A(r,F) = \begin{cases} \mathrm{O}(1) & \textit{if } \alpha > 1/2\ , \\ \mathrm{O}\left(\log\frac{1}{1-r}\right) & \textit{if } \alpha = 1/2, \\ \mathrm{o}\left(\frac{1}{(1-r)^{1-2\alpha}}\right) & \textit{if } 0<\alpha<1/2. \end{cases}$$

Also,

$$(5) \qquad A(r,f)/A(r,F) = \begin{cases} \mathrm{o}\left(\frac{1}{(1-r)}\log\frac{1}{1-r}\right) & \textit{if } F(z) = K \log\frac{1+z}{1-z}, \\ \mathrm{O}\left(\frac{1}{1-r}\right) & \textit{if } F(z) = Kz, \end{cases}$$

PROOF. We first consider the case $\alpha > 1/2$. By Littlewood's subordination theorem and Lemma 1,

$$\int_{-\pi}^{\pi} |f(z)|^2 d\theta \le K \int_{-\pi}^{\pi} \frac{d\theta}{|1-z|^{2\alpha}} \le K(1-r)^{1-2\alpha}.$$

We now use a theorem of Hardy and Littlewood's relating the mean growth of an analytic function with the mean growth of its derivative [**2**, p. 80]. The result is that

$$\int_{-\pi}^{\pi} |f'(z)|^2 d\theta \le K(1-r)^{-2\alpha-1},$$

and hence $A(r,f) \leq K(1-r)^{-2\alpha}$. By Lemma 2, we may divide the left side by $A(r,F)$ and the right side by $(1-r)^{-2\alpha}$, thus giving the desired result.

Now consider the case $\alpha = 1/2$. Applying Lemma 1,

$$\int_{-\pi}^{\pi} |f(z)|^2 d\theta \leq \log\frac{1}{1-r}.$$

By the Cauchy formula

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i}\int_{|\zeta|=p} \frac{f(\zeta)d\zeta}{(\zeta - z)^2} \\ &= \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \frac{f(\rho e^{i(t+\theta)})e^{i(t-\theta)}}{(\rho e^{it} - r)^2} dt, \end{aligned}$$

where $\rho = \frac{1}{2}(1+r)$. Minkowski's inequality (in continuous form) then gives

$$\begin{aligned} M_2(r,f') &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{M_2(\rho,f)dt}{\rho^2 - 2\rho r\cos t + r^2} \\ &= \frac{M_2(\rho,f)}{\rho^2 - r^2} \leq \frac{K(\log\frac{1}{1-r})^{1/2}}{1-r}, \end{aligned} \tag{6}$$

where $M_2(r,f')$ denotes the mean square $\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta\}^{1/2}$. Using (6) and integration by parts gives

$$A(r,f) \leq \left(\frac{K}{1-r}\right)\cdot\log\left(\frac{1}{1-r}\right).$$

Application of Lemma 2 yields

$$A(r,f)/A(r,F) \leq K\log\left(\frac{1}{1-r}\right).$$

We finally consider the case $0 < \alpha < 1/2$. This, and also (5), are easily proved since $f \in H^2$. We may thus use Theorem 1 to obtain $A(r,F) = o(1-r)^{-1}$. Then we divide each side by the approximate relations from Lemma 2.

This completes the proof of Theorem 3. It would be interesting to know whether, in the case $\alpha = 1/2$, the "big O" may be replaced by "little o".

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