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(0.3) INTERPOLATION ON THE ZEROS OF $\pi_n(x)$

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1. Introduction. Balázs' and Turán's work [**1**] on (0,2) interpolation in 1957 led to considerable interest in the general problem of Birkhoff interpolation. However, in spite of the recent classic book on this subject by G.G. Lorentz et al. [**2**], the problem of (0,3) interpolation on the zeros of $\pi_n(x)$ seems to have been ignored. Similarly, although we know [**2**, p. 10] that (0,2,3) interpolation is regular on any real distinct nodes, i.e., is always uniquely solvable, there is no known formula for the explicit expression for the interpolant, except in the trigonometric case on equidistant nodes.

Recently Varma has found some quadrature formulae using values and third derivatives of $\pi_n(x)$ together with values of the first derivatives at ± 1 on using his method in [**3**]. However, his approach is not via interpolatory formulae. In view of this, we propose to show that (0,3) interpolation is regular for $n \geq 4$ on the zeros of $\pi_n(x)$ and to give the explicit formulae for the fundamental polynomials. (For $n < 3$, the problem is not regular because Polya conditions are not satisfied and for $n = 3$, the problem is trivial.) It turns out that the quadrature formula of Varma can be obtained by integrating the polynomial of (0,3) interpolation. The methods used here show that the problem of $(0,1,\ldots,r-3,r)$ on zeros of $\pi_n(x)$ is regular for any positive integral $r \geq 3$.

In §2, we give the preliminaries and state the main results. The proof of Theorem 1 is given in §3 and the fundamental polynomials are derived in §4. §5 comprises the proof of Theorem 2 and the fundamental polynomials for the (0,3) case are given in §6. In §7, we apply the results to derive a quadrature formula.

2. Preliminaries and main results. It is known that the polynomials $\pi_n(x)$ satisfy the differential equation

$$(2.1) \qquad (1-x^2)y'' = -n(n-1)y, \quad n \geq 2.$$

For $n = 0$ and 1, $\pi_0(x) = 1$, $\pi_1(x) = x$ and $\pi_n(x) = (1-x^2)P'_{n-1}(x)$ where $P_n(x)$ denotes the Legendre polynomial of degree n with $P_n(1) = 1$.

We shall require

$$(2.2)\quad \begin{cases} \pi_n'(1) & = -n(n-1) = (-1)^{n+1}\pi_n'(-1), \\ \pi_n''(1) & = -n^2(n-1)^2/2 = (-1)^n\pi_n''(-1), \\ \pi_n'''(1) & = -n^2(n-1)^2(n+1)(n-2)/8 = (-1)^{n+1}\pi_n'''(-1). \end{cases}$$

We recall that $P_{n-1}(1) = 1 = (-1)^{n-1}P_{n-1}(-1)$ and that

$$(2.3)\quad \begin{cases} P_{n-1}'(1) & = n(n-1)/2 = (-1)^n P_{n-1}'(-1), \\ P_{n-1}''(1) & = (n+1)n(n-1)(n-2)/8 = (-1)^{n-1}P_{n-1}''(-1). \end{cases}$$

We shall also make use of the known identities

$$(2.4)\quad \begin{cases} (n+1)P_{n+1}(x) & = (2n+1)xP_n(x) - nP_{n-1}(x), \\ (1-x^2)P_n'(x) & = nP_{n-1}(x) - nxP_n(x), \\ nP_n(x) & = xP_n'(x) - P_{n-1}'(x), \\ (n+1)P_n(x) & = P_{n+1}'(x) - xP_n'(x). \end{cases}$$

The known orthogonal property

$$(2.5)\quad \int_{-1}^{1}(1-x^2)P_{k-1}'(x)P_{j-1}'(x)dx = \frac{2k(k-1)}{2k-1}\delta_{jk},$$

where δ_{jk} denotes the Kronecker delta and the recursion relation

$$(2n-1)xP_{n-1}'(x) = (n-1)P_n'(x) + nP_{n-2}'(x)$$

leads to

$$(2.6)\quad \begin{aligned} &\int_{-1}^{1} t(1-t^2)P_{k-1}'(t)P_{n-1}'(t)dt \\ &= \begin{cases} 2(n-1)n(n+1)/(2n-1)(2n+1), & k = n+1, \\ \frac{2n(n-1)(n-2)}{(2n-1)(2n-3)}, & k = n-1, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

We also note the integrals

$$(2.7)\quad \begin{cases} \int_{-1}^{1} P_{n-1}'(x)P_{k-1}'(x)dx = (1+(-1)^{n+k})\frac{k(k-1)}{2}, & k-1 \le n \\ \int_{-1}^{1} xP_{n-1}'(x)P_{k-1}'(x)dx = (1-(-1)^{n+k})\frac{k(k-1)}{2}, & k \le n. \end{cases}$$

Let $-1 = x_1 < \cdots < x_{n-1} < x_n = 1$ denote the zeros of $\pi_n(x)$. We recall the known identity

$$(2.8) \qquad \frac{P'_{n-1}(t)}{t - x_\nu} = \frac{-1}{P_{n-1}(x_\nu)} \sum_{k=2}^{n-1} \frac{2k-1}{k(k-1)} P'_{k-1}(t) P'_{k-1}(x_\nu).$$

From (2.4), we have

$$\begin{aligned}(1 - x^2)&P'_{n-1}(x) \\ &= \frac{n(n-1)}{2n-1} \left\{ -\frac{P'_{n+1}(x)}{2n+1} + \frac{2(2n-1)}{(2n-3)(2n+1)} P'_{n-1}(x) - \frac{P'_{n-3}(x)}{2n-3} \right\}.\end{aligned}$$

Multiplying both sides by $(1 - x^2)P'_{k-1}(x)$ and using (2.5), we observe that, for $2 \le k \le n-1$, we have
(2.9)

$$\int_{-1}^{1} \pi_n(x)\pi_k(x)dx = \begin{cases} -\frac{n(n-1)}{(2n-1)(2n-3)} \cdot \frac{2(n-2)(n-3)}{2n-5}, & k = n-2 \\ 0, & k \ne n-2. \end{cases}$$

We shall prove that the problem of modified $(0,3)$ interpolation on zeros of $\pi_n(x)$ is regular. More precisely, we shall prove

THEOREM 1. *If $Q(x) \in \pi_{2n-1}$ satisfies the conditions*

$$(2.10) \qquad \begin{cases} Q(x_\nu) = 0 & \nu = 1, 2, \cdots, n, \\ Q'(+1) = Q'(-1) = 0, & \\ Q'''(x_\nu) = 0, & \nu = 2, \ldots, n-1, \end{cases}$$

then $Q(x) \equiv 0$.

As a consequence we will derive

THEOREM 2. *The problem of $(0,3)$ interpolation on the zeros of $\pi_n(x)$ is regular.*

The proofs of Theorems 1 and 2 will depend on the elementary

LEMMA 1. *Let $g(x)$ be a given polynomial of degree $\leq m$ and let $L_g(x)$ denote the linear function interpolating $g(x)$ at ± 1. Then the only polynomial solution of the differential equation*

$$3(1-x^2)y'' - n(n-1)y = g(x)$$

is given by

$$y = -\frac{1}{n(n-1)}L_g(x) + \int_{-1}^{1}(g(t) - L_g(t))K(x,t)dt,$$

where

(2.11)

$$K(x,t) = -\sum_{\nu=2}^{n-1}\frac{(2\nu-1)\pi_\nu(x)P'_{\nu-1}(t)}{2\nu(\nu-1)\lambda_{\nu,n}}, \quad \lambda_{\nu,n} = 3\nu(\nu-1) + n(n-1).$$

The proof of this lemma is a simple consequence of the relation (2.7) and is left out. If we set

$$\ell_j(t) = \frac{\pi_n(t)}{(1-x_j)\pi'_n(x_j)}, \quad j = 1,2,\ldots,n, \tag{2.11a}$$

we have

LEMMA 2. *The following identities are valid:*

$$\begin{aligned}\int_{-1}^{1}&(1+t)(P'_{n-1}(t) - P'_{n-1}(1))K(x,t)dt \\ &= \sum_{k=2}^{n-1}\frac{(n-k)(n+k-1)(2k-1)\pi_k(x)}{2k(k-1)\lambda_{k,n}},\end{aligned} \tag{2.12}$$

$$\begin{aligned}\int_{-1}^{1}&(1-t)(P'_{n-1}(t) - P'_{n-1}(-1))K(x,t)dt \\ &= \sum_{k=2}^{n-1}\frac{(n-k)(n+k-1)(2k-1)\pi_n(x)(-1)^{n+k}}{2k(k-1)\lambda_{k,n}}\end{aligned} \tag{2.13}$$

and
(2.14)

$$\int_{-1}^{1} \ell_j(t)K(x,t)dt = \frac{-1}{n(n-1)P_{n-1}^2(x_j)} \sum_{k=2}^{n-1} \frac{(2k-1)\pi_k(x)P'_{k-1}(x_j)}{k(k-1)\lambda_{k,n}},$$

where $\lambda_{k,n}$ *is given in* (2.11).

PROOF. These identities are easy to verify on using the expression (2.11) for $K(x,t)$ and the formulae (2.7). In particular, from (2.7) we have

$$\int_{-1}^{1} P'_{n-1}(x)dx = 1 + (-1)^n, \quad \int_{-1}^{1} xP'_{n-1}(x)dx = 1 - (-1)^n.$$

We also use the differential equation for $P_{n-1}(x)$, viz.,

$$(1-x^2)P''_{n-1}(x) - 2xP'_{n-1}(x) + n(n-1)P_{n-1}(x) = 0.$$

□

3. Proof of Theorem 1. From (2.10) we see that the polynomial $Q(x) \in \pi_{2n-1}$ must be of the form $\pi_n(x)s(x)$, $s(x) \in \pi_{n-1}$. The conditions $Q'''(x_\nu) = 0, \nu = 2, \ldots, n-1$, after simplification imply that

$$3(1-x_\nu^2)s''(x_\nu) - n(n-1)s(x_\nu) = 0, \quad \nu = 2, \ldots, n-1.$$

Since $s(x) \in \pi_{n-1}$, the above conditions show that $s(x)$ satisfies the differential equation

$$3(1-x^2)s''(x) - n(n-1)s(x) = (Ax+B)P'_{n-1}(x). \tag{3.1}$$

The requirement $Q'(\pm 1) = 0$ implies that $s(\pm 1) = 0$. Putting $x = \pm 1$ in (3.1) we get $A + B = -A + B = 0$, which shows that $A = B = 0$. From Lemma 1, it follows that $s(x) \equiv 0$, which completes the proof. □

4. Fundamental polynomials for modified (0,3) case. We can now find the fundamental polynomials of modified (0,3) interpolation.

We shall denote them by $\{r_\nu(x)\}_1^n$, $\sigma_1(x)$, $\sigma_n(x)$ and $\{\rho_\nu(x)\}_{\nu=2}^{n-1}$ respectively.

a) *The polynomials* $\rho_\nu(x), \nu = 2, \ldots, n-1$. These polynomials are determined by the conditions

$$(4.1)\qquad \begin{cases} \rho_\nu(x_j) &= 0,\ \ j = 1, \ldots, n, \\ \rho'_\nu(x_1) &= \rho'_\nu(x_n) = 0, \\ \rho'''_\nu(x_j) &= \delta_{\nu j},\ \ j = 2, \ldots, n-1. \end{cases}$$

Putting $\rho_\nu(x) = \pi_n(x)s_\nu(x)$, and using (4.1), we see that

$$3(1 - x_j^2)s''_\nu(x_j) - n(n-1)s_\nu(x_j) = \delta_{\nu j}\frac{1 - x_j^2}{\pi'_n(x_j)},\quad j = 2, \ldots, n-1.$$

Equivalently, $s_\nu(x)$ satisfies the differential equation

$$\begin{aligned} &3(1 - x^2)s''_\nu(x) - n(n-1)s_\nu(x) \\ &\quad = \frac{1 - x_\nu^2}{\pi'_n(x_\nu)} \cdot \frac{P'_{n-1}(x)}{(x - x_\nu)P''_{n-1}(x_\nu)} + (Ax + B)P'_{n-1}(x). \end{aligned}$$

From $\rho'_\nu(\pm 1) = 0$, we get $s_\nu(\pm 1) = 0$ so that putting $x = \pm 1$ in the above differential equation, we obtain

$$B = Ax_\nu,\quad A = \frac{-1}{(1 - x_\nu^2)(P''_{n-1}(x_\nu))^2}.$$

Using these values and simplifying, we derive

$$3(1 - x^2)s''_\nu(x) - n(n-1)s_\nu(x) = \frac{\ell_\nu(x)}{P''_{n-1}(x_\nu)},$$

where $\ell_\nu(x)$ is given by (2.11a).

By Lemma 1, we have

$$s_\nu(x) = \frac{1}{P''_{n-1}(x_\nu)}\int_{-1}^{1} \ell_\nu(t)K(x,t)dt,$$

and from (2.14) in Lemma 2 we get

$$(4.2)\qquad s_\nu(x) = \frac{1}{n^2(n-1)^2P^3_{n-1}(x_\nu)}\sum_{k=2}^{n-1}\frac{(2k-1)\pi_k(x)P'_{k-1}(x_\nu)}{k(k-1)\lambda_{k,n}}$$

and $\rho_\nu(x) = \pi_n(x)s_\nu(x)$.

b) *The polynomials* $\sigma_1(x)$, $\sigma_n(x)$. From symmetry it follows that $\sigma_n(x) = -\sigma_1(-x)$ where $\sigma_1(x)$ is determined by the conditions that $\sigma_1(x_\nu) = 0$, $\nu = 1,\dots,n$; $\sigma_1'(x_1) = 1, \sigma_1'(x_n) = 0$ and $\sigma_1'''(x_\nu) = 0$, $\nu = 2,\dots,n-1$. Setting $\sigma_1(x) = \pi_n(x)\alpha(x)$ we see that $\alpha(1) = 0$, $\alpha(-1) = (-1)^n/n(n-1)$. As above, we see that $\alpha(x)$ satisfies the differential equation

$$3(1-x^2)\alpha''(x) - n(n-1)\alpha(x) = (Cx+D)P'_{n-1}(x).$$

Since $\alpha(1) = 0$, we get

$$D = -C = -1/n(n-1).$$

Applying Lemma 1 we get

$$\alpha(x) = \frac{(-1)^n(1-x)}{2n(n-1)} - \frac{1}{n(n-1)}\int_{-1}^{1}(1-t)(P'_{n-1}(t) - P'_{n-1}(-1))K(x,t)dt$$

and using (2.13) in Lemma 1 gives an explicit form for $\alpha(x)$. Indeed, we obtain

(4.3)

$$\alpha(x) = \frac{(-1)^n(1-x)}{2n(n-1)} - \sum_{k=2}^{n-1}\frac{(2k-1)(n-k)(n+k-1)(-1)^{n+k}\pi_k(x)}{2n(n-1)k(k-1)\lambda_{k,n}}.$$

c) *The polynomials* $r_\nu(x), 2 \le \nu \le n-1$. Since $r_\nu(x)$ is determined by the conditions

$$\text{(4.4)} \qquad \begin{cases} r_\nu(x) = \delta_{\nu j}, & j = 1,\dots,n, \\ r_\nu'(x_j) = r_\nu'(x_n) = 0, \\ r_\nu'''(x_j) = 0, & j = 2,\dots,n-1, \end{cases}$$

we set

$$r_\nu(x) = \frac{1-x^2}{1-x_\nu^2}\ell_\nu(x) + \pi_n(x)\beta_\nu(x), \quad \beta_\nu(x) \in \pi_{n-1}.$$

From $r_\nu'(\pm 1) = 0$, it follows that $\beta_\nu(\pm 1) = 0$. As in case (b), we see that $r_\nu'''(x_j) = 0$, $j = 2,\dots,n-1$, implies that

$$3(1-x_j^2)\beta_\nu''(x_j) - n(n-1)\beta_\nu(x_j) = -\frac{1-x_j^2}{\pi_n'(x_j)}\left(\frac{1-x^2}{1-x_\nu^2}\ell_\nu(x)\right)'''_{x=x_j},$$

$$j = 2, \ldots, n-1.$$

Since $\beta_\nu(\pm 1) = 0$ and since $\beta_\nu(x) \in \pi_{n-1}$, it follows from the above that $\beta_\nu(x)$ satisfies the differential equation

$$3(1-x^2)\beta_\nu''(x) - n(n-1)\beta_\nu(x) = \gamma_\nu(x) \tag{4.5}$$

where $\gamma_\nu(x)$ is a polynomial of degree $n-1$ which satisfies the interpolating conditions

$$\gamma_\nu(\pm 1) = 0$$

$$\gamma_\nu(x_j) = -\frac{1-x_j^2}{\pi_n'(x_j)} + \left(\frac{1-x^2}{1-x_\nu^2}\ell_\nu(x)\right)'''_{x=x_j}, \quad j = 2, \ldots, n-1.$$

Some elementary calculations show that $\gamma_\nu(x)$ can be explicitly given by

(4.6)

$$\gamma_\nu(x) = \frac{n(n-1)}{(1-x_\nu^2)\pi_n'(x_\nu)}\left\{\frac{1-x^2-(1-x_\nu^2)\ell_\nu(x)}{x-x_\nu}\right\} - \frac{1-x^2}{\pi_n'(x_\nu)} \cdot \frac{\{6(1-\ell_\nu(x)) + 2(x-x_\nu)\ell_\nu'(x) + (x-x_\nu)^2\ell_\nu''(x)\}}{(x-x_\nu)^3}$$

From Lemma 1, we now get

$$\beta_\nu(x) = \int_{-1}^{1} \gamma_\nu(t)K(x,t)dt$$

where $K(x,t)$ is given by (2.11) and $\gamma_\nu(t)$ is given by (4.6).

d) *The polynomials* $r_1(x), r_n(x)$. These polynomials are similar to those in (c) above. They also satisfy (4.6) with ν replaced by 1 and n respectively. It is then clear that $r_n(x) = r_1(-x)$. We shall find $r_1(x)$ explicitly. To do so, we set

$$r_1(x) = (1-x)(Ax+B)\ell_1(x) + \pi_n(x)\beta_1(x), \quad \beta_1(x) \in \pi_{n-1}, \tag{4.7}$$

where we choose A and B such that $r_1(x_1) = 1$ and $\beta_1(x_1) = 0$ when $r_1'(x_1) = 0$. Then

$$2(B-A) = 1 \text{ and } 2(B-A)\ell_1'(-1) + (3A-B)\ell_1(-1) = 0.$$

Since $\ell(-1) = 1$ and $\ell_1'(-1) = \frac{1}{2}\frac{\pi_n''(-1)}{\pi'(-1)} = -\frac{n(n-1)}{4}$, it follows easily that

$$A = \frac{n^2 - n + 2}{8}, \qquad B = \frac{n^2 - n + 6}{8}. \tag{4.8}$$

From $r'''(x_j) = 0$, we get

(4.9)

$$3(1 - x_j^2)\beta_1''(x_j) - n(n-1)\beta_1(x_j) = -\frac{1 - x_j^2}{\pi_n'(x_j)}\Lambda'''(x_j), \quad j = 2, \ldots, n-1$$

where $\Lambda(x) = (1 - x)(Ax + B)\ell_1(x)$ and A, B are given by (4.8).

Now

$$\begin{aligned}
&-\frac{1 - x_j^2}{\pi_n'(x_j)}\Lambda'''(x_j) \\
&\quad = -\frac{1 - x_j^2}{\pi_n'(-1)}\left\{\left(-\frac{n(n-1)}{(1 + x_j)^2} + \frac{6(1 - x_j)}{(1 + x_j)^3}\right)(Ax_j + B)\right. \\
&\qquad\qquad \left. - \frac{6}{(1 + x_j)^2}(-Ax_j + 2A - B)\right\}.
\end{aligned}$$

Elementary calculation shows that the unique polynomial $\gamma_1(x)$ which satisfies the conditions

$$\gamma_1(\pm 1) = 0, \quad \gamma_1(x_j) = -\frac{1 - x_j^2}{\pi_n'(x_j)}\Lambda'''(x_j), \quad j = 2, \ldots, n-1,$$

is given by

$$\begin{aligned}
\gamma_1(x) = -\frac{1 - x^2}{\pi_n'(-1)}&\left\{n(n-1)\frac{1 + (1 + x)\ell_1'(-1) - \ell_1(x)}{(1 + x)^2}\right. \\
&\left. + \frac{6\{1 + (1 + x)\ell_1'(-1) + \frac{1}{2}(1 + x)^2\ell_1''(-1) - \ell_1(x)\}}{(1 + x)^3}\right\}.
\end{aligned}$$

Then from (4.9), we see the differential equation for $\beta_1(x)$ to be

$$3(1 - x^2)\beta_1''(x) - n(n-1)\beta_1(x) = \gamma_1(x).$$

By Lemma 1, we then have

$$\beta_1(x) = \int_0^1 K(x,t)\gamma_1(t)dt$$

and $r_1(x)$ is given by (4.7).

5. Proof of Theorem 2. We shall show that if $Q(x) \in \pi_{2n-1}$ and satisfies

$$Q(x_\nu) = 0, \quad Q'''(x_\nu) = 0, \quad \nu = 1,2,\ldots,n, \tag{5.1}$$

then $Q(x)$ is identically zero.

By Theorem 1, there exists a unique polynomial $Q(x) \in \pi_{2n-1}$ such that

$$\begin{aligned} &Q(x_\nu) = 0, \quad \nu = 1,\ldots,n \\ &Q'(-1) = C, \quad Q'(1) = D, \quad C^2 + D^2 \neq 0 \\ &Q'''(x_j) = 0, \quad j = 2,\ldots,n-1. \end{aligned}$$

From the fundamental polynomials of modified $(0,3)$ interpolation we have

$$Q(x) = C\sigma_1(x) + D\sigma_n(x).$$

If we now impose the requirement that $Q'''(-1) = Q'''(1) = 0$, then we get a homogeneous system of two equations whose determinant Δ is given by

$$\Delta = \begin{vmatrix} \sigma_1'''(-1) & \sigma_1'''(-1) \\ \sigma_1'''(1) & \sigma_n'''(1) \end{vmatrix}.$$

Since $\sigma_n(x) = -\sigma_1(-x)$, we have

$$\Delta = -\{\sigma_1'''(1) - \sigma_1'''(-1)\}\{\sigma_1'''(+1) + \sigma_1'''(-1)\}.$$

From the explicit formula for $\sigma_1(x)$ in §3, we have

$$\begin{aligned} &\sigma_1(x) \pm \sigma(-x) \\ &\quad = \pi_n(x)\alpha(x) \pm \pi_n(-x)\alpha(-x) = \pi_n(x)(\alpha(x) \pm (-1)^n\alpha(-x)) \\ &\quad = \frac{(\alpha x + \beta)\pi_n(x)}{2n(n-1)} \\ &\quad - \sum_{k=2}^{n-1} \frac{(2k-1)(n-k)(n+k-1)((-1)^{n+k} \pm 1)\pi_n(x)\pi_k(x)}{2k(k-1)n(n-1)\lambda_{k,n}} \end{aligned}$$

where $\alpha = (-1)^{n+1} \pm 1$, $\beta = (-1)^n \pm 1$ (i.e., $\alpha + \beta = \pm 2$). Since

$$\begin{aligned}
((\alpha x + \beta)\pi_n(x))'''_{x=1} &= (\alpha + \beta)\pi_n'''(1) + 3\alpha\pi_n''(1) \\
&= \pm 2\pi_n'''(1) + 3\alpha\pi_n''(1) \\
(\pi_n(x)\pi_k(x))'''_{x=1} &= 3(\pi_n''(1)\pi_k'(1) + \pi_n'(1) + \pi_k''(1)) \\
&= \frac{3n(n-1)k(k-1)}{2}(n^2 - n + k^2 - k),
\end{aligned}$$

it is easy to check that, when n is even, $\sigma_1'''(1) + \sigma_1'''(-1) < 0$ and $\sigma_1''' - \sigma_1'''(-1) > 0$ and that the same holds when n is odd.

Thus $\Delta \neq 0$ which shows that $C = D = 0$. By Theorem 1 this implies that $Q(x) \equiv 0$, contrary to our hypothesis. This completes the proof of Theorem 2.

6. Fundamental polynomials of the $(0, 3)$ case. The fundamental polynomials for the $(0, 3)$ case will be denoted by $\{r_\nu^*\}_1^n$ and $\{\rho_\nu^*(x)\}_1^n$. They are characterized by theorem properties, viz.,

$$\text{(6.1)} \qquad \begin{cases} r_\nu^*(x_j) = \delta_{\nu_j}, & r_\nu^{*'''}(x_j) = 0, \quad \nu, j = 1, \ldots, n \\ \rho_\nu^*(x_j) = 0 & \rho_\nu^{*'''}(x_j) = \rho_{\nu j}, \quad \nu, j = 1, \ldots, n. \end{cases}$$

It is easy to check that

$$r_\nu^*(x) = \frac{1}{\Delta}\begin{vmatrix} r_\nu(x) & \sigma_1(x) & \sigma_n(x) \\ r_1'''(1) & \sigma_1'''(1) & \sigma_n'''(1) \\ r_1'''(-1) & \sigma_1'''(-1) & \sigma_n'''(-1) \end{vmatrix}, \quad \nu = 1, \ldots, n,$$

$$\rho_\nu^*(x) = \frac{1}{\Delta}\begin{vmatrix} \rho_\nu(x) & \sigma_1(x) & \sigma_n(x) \\ \rho_\nu'''(1) & \sigma_1'''(1) & \sigma_n'''(1) \\ \rho_\nu'''(-1) & \sigma_1'''(-1) & \sigma_n'''(-1) \end{vmatrix}, \quad \nu = 2, \ldots, n-1,$$

where

$$\Delta = (\sigma_1'''(1) - \sigma_1'''(-1))(\sigma_1'''(1) + \sigma_1'''(-1)).$$

The expressions for $\rho_1^*(x)$, $\rho_n^*(x)$ are simpler. Indeed, we have

$$\rho_1^*(x) = -\frac{1}{\Delta}\begin{vmatrix} \sigma_1(x) & \sigma_n(x) \\ \sigma_1'''(1) & \sigma_n'''(1) \end{vmatrix}, \quad \rho_n^*(x) = \frac{1}{\Delta}\begin{vmatrix} \sigma_1(x) & -\sigma_1(-x) \\ \sigma_1'''(-1) & \sigma_1'''(1) \end{vmatrix}.$$

7. Application to quadrature. For a given function $f \in C^3[-1,1]$, we denote by $R_n(f,x)$ the unique polynomial interpolant of modified $(0,3)$ interpolation of $f(x)$ on the zeros of $\pi_n(x)$. Thus

$$(6.1)\qquad \begin{cases} R_n(f,x_\nu) = f(x_\nu), \quad \nu = 1,\cdots,n, \\ R_n'(f,x_1) = f'(x_1), \quad R_n'(f,x_n) = f'(x_n), \\ R_n'''(f,x_j) = f'''(x_j), \quad j = 2,\ldots,n-1. \end{cases}$$

By Theorem 1, we have

$$R_n(f,x) = \sum_{\nu=1}^{n} f(x_\nu)r_\nu(x) + f'(x_1)\sigma_1(x) + f'(x_n)\sigma_n(x) + \sum_{\nu=2}^{n-1} f'''(x_\nu)\rho_\nu(x).$$

Integrating both sides from -1 to 1 we get a quadrature formula, exact for polynomials of degree $2n-1$. On simplifying, it turns out that

$$\begin{aligned}\int_{-1}^{1} f(x)dx =& A_n(f(1)+f(-1)) + B_n \sum_{k=2}^{n-1} \frac{f(x_k)}{P_{n-1}^2(x_k)} \\ &+ C_n(f'(1)-f'(-1)) + D_n \sum_{k=2}^{n-1} \frac{x_k(1-x_k^2)}{P_{n-1}^2(x_k)} f'''(x_k),\end{aligned}$$

where

(6.2)

$$\begin{cases} A_n & = \int_{-1}^{1} r_1(x)dx = \int_{-1}^{1} r_n(x)dx = \frac{8n^2-25n+24}{n(2n-1)(2n^2-8n+9)}, \\ \frac{B_n}{P_{n-1}^2(x_\nu)} & = \int_{-1}^{1} r_\nu(x)dx, \ \nu = 2,\ldots,n-1, \\ & \qquad \text{i.e., } B_n = \frac{4(n-2)(n-3)}{n(2n-1)(2n^2-8n+9)}, \\ C_n & = \int_{-1}^{1} \sigma_1(x)dx = -\int_{-1}^{1} \sigma_n(x)dx = -\frac{1}{(2n-1)(2n^2-8n+9)}, \\ \frac{D_n x_\nu(1-x_\nu^2)}{P_{n-1}^2(x_\nu)} & = \int_{-1}^{1} \rho_\nu(x)dx, \ \nu = 2,\ldots,n-1, \\ D_n & = \frac{1}{n(n-1)(2n-1)(2n^2-8n+9)}. \end{cases}$$

These formulae were obtained by Varma in a very nice simple way without the use of the fundamental polynomials of modified $(0,3)$ interpolation. But he could not obtain the quadrature formula without using $f'(1)$ and $f'(-1)$. But in view of Theorem 2, we can give such a quadrature formula. Indeed, we have

$$\int_{-1}^{1} f(x)dx = \sum_{1}^{n} A_\nu^* f(x_\nu) + \sum_{1}^{n} B_\nu^* f'''(x_\nu),$$

where

$$A_\nu^* = \frac{B_n}{P_{n-1}^2(x_\nu)} - C_n \cdot \frac{r_1'''(1) - r_1'''(-1)}{\sigma_1'''(1) - \sigma_1'''(-1)}, \quad \nu = 1, \ldots, n,$$

and

$$B_\nu^* = \frac{D_n x_\nu (1 - x_\nu^2)}{P_{n-1}^2(x_\nu)} - C_n \cdot \frac{r_1'''(1) - r_1'''(-1)}{\sigma_1'''(1) - \sigma_1'''(-1)}, \quad \nu = 2, \ldots, n-1.$$

Moreover,

$$B_1^* = -B_n^* = -\frac{C_n}{\sigma_1'''(1) - \sigma_1'''(-1)},$$

where A_n, B_n, C_n and D_n are given by (6.2).

It is interesting to note that the method used above can be adapted to derive the fundamental polynomials of $(0, 2, 3)$ interpolation on zeros of $\pi_n(x)$. We propose to return to this later.

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