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DEGREE OF APPROXIMATION BY LACUNARY INTERPOLATORS: $(0, \ldots, R-2, R)$ INTERPOLATION

HEINZ H. GONSKA

DEDICATED TO PROFESSOR AMBIKESHWAR SHARMA

ABSTRACT. We propose a modified general technique to derive quantitative assertions on lacunary interpolation by algebraic polynomials. The inequalities obtained are consequences of a smoothing approach. An application to various classical cases such as modified and non-modified $(0,2)$ and $(0,1,3)$ interpolations yields improvements of earlier results.

1. Introduction. "Lacunary interpolation" describes a certain subset of Hermite-Birkhoff interpolation problems and was first initiated by Surányi and Turán [**10**] in 1955. They used the term to describe their $(0,2)$ interpolation problem; here the values and the second derivatives of a function are prescribed on some special given nodes. Historically, the paper of Surányi and Turán was the starting-point for a large number of articles dealing with related questions. Partial surveys of the corresponding results are given in a paper by Sharma [**9**] and in the recent book by Lorentz, Jetter, and Riemenschneider [**5**, Chapters 11 and 12].

The work of the Hungarian school has to be mentioned in particular. In a series of papers they solved several problems concerning (i) existence and uniqueness, (ii) explicit representation, (iii) uniform convergence, and (iv) applications to Markoff-type inequalities. Their work was followed by numerous papers from around the world in which similar problems were considered. For instance, as early as 1958 Saxena and Sharma started their investigations on $(0,1,3)$ interpolation, and in 1961 the first paper on a modified $(0,2)$ problem with roots different from those used by Surányi and Turán was published by Varma and Sharma [**14**]. All papers mentioned focused on problems (i) through (iv).

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From a *quantitative* point of view, the degree of approximation by such interpolation procedures is of interest. However, with the exception of Vértesi's papers [**15, 16**], not very much appears to have been done in this direction. In the present paper we shall show that the now standard smoothing technique serves this purpose quite well.

2. Problem description and results. As mentioned in the title, we shall deal with the so-called $(0, 1, \ldots, R-2, R)$ case, $R \geq 2$, further assuming that the interpolation problem is regular. A description follows. Let $\mathbf{N}'$ denote an infinite subset of $\mathbf{N}\backslash\{0\}$, and, for $N \in \mathbf{N}'$, let interpolation nodes $x_{i,N}$ be given by

$$1 \geq x_{1,N} > x_{2,N} > \cdots > x_{N-1,N} > x_{N,N} \geq -1.$$

This defines a bi-infinite matrix of nodes, and we make the (strong) assumption that, for each row of this matrix, there is a uniquely determined polynomial p of degree

$$\deg p \leq R - 1 + \sum_{\substack{j=0 \\ j \neq R-1}}^{R} (u_{j,N} - l_{j,N}) =: d(N)$$

satisfying the following $d(N) + 1$ interpolation conditions of the

$(0, 1, \ldots, R-2, R)$-type:

$$p^{(j)}(x_{i,N}) = a_{i,N}^{(j)}, \ 1 \leq l_{j,N} \leq i \leq u_{j,N} \leq N, \ 0 \leq j \leq R, j \neq R-1.$$

In the sequel we shall drop the subscript N in $x_{i,N}$, etc., if it is clear what N is. The introduction of l_j ("lower bound for j-th derivative") and of u_j ("upper ...") allows us to modify the so-called unmodified problem where $l_j = 1$ and $u_j = N$ for $0 \leq j \leq R$, $j \neq R-1$. If, for a fixed value of N, one thinks of the above problem as being described by an incidence matrix $E = (e_{ij})$, then the l_j's and the u_j's serve the purpose of altering the incidence matrix describing the unmodified problem in certain upper and lower rows, respectively. Furthermore, we assume $u_R - l_R \geq 0$ so that this describes indeed a lacunary interpolation problem. One additional assumption is that there should be at least one row of E with at least two non-zero entries in it. This guarantees that $d(N) \geq N$. Under the above assumptions of existence and unicity, there exist fundamental polynomials $A_{i,j}$ in $\Pi_{d(N)}$ (algebraic polynomials of degree $\leq d(N)$) satisfying the conditions

$$A_{i,j}^{(j')}(x_{i'}) = \delta_{i,i'} \cdot \delta_{j,j'},$$

with i, i', j, j' in the proper ranges. Hence p can be written as

$$p(x) = \sum_{\substack{j=0 \\ j \neq R-1}}^{R} \sum_{i=l_j}^{u_j} a_i^{(j)} \cdot A_{i,j}(x), \text{ where } a_i^{(j)} \to \sum_{\substack{j=0 \\ j \neq R-1}}^{R} \sum_{i=l_j}^{u_j} a_i^{(j)} \cdot A_{i,j}(x)$$

describes a linear mapping. Now let $R', 0 \leq R' \leq R, R' \neq R-1$, be fixed and let $f \in C^{R'}[-1,1]$. Then we may choose the $a_i^{(j)}$'s from above as

$$a_i^{(j)} = \begin{cases} f^{(j)}(x_i), & \text{for } 0 \leq j \leq R', j \neq R-1, \text{ and } l_j \leq i \leq u_j, \\ 0, & \text{for } R'+1 \leq j \leq R, j \neq R-1, \text{ and } l_j \leq i \leq u_j. \end{cases}$$

In this case the polynomial p from above is the result of applying a linear operator $L_{R',d(N)} : C^{R'}[-1,1] \to \Pi_{d(N)}$ to the given function f, and can thus be written as

$$p(x) = L_{R',d(N)}(f;x) = \sum_{\substack{j=0 \\ j \neq R-1}}^{R'} \sum_{i=l_j}^{u_j} f^{(j)}(x_i) \cdot A_{i,j}(x).$$

Note that the subscript R' indicates that the derivatives of order $R'+1, \ldots, R-2, R$ are forced to be equal to zero. In particular, for $R' = R$, there is no "zeroing" of derivatives at all. As mentioned above, $d(N) \geq N$ so that, for the special choice $R' = R$, $L_{R,d(N)}(p) = p$ for all $p \in \Pi_N$, i.e., $L_{R,d(N)}$ is idempotent.

In this note we propose a general method to give upper bounds for the distance between $L_{R',d(N)}f$ and f in the case that f is continuously differentiable of order R'. The main result of §3 is the extension of a lemma of Müller which is needed in §4. In that section we describe our modified smoothing approach. The applications of §5 show that our approach, in combination with earlier estimates for the fundamental functions of various $(0, \ldots, R-2, R)$ interpolators, is capable of giving quite elegant quantitative assertions concerning these processes, and of improving some known inequalities.

The constants c figuring in this paper will exclusively depend on the variables explicitly indicated (such as an order of differentiability r or the order s of a modulus of smoothness), but never on the function f, the point x, or the degree of the approximating polynomials. Otherwise, the value of c may vary even within the same line.

3. Smoothing by smooth functions and by polynomials. This section contains two assertions concerning the smoothing of functions by smoother ones. The first result extends a statement of Müller [**6**] and is a further refinement of the so-called Freud-Popov lemma. It constitutes an extension of Müller's result insofar as our set of inequalities contains estimates for *all* derivatives of the smoothing functions. The symbol $||\cdot||$ will always denote the sup norm. Occasionally the norms and moduli of smoothness will carry subscripts to indicate the interval over which they are taken.

LEMMA 3.1. *Let $I = [0,1]$ and $f \in C^r(I)$, $r \in \mathbf{N}_0$. For any $h \in (0,1]$ and $s \in \mathbf{N}$ there exists a function $f_{h,r+s} \in C^{2r+s}(I)$ with*

(i) $||f^{(j)} - f^{(j)}_{h,r+s}|| \leq c \cdot \omega_{r+s}(f^{(j)}, h)$, for $0 \leq j \leq r$,

(ii) $||f^{(j)}_{h,r+s}|| \leq c \cdot h^{-j} \cdot \omega_j(f, h)$, for $0 \leq j \leq r+s$,

(iii) $||f^{(j)}_{h,r+s}|| \leq c \cdot h^{-(r+s)} \cdot \omega_{r+s}(f^{(j-r-s)}, h)$, for $r+s \leq j \leq 2r+s$.

Here, the constant c depends only on r and s.

PROOF. The functions $f_{h,r+s}$ are the same as in Müller's paper. Therefore we skip details and only recall that one has the representation

$$(3.1) \qquad f_{h,r+s}(x) = \sum_{i=1}^{r+s} \binom{r+s}{i} (-1)^{i+1} (ih)^{-(r+s)} \cdot \Delta_{ih}^{r+s}(T_{r+s}; x)$$

for all $x \in I$. Here, $\Delta_\varepsilon^k(f;x), k \in \mathbf{N}, \varepsilon > 0$, denotes the k-th forward difference of the function f with step size ε at the point x. Furthermore, T is the linear Whitney extension operator mapping $C(I)$ into $C(J)$, $J = [0, 1+(r+s)^2]$, such that

$$(3.2) \qquad \omega_{r+s}((Tf)^{(j)}; h)_J \leq c \cdot \omega_{r+s}(f^{(j)}; h)_I, \quad 0 \leq j \leq r,$$

for $f \in C^r(I)$ (see Johnen [**4**], Müller [**6**]), and T_{r+s} denotes an

$(r+s)$-th primitive to Tf.

Note that for the proofs of (i) through (iii) it is convenient to consider first a sufficiently small h and then to extend the statement to all $h \in (0,1]$.

(i) was already shown in Müller's paper [**6**]. To obtain (ii), differentiation gives

$$f_{h,r+s}^{(j)}(x) = \sum_{i=1}^{r+s} \binom{r+s}{i} (-1)^{i+1} (ih)^{-(r+s)} \cdot \Delta_{ih}^{r+s}(T_{r+s-j}; x),$$
$$0 \le j \le r+s.$$

Thus

$$\begin{aligned} |f_{h,r+s}^{(j)}(x)| &\le \sum_{i=1}^{r+s} \binom{r+s}{i} \cdot h^{-(r+s)} \cdot \max_{1 \le i \le r+s} |\Delta_{ih}^{r+s}(T_{r+s-j}; x)| \\ &\le (2^{r+s}-1) \cdot h^{-(r+s)} \cdot \omega_{r+s}(T_{r+s-j}; (r+s)\cdot h)_J \\ &\le c_{r,s} \cdot h^{-(r+s)} \cdot h^{r+s-j} \cdot \omega_{r+s-(r+s-j)}(T_{r+s-j}^{(r+s-j)}; h)_J \\ &= c_{r,s} \cdot h^{-j} \cdot \omega_j(Tf; h)_J, \quad \text{for } 0 \le j \le r+s. \end{aligned}$$

To complete the proof of (ii), we need estimates for $\omega_j(Tf;h)_J$. To this end observe that, for $0 \le j \le r+s$ and $g \in C^j(I)$, one has

$$||(Tg)^{(j)}||_J \le c \cdot ||g^{(j)}||_I.$$

This is a consequence of the construction of T as given in Johnen's paper. Hence in view of the linearity of T and of certain elementary properties of ω_j we may write

$$\begin{aligned} \omega_j(Tf;h)_J &\le \omega_j(T(f-g);h)_J + \omega_j(Tg;h)_J \\ &\le 2^j \cdot ||T(f-g)||_J + h^j \cdot ||(Tg)^{(j)}||_J \\ &\le c \cdot (||f-g||_I + h^j \cdot ||g^{(j)}||_I). \end{aligned}$$

This implies

$$\omega_j(Tf;h)_J \le c \cdot \inf\{||f-g||_I + h^j ||g^{(j)}||_I : g \in C^j(I)\} \le c \cdot \omega_j(f;h)_I.$$

(We note that the last inequality can be proved *without* using the extension technique, see DeVore [**2**]). Hence we have, for $0 \le j \le r+s$,

$$||f_{h,r+s}^{(j)}||_I \le c_{r,s} \cdots h^{-j} \cdot \omega_j(f;h)_I.$$

For the proof of (iii) let $r + s \leq j \leq 2r + s$. Using (3.1) again it is seen that

$$f_{h,r+s}^{(j)}(x) = \sum_{i=1}^{r+s} \binom{r+s}{i} (-1)^{i+1} (ih)^{-(r+s)} \cdot \Delta_{ih}^{r+s}((Tf)^{(j-r-s)}; x),$$

hence

$$\begin{aligned}|f_{h,r+s}^{(j)}(x)| &\leq \sum_{i=1}^{r+s} \binom{r+s}{i} \cdot h^{-(r+s)} \cdot \max_{1\leq i\leq r+s} |\Delta_{ih}^{r+s}((Tf)^{(j-r-s)}; x)| \\ &\leq (2^{r+s} - 1) \cdot h^{-(r+s)} \cdot \omega_{r+s}((Tf)^{(j-r-s)}, (r+s)h) \\ &\leq c_{r,s} \cdot h^{-(r+s)} \cdot \omega_{r+s}((Tf)^{(j-r-s)}, h).\end{aligned}$$

Because $0 \leq j - r - s \leq r$, we can use inequality (3.2) again to arrive at

$$|f_{h,r+s}^{(j)}(x)| \leq c \cdot h^{-(r+s)} \cdot \omega_{r+s}(f^{(j-r-s)}, h).$$

□

REMARK 3.2. Note that the statement of Lemma 3.1 can be carried over to any finite interval $[a, b]$ by using the suitable linear transformation, and that the impact of this transformation will only be on the constant c figuring in the lemma.

We need also the following result concerning the degree of simultaneous approximation of continuously differentiable functions by polynomials in Π_n.

THEOREM 3.3. (TRIGUB [**11**, LEMMA 1]) *Let $r \geq 0$ and $n \geq r$. Then there exists a linear operator $Q_n = Q_{n,r} : C^r[-1,1] \to \Pi_n$ such that, for all $f \in C^r[-1,1]$, all $|x| \leq 1$ and $0 \leq k \leq r$, one has*

$$|(Q_n f - f)^{(k)}(x) \leq c_r \cdot \Delta_n(x)^{r-k} \cdot ||f^{(r)}||_{[-1,1]}.$$

Here $\Delta_n(x) = \sqrt{1-x^2} \cdot n^{-1} + n^{-2}$, and the constant c_r depends only on r.

4. A general inequality on approximation by lacunary interpolators. We shall first outline the approach used in earlier investigations concerning the convergence properties of $L_{R',d(N)}$. Note that in the sequel $|| \cdot ||$ will always denote the sup norm on $[-1,1]$.

THEOREM 4.1. *For any $f \in C^{R'}[-1,1]$ and any $\Phi \in \Pi_{d(N)}$ there holds*

$$||L_{R',d(N)}f - f|| \leq \sum_{\substack{j=0 \\ j\neq R-1}}^{R'} ||f^{(j)} - \Phi^{(j)}|| \cdot \sum_{i=l_j}^{u_j} ||A_{i,j}|| + \sum_{\substack{j=R'+1 \\ j\neq R-1}}^{R} ||\Phi^{(j)}|| \cdot \sum_{i=l_j}^{u_j} ||A_{i,j}|| + ||\Phi - f||.$$

Note that the second sum may be empty (equal to 0).

PROOF. The proof is obtained by "polynomial smoothing":

$$\begin{aligned}|L_{R',d(N)}f(x) - f(x)| &= |L_{R',d(N)}(f,x) - L_{R,d(N)}(\Phi,x) + \Phi(x) - f(x)| \\ &\leq \sum_{\substack{j=0 \\ j\neq R-1}}^{R'} \sum_{i=l_j}^{u_j} |f^{(j)}(x_i) - \Phi^{(j)}(x_i)| \cdot |A_{i,j}(x)| \\ &\quad + \sum_{\substack{j=R'+1 \\ j\neq R-1}}^{R} \sum_{i=l_j}^{u_j} |\Phi^{(j)}(x_i)| \cdot |A_{i,j}(x)| + |\Phi(x) - f(x)|.\end{aligned}$$

Passing to the sup norm yields Theorem 4.1. □

Note that it is essential for the proof of Theorem 4.1 that, for each $\Phi \in \Pi_{d(N)}$, one has $L_{R,d(N)}\Phi = \Phi$. Furthermore the problem of estimating $||L_{R',d(N)}f - f||$ is reduced to finding polynomials Φ with "good" derivatives and to giving bounds for the fundamental polynomials $A_{i,j}$. While our approach below is similar, it will be different in the sense that we shall make essential use of the smoothing functions $f_{h,r+s}$ from Lemma 3.1. This allows us to derive an upper bound which involves various moduli of smoothness of f (with extra flexibility gained through the use of the undetermined value of h). As far as employing Jackson-type inequalities on simultaneous approximation is concerned, the use of the above version of Trigub's result will suffice.

THEOREM 4.2. *Let $f \in C^{R'}[-1,1]$ and $L_{R',d(N)}$ be given as above. Then we have, for $x \in [-1,1], 0 < h \leq 1, s \geq \max\{R - R', 1\}$ and*

$N \geq R' + s$, that

$$
\begin{aligned}
&|L_{R',d(N)}(f;x) - f(x)| \\
&\quad \leq c \cdot \Big[\sum_{\substack{j=0 \\ j \neq R-1}}^{R'} \{h^{R'-j} \cdot \omega_{s+j}(f^{(R')}, h) \\
&\qquad\qquad + N^{-R'-s+j} \cdot h^{-s} \cdot \omega_s(f^{(R')}, h)\} \cdot \sum_{i=l_j}^{u_j} |A_{i,j}(x)| \\
&\quad + \sum_{\substack{j=R'+1 \\ j \neq R-1}}^{R} \{N^{-R'-s+j} \cdot h^{-s} \cdot \omega_s(f^{(R')}, h) \\
&\qquad\qquad + h^{R'-j} \cdot \omega_{j-R'}(f^{(R')}, h)\} \cdot \sum_{i=l_j}^{u_j} |A_{i,j}(x)| \\
&\quad + \{h^{R'} + N^{-R'-s} \cdot h^{-s}\} \cdot \omega_s(f^{(R')}, h)\Big].
\end{aligned}
$$

The constant c depends only on R' and s.

PROOF. For $R' \geq 0, s \geq 1$ and $0 < h \leq 1$ we choose functions $f_h = f_{h,R'+s} \in C^{R'+s}[-1,1]$ according to Lemma 3.1, that is,

$$
\begin{aligned}
&||f^{(j)} - f_h^{(j)}|| \leq c \cdot \omega_{R'+s}(f^{(j)}, h) \ \text{ for } 0 \leq j \leq R', \\
&||f_h^{(j)}|| \leq c \cdot h^{-j} \cdot \omega_j(f, h) \ \text{ for } 0 \leq j \leq R' + s.
\end{aligned}
$$

As a further tool we use the operators $Q_N = Q_{N,R'+s}$ from Theorem 3.3 so that

$$
|(Q_N g - g)^{(j)}(x)| \leq c \cdot N^{-R'-s+j} \cdot ||g^{(R'+s)}||, \qquad 0 \leq j \leq R' + s,
$$

for all $g \in C^{R'+s}[-1,1]$ and $N \geq R' + s$. Because $d(N) \geq N \geq R' + s$ we obtain from Theorem 4.1 with $\Phi = Q_n f_h = Q_n f_{h,R'+s}$ that

$$
\begin{aligned}
&|L_{R',d(N)}(f,x) - f(x)| \\
&\quad \leq \sum_{\substack{j=0 \\ j \neq R-1}}^{R'} ||f^{(j)} - (Q_N f_h)^{(j)}|| \cdot \sum_{i=l_j}^{u_j} ||A_{i,j}|| \\
&\qquad + \sum_{\substack{j=R'+1 \\ j \neq R-1}}^{R} ||(Q_N f_h)^{(j)}|| \cdot \sum_{i=l_j}^{u_j} ||A_{i,j}|| + ||Q_N f_h - f||.
\end{aligned}
$$

To investigate $||f^{(j)} - (Q_N f_h)^{(j)}||$, $0 \le j \le R'$, we write

$$\begin{aligned}
&||f^{(j)} - (Q_N f_h)^{(j)}|| \\
&\le ||f^{(j)} - f_h^{(j)}|| + ||f_h^{(j)} - (Q_N f_h)^{(j)}|| \\
&\le c \cdot \omega_{R'+s}(f^{(j)}, h) + c \cdot N^{-(R'+s-j)} \cdot ||f_h^{(R'+s)}|| \\
&\le c \cdot \omega_{R'+s}(f^{(j)}, h) + c \cdot N^{-(R'+s-j)} \cdot h^{-(R'+s)} \cdot \omega_{R'+s}(f, h) \\
&\le c \cdot h^{R'-j} \cdot \omega_{R'+s-R'+j}(f^{(R')}, h) \\
&\quad + c \cdot N^{-(R'+s-j)} \cdot h^{-(R'+s)} \cdot h^{R'} \cdot \omega_{R'+s-R'}(f^{(R')}, h) \\
&\le c \cdot \{h^{R'-j} \cdot \omega_{s+j}(f^{(R')}, h) + N^{-(R'+s-j)} \cdot h^{-s} \cdot \omega_s(f^{(R')}, h)\}.
\end{aligned}$$

In particular, for $j = 0$, this reads

$$||f - Q_N f_h|| \le c \cdot \{h^{R'} \cdot \omega_s(f^{(R')}, h) + N^{-(R'+s)} \cdot h^{-s} \cdot \omega_s(f^{(R')}, h)\}.$$

We also need estimates for $||(Q_N f_h)^{(j)}||$, for $R' + 1 \le j \le R \le R' + s$.

$$\begin{aligned}
||(Q_N f_h)^{(j)}|| &\le ||(Q_N f_h)^{(j)} - f_h^{(j)}|| + ||f_h^{(j)}|| \\
&\le c \cdot N^{-(R'+s-j)} \cdot ||f_h^{(R'+s)}|| + c \cdot h^{-j} \cdot \omega_j(f, h) \\
&\le c \cdot N^{-(R'+s-j)} \cdot h^{-(R'+s)} \cdot \omega_{R'+s}(f, h) + c \cdot h^{-j} \cdot \omega_j(f, h) \\
&\le c \cdot N^{-(R'+s-j)} \cdot h^{-(R'+s)} \cdot h^{R'} \cdot \omega_{R'+s-R'}(f^{(R')}, h) \\
&\qquad + c \cdot h^{-j} \cdot h^{R'} \cdot \omega_{j-R'}(f^{(R')}, h) \\
&\le c \cdot \{N^{-(R'+s-j)} \cdot h^{-s} \cdot \omega_s(f^{(R')}, h) \\
&\qquad + h^{R'-j} \cdot \omega_{j-R'}(f^{(R')}, h)\}.
\end{aligned}$$

Note that, in the above, we have made use of the assumption that $R \le R' + s$, i.e., $R - R' \le s$. Combining the above observations yields the inequality of the theorem. □

COROLLARY 4.3. *Let $f \in C^{R'}[-1, 1]$ and $L_{R'.d(N)}$ be given as above.*

For the choice $h = N^{-1}$ the inequality of Theorem 4.2 simplifies to

$$
\begin{aligned}
&|L_{R',d(N)}(f;x) - f(x)| \\
&\quad \leq c \cdot \Big[\sum_{\substack{j=0 \\ j\neq R-1}}^{R'} N^{-R'+j} \cdot \omega_s(f^{(R')}, N^{-1}) \cdot \sum_{i=l_j}^{u_j} |A_{i,j}(x)| \\
&\qquad + \sum_{\substack{j=R'+1 \\ j\neq R-1}}^{R} N^{-R'+j} \cdot \{\omega_s(f^{(R')}, N^{-1}) \\
&\qquad + \omega_{j-R'}(f^{(R')}, N^{-1})\} \cdot \sum_{i=l_j}^{u_j} |A_{i,j}(x)| \\
&\qquad + N^{-R'} \cdot \omega_s(f^{(R')}, N^{-1})\Big].
\end{aligned}
$$

A critical term in Theorem 4.2 and in Corollary 4.3 is the one involving $\omega_{j-R'}(f^{(R')}, \dots)$, $j = R'+1, \dots, R$; $j \neq R-1$. However, if $j - R' \geq s$ for all j's in question, then this quantity is bounded from above by $c \cdot \omega_s(f^{(R')}, \dots)$, and our upper bound is solely given in terms of $\omega_s(f^{(R')}, \dots)$. Note further that the critical term is not present at all if $R' = R$. Facts of this type are summarized in

COROLLARY 4.4. *Let $f \in C^{R'}[-1,1], L_{R',d(N)}$ and h be given as in Corollary* 4.3.

(i) *For $R' = R-2$, any $s \geq 2$, and all $N \geq R-2+s$ we have*

$$
\begin{aligned}
&|L_{R',d(N)}(f;x) - f(x)| \\
&\quad \leq c \cdot \Big[\sum_{\substack{j=0 \\ j\neq R-1}}^{R-2} N^{-R+2+j} \cdot \omega_s(f^{(R-2)}, N^{-1}) \cdot \sum_{i=l_j}^{u_j} |A_{i,j}(x)| \\
&\qquad + N^2 \cdot \{\omega_s(f^{(R-2)}, N^{-1}) + \omega_2(f^{(R-2)}, N^{-1})\} \cdot \sum_{i=l_R}^{u_R} |A_{i,R}(x)| \\
&\qquad + N^{-R+2} \cdot \omega_s(f^{(R-2)}, N^{-1})\Big].
\end{aligned}
$$

(ii) *For $R' = R$, an arbitrary $s \geq 1$, and for all $N \geq R + s$ there holds*

$$\begin{aligned}|L_{R,d(N)}(f;x) - f(x)| \\ \leq c \cdot \Big[\sum_{\substack{j=0 \\ j \neq R-1}}^{R} N^{-R+j} \cdot \omega_s(f^{(R)}, N^{-1}) \cdot \sum_{i=l_j}^{u_j} |A_{i,j}(x)| \\ + N^{-R} \cdot \omega_s(f^{(R)}, N^{-1}) \Big].\end{aligned}$$

In both cases the constant c depends only on R' and s.

5. Examples. Here we consider various classical cases which have been investigated before; however, mostly from a non-quantitative point of view.

5.1. *The case $R = 2$.* This is the most classical one which was first investigated by Surányi and Turán [**10**] (see, for example, also Balázs and Turán [**1**]). Choose $x_i, 1 \leq i \leq N$, to be the zeros of $(1 - x^2) \cdot P'_{n-1}(x), n$ even, where P_{n-1} is the Legendre polynomial of degree $n - 1$. Hence $N = n$ and, for the *non-modified* ("pure"), case one has $l_0 = l_2 = 1, u_0 = u_2 = n$. Hence, $d(N) = 2n - 1$. The following lemma is essential.

LEMMA 5.1. (BÁZS and TURÁN [**1**]) *For the fundamental polynomials $A_{i,j}$ of pure $(0,2)$ interpolation at the roots of $(1 - x^2) \cdot P'_{n-1}(x)$ one has, for $n = 4, 6, 8, \ldots$,*

(i) $\sum_{i=1}^{n} ||A_{i,0}|| \leq 249 \cdot \pi \cdot n$,

(ii) $\sum_{i=1}^{n} ||A_{i,2}|| \leq 38 \cdot \pi \cdot n^{-1}$.

For the case $R' = 0$ the following proposition is an immediate consequence of Corollary 4.4 (i) and Lemma 5.1. It is a full quantitative version of a result of Freud [**3**]. Note that the function $\varepsilon(h)$ figuring there can attain at most the value $\omega_2(f, h)$. The theorem below also improves a quantitative version of Freud's result due to Vértesi [**15, 16**]. Although formally his inequality is the same as ours, he needs the somewhat unnatural condition (A_2). That this is superfluous is shown in

PROPOSITION 5.2. *For $f \in C[-1,1]$ and $n = 4, 6, 8, \ldots$, there holds*

$$||L_{0,2n-1}f - f|| \leq c \cdot n \cdot \omega_2(f, n^{-1}),$$

where the constant c does not depend on n and f.

PROOF. Corollary 4.4 (i) shows that, for an arbitrary $s \geq 2$ and all even $n \geq \max\{4, s\}$, we obtain an inequality involving $\omega_s(f, \cdot)$ and $\omega_2(f, \cdot)$. In view of the relationship $\omega_s(f, n^{-1}) \leq c \cdot \omega_2(f, n^{-1}), s \geq 2$, we arrive at the above result. □

For the choice $R' = R = 2$ the situation is somewhat different.

PROPOSITION 5.3. *For $f \in C^2[-1,1]$, $s \geq 1$ and even $n \geq \max\{4, s+2\}$, we have, with a constant c neither depending on n nor on f, that*

$$||L_{2,2n-1}f - f|| \leq c \cdot n^{-1} \cdot \omega_s(f'', n^{-1}).$$

The proof is a consequence of Corollary 4.4 (ii) and of Lemma 5.1.

REMARK 5.4. Proposition 5.3 implies that, for any $f \in C^2[-1,1]$, the polynomials $L_{2,2n-1}f$ converge uniformly to f (cf. Lorentz-Jetter-Riemenschneider [**5**, Corollary 12.15]). However, due to the arbitrary choice of $s \geq 2$, it also expresses the fact that, for any $f \in C^k[-1,1]$, $k \geq 2$, the degree of uniform convergence is $0(n^{-k+1}), n \to \infty$.

A *modified* case of the $(0,2)$ problem was studied by Varma and Sharma [**12, 13, 14**]. Here x_i, $1 \leq i \leq N$, are the zeros of $(1-x^2) \cdot T_n(x)$ (T_n denoting the n-th Čebyšev polynomial of the first kind), and $l_0 = 1, l_2 = 2, u_0 = n+2, u_2 = n+1$, hence $N = n+2$ and $d(N) = 2n+1$. Again, for even $n \in \mathbf{N}$, the problem is regular, and the following holds.

LEMMA 5.5. (VARMA [**12, 13**]). *The fundamental polynomials of modified $(0,2)$ interpolation at the roots of $(1-x^2) \cdot T_n(x), n$ even, satisfy the inequalities*

(i) $\sum_{i=1}^{n+2} ||A_{i,0}|| \leq 140 \cdot n^{3/2}$,

(ii) $\sum_{i=2}^{n+1} ||A_{i,2}|| \leq 45 \cdot n^{-1/2}$.

For the case $R' = 0$, Corollary 4.4 (i) yields the following quantitative version of a result of Varma [**12**, Theorem 3.1]. Note that, for $f' \in \operatorname{Lip} \alpha, \alpha > \frac{1}{2}$, one has $\omega_2(f, n^{-1}) \leq n^{-1} \cdot \omega_1(f', n^{-1}) = o(n^{-3/2})$.

PROPOSITION 5.6. *Let $f \in C[-1,1], n \geq 2$, be even, and let $L_{0,2n+1}$ denote the lacunary interpolation operator corresponding to the modified problem described above. Then*

$$||L_{0,2n+1}f - f|| \leq c \cdot n^{3/2} \cdot \omega_2(f, n^{-1})$$

with c independent of n and f.

For $R' = R = 2$ we have

PROPOSITION 5.7. *Let $s \geq 2$. There is a constant c depending only on s such that, for all even $n \geq s$ and all $f \in C^2[-1,1]$, one has*

$$||L_{2,2n+1}f - f|| \leq c \cdot n^{-1/2} \cdot \omega_s(f'', n^{-1}).$$

As was the case in Proposition 5.3, this assertion shows that for the operators $L_{2,2n+1}$ no saturation occurs.

5.2. *The case $R = 3$.* We consider the non-modified ("pure") $(0,1,3)$ interpolation at the roots of $(1 - x^2) \cdot P'_{n-1}(x)$. Here $N = n, l_0 = l_1 = l_3 = 1$ and $u_0 = u_1 = u_3 = n$ so that $d(N) = 3n - 1$. This problem has also a certain tradition: It was already considered in 1958 by Saxena and Sharma [**7**] (see also Saxena and Sharma [**8**], and Vértesi [**15**]). Estimates for the fundamental polynomials of this interpolation process are given in

LEMMA 5.8. (SAXENA and SHARMA [**8**]). *For the fundamental polynomials of pure $(0,1,3)$ interpolation at the roots of $(1 - x^2)P'_{n-1}(x)$, one has, for $n = 4, 6, 8, \ldots$, the inequalities*

(i) $\sum_{i=1}^{n} ||A_{i,0}|| \leq 10^5 \cdot n$,

(ii) $\sum_{i=1}^{n} ||A_{i,1}|| \leq 2333$,

(iii) $\sum_{i=1}^{n} ||A_{i,3}|| \leq 71\pi \cdot n^{-2}$.

For the case $R' = 0$, an application of Corollary 4.3 only yields an assertion which implies uniform convergence for constant functions.

For the case $R' = 1$, Corollary 4.4 (i) leads to

PROPOSITION 5.9. *Let $f \in C^1[-1,1]$. Then, for all even $n \geq 4$, the $(0,1,3)$ interpolation operator $L_{1,3n-1}$ corresponding to the zeros of $(1-x^2)P'_{n-1}(x)$ satisfies the inequality*

$$||L_{1,3n-1}f - f|| \leq c \cdot \omega_2(f', n^{-1}).$$

We note that the latter inequality implies uniform convergence for any $f \in C^1[-1,1]$ and hence contains the main result in [**8**]. Furthermore, the estimate also improves Vértesi's Theorem 3.2 in [**15**], where the upper bound is of the form $O(n \cdot \omega_2(f', n^{-1}))$ and an extra condition for $\omega_2(f', \dots)$ is needed.

Again, if we do not force derivatives to be equal to zero, we obtain, for $R' = R = 3$:

PROPOSITION 5.10. *There is a constant c depending only on $s \geq 1$ such that, for all $f \in C^3[-1,1]$ and all even $n \geq \max\{4, s+3\}$, there holds*

$$||L_{3,3n-1}f - f|| \leq c \cdot n^{-2} \cdot \omega_s(f^{(3)}, n^{-1}).$$

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DREXEL UNIVERSITY, PHILADELPHIA, PA 19104

FACHBEREICH MATHEMATIK, UNIVERSITÄT DUISBURG, D-4100 DUISBURG 1, WEST GERMANY