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ON AN ERROR TERM OF LANDAU - II

R. SITARAMACHANDRARAO

In memory of Ernst Straus and Robert A. Smith

ABSTRACT. Let $\phi(n)$ denote the Euler totient function. In 1900, E. Landau proved that $\sum_{n\leq x} 1/\phi(n) = A(\log x + B) + E_0(x)$ where $A > 0$ and B are constants and $E_0(x) = O(\log x/x)$. In an earlier paper, we sharpened this result and in the present paper prove, in particular, some average and Ω-type theorems for the error function $E_0(x)$.

1. Introduction. Let $\phi(n)$ denote the Euler totient function defined to be the number of positive integers $\leqq n$ and prime to n. We write

$$\sum_{n\leqq x} 1/\phi(n) = A(\log x + B) + E_0(x) \tag{1.1}$$

and

$$\sum_{n\leqq x} n/\phi(n) = Ax - \frac{1}{2}\log x + E_1(x) \tag{1.2}$$

where

$$A = \frac{315\zeta(3)}{2\pi^4} \text{ and } B = \gamma - \sum_p \frac{\log p}{p^2 - p + 1}. \tag{1.3}$$

Here ζ denotes the Riemann zeta function, γ denotes the Euler-Mascheroni constant and the sum on the extreme right of (1.3) extends over all primes p. In 1900, E. Landau (cf. [3, p. 184]) proved that, as $x \to \infty$,

$$E_0(x) = 0\,(\log x/x). \tag{1.4}$$

A systematic study of the error function $E_0(x)$ does not appear to have been made by later authors. In an earlier paper [5], using a theorem of A. Walfisz based on Weyl's inequality, we were able to improve upon (1.4) by proving that, as $x \to \infty$,

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(1.5) $$E_0(x) = 0\,((\log x)^{2/3}/x).$$

In the present paper, we continue our study and consider the averages of the error function $E_0(x)$ and also of the closely related function $E_1(x)$. In fact, we prove the following equations:

(1.6) $$\sum_{n \leqq x} E_0(n) = \frac{(A+1)}{2} \log x + 0(1);$$

(1.7) $$\int_1^x E_0(t)dt = \frac{1}{2} \log x + 0(1);$$

(1.8) $$\sum_{n \leqq x} E_1(n) = \frac{(A-D)}{2} x + 0(x^{4/5});$$

and

(1.9) $$\int_1^x E_1(t)dt = -\frac{D}{2} x + 0(x^{4/5}),$$

where A is given by (1.3) and;

(1.10) $$D = \gamma + \log(2\pi) + \sum_p \frac{\log p}{p(p-1)}.$$

We also refer to Remark 2.1 for some Ω-results.

Throughout the paper, the constants implied by the symbols 0 and $\ll$ are absolute.

2. Preliminaries. Let μ denote the Möbius function, ψ be the Dedekind's ψ-function defined by $\psi(n) = \sum_{d\delta=n} \mu^2(d)\delta$ and $\theta(n) = \sum_{d\delta=n} \mu^2(d)$. Clearly

$$\psi(n) = n \prod_{p \mid n} \left(1 + \frac{1}{p}\right) \text{ and } \theta(n) = 2^{\omega(n)}$$

where the product extends over all distinct prime factors of n and $\omega(n)$ denotes the number of distinct prime divisors of n. Let $[x]$ denote the largest integer $\leqq x$, $\{x\} = x - [x]$ be the fractional part of x and $P(x) = \{x\} - 1/2$.

LEMMA 2.1. (cf. [3], Lemma). *Let $f(n)$ be an arithmetical function and suppose $\sum_{n \leqq x} f(n) = g(x) + E(x)$ where g is twice continuously differentiable and $g''(x)$ is of constant sign for $x \geqq 1$. Then*

$$\sum_{n \leqq x} E(n) = \frac{1}{2} g(x) + \left(\frac{1}{2} - P(x)\right) E(x) + \int_1^x E(t)dt + 0(|g'(x)| + 1).$$

LEMMA 2.2. (cf. [1], Lemma 5.2).

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \mu^2(m) = \frac{nx}{\zeta(2)\psi(n)} + 0(\theta(n)\, x^{1/2}).$$

LEMMA 2.3.

$$\sum_{n\leq x}\frac{\mu^2(n)n}{\phi(n)} = x + 0(x^{1/2}); \tag{2.1}$$

$$\sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)} = \log x + \gamma + \sum_{p}\frac{\log p}{p(p-1)} + 0\left(\frac{1}{\sqrt{x}}\right);\text{ and} \tag{2.2}$$

$$\sum_{n>x}\frac{\mu^2(n)}{n\,\phi(n)} = \frac{1}{x} + 0\left(\frac{1}{x^{3/2}}\right). \tag{2.3}$$

PROOF. Since

$$\frac{n}{\phi(n)} = \prod_{p\mid n}\left(1-\frac{1}{p}\right)^{-1} = \prod_{n\mid p}\left(1+\frac{1}{p-1}\right) = \sum_{d\delta=n}\frac{\mu^2(d)}{\phi(d)}, \tag{2.4}$$

we have by Lemma 2.2

$$\begin{aligned}
\sum_{n\leq x}\frac{\mu^2(n)n}{\phi(n)} &= \sum_{n\leq x}\mu^2(n)\sum_{d\delta=n}\frac{\mu^2(d)}{\phi(d)} = \sum_{d\delta\leq x,\ (d,\delta)=1}\frac{\mu^2(d)\mu^2(\delta)}{\phi(d)}\\
&= \sum_{d\leq x}\frac{\mu^2(d)}{\phi(d)}\sum_{\substack{\delta\leq x/d\\(\delta,d)=1}}\mu^2(\delta)\\
&= \sum_{d\leq x}\frac{\mu^2(d)}{\phi(d)}\left\{\frac{x}{\zeta(2)\,\psi(d)} + 0\left(\theta(d)\left(\frac{x}{d}\right)^{1/2}\right)\right\} \qquad (2.5)\\
&= \frac{x}{\zeta(2)}\sum_{n=1}^{\infty}\frac{\mu^2(n)}{\phi(n)\psi(n)} + 0\left(x\sum_{n>x}\frac{1}{\phi(n)\psi(n)}\right)\\
&\quad + 0\left(x^{1/2}\sum_{n\leq x}\frac{\theta(n)}{n^{1/2}\phi(n)}\right).
\end{aligned}$$

Now, by Euler's infinite product factorization theorem (cf. [**2**, Theorem 264]),

$$\sum_{n=1}^{\infty}\frac{\mu^2(n)}{\phi(n)\psi(n)} = \prod_{p}\left(1+\frac{1}{p^2-1}\right) = \prod_{p}\left(1-\frac{1}{p^2}\right)^{-1} = \zeta(2).$$

Also since $\phi(n)\psi(n) = n^2\prod_{p\mid n}(1-(1/p^2) > n^2\prod_p(1-1/p^2)$, the first 0-term on the right of (2.5) is $0(x\sum_{n>x}1/n^2) = 0(1)$. Further, since for each $\varepsilon > 0$, $\theta(n) \ll_\varepsilon n^\varepsilon$ and $n^{1-\varepsilon} \ll \phi(n)$ as $n\to\infty$, we see that the series $\sum_{n=1}^{\infty}(\theta(n))/(n^{1/2}\phi(n))$ converges. Thus (2.1) follows from (2.5).

To prove (2.2), we write

$$\sum_{n\leq x}\frac{\mu^2(n)n}{\phi(n)} = x + \Delta(x).$$

Then, by the theorem of partial summation,

$$\sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)} = \frac{x+\Delta(x)}{x} + \int_1^x \frac{(t+\Delta(t))}{t^2}\,dt$$
$$= \log x + 1 + \int_1^\infty \frac{\Delta(t)}{t^2}\,dt + 0\left(\frac{1}{\sqrt{x}}\right)$$

since $\Delta(x) = 0(x^{1/2})$ by (2.1). Thus (2.2) follows if we prove that

$$\int_1^\infty \frac{\Delta(t)}{t^2}\,dt = -1 + \gamma + \sum_p \frac{\log p}{p(p-1)}.$$

To see this, we have, for complex s with $\operatorname{Re} s > 1$, by partial summation

$$\sum_{n=1}^\infty \frac{\mu^2(n)n}{\phi(n)}\cdot\frac{1}{n^s} = \frac{s}{s-1} + s\int_1^\infty \frac{\Delta(t)}{t^{s+1}}\,dt.$$

But, by Euler's infinite product factorization theorem for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^\infty \frac{\mu^2(n)n}{\phi(n)}\cdot\frac{1}{n^s} = \prod_p\left\{1 + \frac{1}{(p-1)p^{s-1}}\right\}.$$

Thus on comparing the above two, we obtain for $\operatorname{Re} s > 1$

$$\text{(2.6)}\qquad \begin{aligned}\int_1^\infty \frac{\Delta(t)}{t^{s+1}}\,dt &= -\frac{1}{s-1} + \frac{1}{s}\prod_p\left\{1+\frac{1}{(p-1)p^{s-1}}\right\}\\ &= -\frac{1}{s-1} + \frac{1}{s}\zeta(s)\,h(s),\end{aligned}$$

where

$$h(s) = \prod_p\left\{1 + \frac{1}{(p-1)p^s} - \frac{1}{(p-1)p^{2s-1}}\right\}.$$

Clearly, the product defining $h(s)$ converges uniformly and absolutely on compact subsets of $\operatorname{Re} s > 1/2$, and thus h is analytic in $\operatorname{Re} s > 1/2$ and $h(1) = 1$. Since $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, the function on the right of (2.6) is analytic in the half-plane $\operatorname{Re} s > 1/2$. Also, since by (2.1), $\Delta(x) = 0(x^{1/2})$, the integral on the left of (2.6) converges uniformly and absolutely on compact subsets of $\operatorname{Re} s > 1/2$ and as such defines an analytic function there. Thus (2.6) holds good in the half-plane $\operatorname{Re} s > 1/2$. Since $\zeta(s) - 1/(s-1) \to \gamma$ as $s \to 1$, a simple calculation yields the asserted value of $\int_1^\infty \Delta(t)t^{-2}\,dt$. And (2.3) follows from (2.2) via partial summation.

LEMMA 2.4. $E_1(x) = x\,E_0(x) + 0(1)$.

PROOF. This follows readily from

$$\text{(2.7)}\qquad E_0(x) = -\frac{1}{x}\sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)}\,P\left(\frac{x}{n}\right) + 0\left(\frac{1}{x}\right) \text{ and}$$

$$(2.8)\qquad E_1(x) = -\sum_{n\le x}\frac{\mu^2(n)}{\phi(n)}P\Big(\frac{x}{n}\Big) - \Big(1+\frac{\gamma}{2}+\frac{1}{2}\sum_p\frac{\log p}{p(p-1)}\Big) + 0\Big(\frac{1}{\sqrt{x}}\Big).$$

To prove (2.7), we first recall an easy consequence (cf. [**5**, Lemma 2.1]) of Euler-Maclaurin summation formula

$$(2.9)\qquad \sum_{n\le x}\frac{1}{n} = \log x + \gamma - P(x)/x + 0(1/x^2).$$

Hence, by (2.4),

$$\sum_{n\le x}\frac{1}{\phi(n)} = \sum_{n\le x}\frac{1}{n}\sum_{d\delta=n}\frac{\mu^2(d)}{\phi(d)} = \sum_{d\le x}\frac{\mu^2(d)}{\phi(d)}\sum_{\delta\le x/d}\frac{1}{\delta}$$

$$(2.10)\qquad \begin{aligned} &= \sum_{d\le x}\frac{\mu^2(d)}{d\phi(d)}\Big\{\log\frac{x}{d} + \gamma - \frac{P(x/d)}{x/d} + 0\Big(\frac{d^2}{x^2}\Big)\Big\}\\ &= \sum_{n\le x}\frac{\mu^2(n)}{n\phi(n)}\log\frac{x}{n} + \gamma\sum_{n=1}^{\infty}\frac{\mu^2(n)}{n\phi(n)} + 0\Big(\sum_{n>x}\frac{1}{n\phi(n)}\Big)\\ &\quad - \frac{1}{x}\sum_{n\le x}\frac{\mu^2(n)}{\phi(n)}P\Big(\frac{x}{n}\Big) + 0\Big(\frac{1}{x^2}\sum_{n\le x}\frac{n}{\phi(n)}\Big). \end{aligned}$$

If $\sigma(n) = \sum_{d|n} d$, then $\sigma(n)\phi(n) > n^2\prod_p(1-p^{-2})$ for all n and hence

$$(2.11)\qquad \sum_{n>x}\frac{1}{n\phi(n)} \ll \sum_{n>x}\frac{\sigma(n)}{n^3} \ll \frac{1}{x}$$

on account of $\sum_{n\le x}\sigma(n) \ll x^2$ and partial summation. Also

$$(2.12)\qquad \sum_{n\le x}\frac{n}{\phi(n)} \ll \sum_{n\le x}\frac{\sigma(n)}{n} = \sum_{d\delta\le x}\frac{1}{\delta} = \sum_{\delta\le x}\frac{1}{\delta}\sum_{d\le x/\delta}1 \le \sum_{\delta\le x}\frac{1}{\delta}\cdot\frac{x}{\delta} \ll x.$$

Further, on writing

$$(2.13)\qquad \alpha := \sum_{n=1}^{\infty}\frac{\mu^2(n)}{n\phi(n)},$$

we have, by (2.3),

$$\sum_{n\le x}\frac{\mu^2(n)}{n\phi(n)}\log\frac{x}{n} = \sum_{n\le x}\frac{\mu^2(n)}{n\phi(n)}\int_n^x\frac{dt}{t} = \int_1^x\frac{1}{t}\sum_{n\le t}\frac{\mu^2(n)}{n\phi(n)}dt$$

$$(2.14)\qquad \begin{aligned} &= \int_1^x\frac{1}{t}\Big(\alpha - \sum_{n>t}\frac{\mu^2(n)}{n\phi(n)}\Big)dt\\ &= \alpha\log x - \int_1^{\infty}\frac{1}{t}\sum_{n>t}\frac{\mu^2(n)}{n\phi(n)}dt + 0\Big(\int_x^{\infty}\frac{dt}{t^2}\Big)\\ &= \alpha\log x - \int_1^{\infty}\frac{1}{t}\sum_{n>t}\frac{\mu^2(n)}{n\phi(n)}dt + 0\Big(\frac{1}{x}\Big). \end{aligned}$$

Thus, from (2.10) through (2.14), we obtain

$$\text{(2.15)}\qquad \begin{aligned}\sum_{n\leq x} 1/\phi(n) = \alpha \log x + \alpha\gamma - \int_1^\infty \frac{1}{t}\sum_{n>t}\frac{\mu^2(n)}{n\phi(n)}\,dt \\ - \frac{1}{x}\sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) + 0\left(\frac{1}{x}\right).\end{aligned}$$

Since, trivially,

$$\frac{1}{x}\sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) \ll \frac{1}{x}\sum_{n\leq x}\frac{1}{\phi(n)} \ll \frac{1}{x}\sum_{n\leq x}\frac{\sigma(n)}{n^2} \ll \frac{\log x}{x},$$

a comparison of (2.15) with (1.1) and (1.4) yields

$$\text{(2.16)}\qquad \alpha = A \text{ and } \alpha\gamma - \int_1^\infty \frac{1}{t}\sum_{n>t}\frac{\mu^2(n)}{n\phi(n)}\,dt = AB.$$

On substituting these values in (2.15) and again comparing it with (1.1), we obtain (2.7).

To prove (2.8), we have by (2.2), (2.3), (2.13), and (2.16),

$$\begin{aligned}\sum_{n\leq x}\frac{n}{\phi(n)} &= \sum_{d\delta\leq x}\frac{\mu^2(d)}{\phi(d)} = \sum_{d\leq x}\frac{\mu^2(d)}{\phi(d)}\left[\frac{x}{d}\right] \\ &= \sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)}\left(\frac{x}{n} - P\left(\frac{x}{n}\right) - \frac{1}{2}\right) \\ &= x\left(A - \sum_{n>x}\frac{\mu^2(n)}{n\phi(n)}\right) - \sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) - \frac{1}{2}\sum_{n\leq x}\frac{\mu^2/n)}{\phi(n)} \\ &= x\left(A - \frac{1}{x} + 0\left(\frac{1}{x^{3/2}}\right)\right) - \sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) \\ &\qquad - \frac{1}{2}\left(\log x + \gamma + \sum_p \frac{\log p}{p(p-1)} + 0\left(\frac{1}{\sqrt{x}}\right)\right) \\ &= Ax - \frac{1}{2}\log x - \left(1 + \frac{\gamma}{2} + \frac{1}{2}\sum_p \frac{\log p}{p(p-1)}\right) \\ &\qquad - \sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) + 0\left(\frac{1}{\sqrt{x}}\right).\end{aligned}$$

On comparing this with (1.2), we obtain (2.8) and this completes the proof of the lemma.

REMARK 2.1. Since $\underline{\lim}(\phi(n) \log\log n)/n = e^{-\gamma}$ (cf. **[2]**, Theorem 382]), it is clear that

$$\text{(2.17)}\qquad E_0(x) = \Omega\left(\frac{\log\log x}{x}\right)$$

which should be compared with (1.5). Now Lemma 2.4 yields

$$\text{(2.18)}\qquad E_1(x) = \Omega(\log\log x) \text{ and}$$

$$E_1(x) = 0((\log x)^{2/3}). \tag{2.19}$$

It may be of interest to obtain sharper Ω and 0-estimates.

3. Proofs of (1.6) and (1.7). Since, for integral x,

$$\sum_{n\leq x} \log n = x\log x - x + \frac{1}{2}\log x + \frac{1}{2}\log 2\pi + 0\left(\frac{1}{x}\right), \tag{3.1}$$

we have, by (1.1), (1.2), and Lemma 2.4,

$$\begin{aligned}
\sum_{n\leq x} E_0(n) &= \sum_{n\leq x}\left(\sum_{m\leq n}\frac{1}{\phi(m)} - A\log m - AB\right) \\
&= \sum_{n\leq x}(x+1-n)\frac{1}{\phi(n)} - A\sum_{n\leq x}\log n - ABx \\
&= (x+1)(A\log x + AB + E_0(x)) - \left(AX - \frac{1}{2}\log x + E_1(x)\right) \\
&\qquad - A\left(x\log x - x + \frac{1}{2}\log x + 0(1)\right) - ABx. \\
&= \left(\frac{A+1}{2}\right)\log x + x\,E_0(x) - E_1(x) + 0(1) \\
&= \left(\frac{A+1}{2}\right)\log x + 0(1)
\end{aligned}$$

which is (1.6).

To prove (1.7), it is enough if we show that

$$\int_1^x E_0(t)dt = \sum_{n\leq x} E_0(n) - \frac{A}{2}\log x + 0(1).$$

This follows readily on specializing Lemma 2.1 with $f(n) = 1/\phi(n)$, $g(x) = A\log x + B$ and $E(x) = E_0(x)$ and noting that $E_0(x) = 0(1)$, a consequence of (1.4).

4. Proofs of (1.8) and (1.9). Let $0 < \rho = \rho(x) < 1$ be a function of x to be chosen suitably later but subject to the condition that ρ^{-1} is an integer. Then, by (2.4),

$$\begin{aligned}
\sum_{n\leq x}\frac{n}{\phi(n)} &= \sum_{d\delta\leq x}\frac{\mu^2(d)}{\phi(d)} \\
&= \sum_{\substack{d\delta\leq x,\\ d\leq\rho x}}\frac{\mu^2(d)}{\phi(d)} + \sum_{\substack{d\delta\leq x\\ \delta\leq\rho^{-1}}}\frac{\mu^2(d)}{\phi(d)} - \sum_{\substack{d\leq\rho x\\ \delta\leq\rho^{-1}}}\frac{\mu^2(d)}{\phi(d)} \qquad (4.1)\\
&= S_1 + S_2 - S_3,
\end{aligned}$$

say. Now, by (2.3), (2.13), (2.16) and (2.2),

$$\begin{aligned}
S_1 &= \sum_{d\leq\rho x} \frac{\mu^2(d)}{\phi(d)} \sum_{\delta\leq x/d} 1 = \sum_{d\leq\rho x} \frac{\mu^2(d)}{\phi(d)} \left[\frac{x}{d}\right] \\
&= \sum_{n\leq\rho x} \frac{\mu^2(n)}{\phi(n)} \left(\frac{x}{n} - P\left(\frac{x}{n}\right) - \frac{1}{2}\right) \\
&= x \sum_{n\leq\rho x} \frac{\mu^2(n)}{n\phi(n)} - \frac{1}{2} \sum_{n\leq\rho x} \frac{\mu^2(n)}{\phi(n)} - \sum_{n\leq\rho x} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) \\
&= x\left(A - \frac{1}{\rho x} + 0\left(\frac{1}{(\rho x)^{3/2}}\right)\right) \qquad (4.2)\\
&\quad - \frac{1}{2}\left(\log \rho x + \gamma + \sum_p \frac{\log p}{p(p-1)} + 0\left(\frac{1}{(\rho x)^{1/2}}\right)\right) - \sum_{n\leq\rho x} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) \\
&= Ax - \rho^{-1} - \frac{1}{2} \log \rho x - \frac{1}{2}\left(\gamma + \sum_p \frac{\log P}{p(p-1)}\right) \\
&\quad - \sum_{n\leq\rho x} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) + 0(\rho^{-3/2} x^{-1/2}).
\end{aligned}$$

Also, by (2.2) and (3.1),

$$\begin{aligned}
S_2 &= \sum_{\delta\leq\rho^{-1}} \sum_{d\leq x/\delta} \frac{\mu^2(d)}{\phi(d)} \\
&= \sum_{\delta\leq\rho^{-1}} \left\{\log x - \log \delta + \gamma + \sum_p \frac{\log p}{p(p-1)} + 0\left(\left(\frac{\delta}{x}\right)^{1/2}\right)\right\} \\
&= \rho^{-1} \log x - \left(\rho^{-1} \log \rho^{-1} - \rho^{-1} + \frac{1}{2} \log \rho^{-1} + \frac{1}{2} \log 2\pi + 0(\rho)\right) \\
&\quad + \left(\gamma + \sum_p \frac{\log p}{p(p-1)}\right)\rho^{-1} + 0(\rho^{-3/2} x^{-1/2}) \\
&= \rho^{-1} \log(\rho x) + \left(1 + \gamma + \sum_p \frac{\log p}{p(p-1)}\right)\rho^{-1} - \frac{1}{2} \log \rho^{-1} \qquad (4.3)\\
&\quad - \frac{1}{2} \log 2\pi + 0(\rho^{-3/2} x^{-1/2} + \rho).
\end{aligned}$$

Further,

$$\begin{aligned}
S_3 &= \left(\log \rho x + \gamma + \sum_p \frac{\log p}{p(p-1)} + 0(\rho^{-1/2} x^{-1/2})\right) \rho^{-1} \\
&= \rho^{-1} \log \rho x + \left(\gamma + \sum_p \frac{\log p}{p(p-1)}\right)\rho^{-1} + 0(\rho^{-3/2} x^{-1/2})
\end{aligned} \qquad (4.4)$$

Thus, from (4.1) through (4.4),

$$\begin{aligned}
\sum_{n\leq x} \frac{n}{\phi(n)} &= Ax - \frac{1}{2} \log x - \frac{1}{2}\left(\gamma + \log 2\pi + \sum_p \frac{\log p}{p(p-1)}\right) \\
&\quad - \sum_{n\leq\rho x} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{x}{n}\right) + 0(\rho^{-3/2} x^{-1/2} + \rho).
\end{aligned} \qquad (4.5)$$

Again, by (2.4),

$$\sum_{n\le x}\frac{n^2}{\phi(n)}=\sum_{d\delta\le x}\frac{\mu^2(d)d\delta}{\phi(d)}$$

$$(4.6)\qquad =\sum_{\substack{d\delta\le x\\ d\le\rho x}}\frac{\mu^2(d)d\delta}{\phi(d)}+\sum_{\substack{d\delta\le x\\ \delta\le\rho^{-1}}}\frac{\mu^2(d)d\delta}{\phi(d)}-\sum_{\substack{d\le\rho x\\ \delta\le\rho^{-1}}}\frac{\mu^2(d)d\delta}{\phi(d)}$$

$$=S_1'+S_2'-S_3',$$

say. Then, by Lemma 2.3, (2.13) and (2.16),

$$S_1'=\sum_{d\le\rho x}\frac{\mu^2(d)d}{\phi(d)}\sum_{\delta\le x/d}\delta=\frac{1}{2}\sum_{d\le\rho x}\frac{\mu^2(d)d}{\phi(d)}\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)$$

$$=\frac{1}{2}\sum_{n\le\rho x}\frac{\mu^2(n)n}{\phi(n)}\left(\frac{x}{n}-P\left(\frac{x}{n}\right)-\frac{1}{2}\right)\left(\frac{x}{n}-P\left(\frac{x}{n}\right)+\frac{1}{2}\right)$$

$$(4.7)\qquad =\frac{1}{2}\sum_{n\le\rho x}\frac{\mu^2(n)n}{\phi(n)}\left(\frac{x^2}{x^2}-2P\left(\frac{x}{n}\right)\frac{x}{n}+P^2\left(\frac{x}{n}\right)-\frac{1}{4}\right)$$

$$=\frac{x^2}{2}\sum_{n\le\rho x}\frac{\mu^2(n)}{n\phi(n)}-x\sum_{n\le\rho x}\frac{\mu^2(n)}{\phi(n)}P\left(\frac{x}{n}\right)+0\left(\sum_{n\le\rho x}\frac{\mu^2(n)n}{\phi(n)}\right)$$

$$=\frac{Ax^2}{2}-\frac{\rho^{-1}x}{2}-x\sum_{n\le\rho x}\frac{\mu^2(n)}{\phi(n)}P\left(\frac{x}{n}\right)+0(\rho^{-3/2}x^{1/2}+\rho x).$$

Also, by (2.1),

$$(4.8)\qquad S_2'=\sum_{\delta\le\rho^{-1}}\delta\sum_{d\le x/\delta}\frac{\mu^2(d)d}{\phi(d)}=\sum_{\delta\le\rho^{-1}}\delta\left(\frac{x}{\delta}+0\left(\left(\frac{x}{\delta}\right)^{1/2}\right)\right)$$

$$=\rho^{-1}x+0(\rho^{-3/2}x^{1/2}).$$

Further,

$$(4.9)\qquad S_3'=\left(\sum_{n\le\rho x}\frac{\mu^2(n)}{n\phi(n)}\right)\left(\sum_{n\le\rho^{-1}}n\right)=\frac{1}{2}(\rho x+0((\rho x)^{1/2}))(\rho^{-2}+\rho^{-1})$$

$$=\frac{\rho^{-1}x}{2}+\frac{x}{2}+0(\rho^{-3/2}x^{1/2}).$$

Thus, from (4.6) through (4.9),

$$(4.10)\qquad \sum_{n\le x}\frac{n}{\phi(n)}=\frac{Ax^2}{2}-\frac{x}{2}-x\sum_{n\le\rho x}\frac{\mu^2(n)}{\phi(n)}P\left(\frac{x}{n}\right)+0(\rho^{-3/2}x^{1/2}+\rho x).$$

Hence, for integral x, we have, by (3.1), (4.5) and (4.10),

$$\sum_{n\le x} E_1(n) = \sum_{n\le x}\left(\sum_{m\le n}\frac{m}{\phi(m)} - An + \frac{1}{2}\log n\right)$$
$$= \sum_{n\le x}(x+1-n)\frac{n}{\phi(n)} - A\sum_{n\le x} n + \frac{1}{2}\sum_{n\le x}\log n$$
$$= (x+1)\Bigg\{Ax - \frac{1}{2}\log x - \frac{1}{2}\left(\gamma + \log 2\pi + \sum_p \frac{\log p}{p(p-1)}\right)$$
$$- \sum_{n\le\rho x}\frac{\mu^2(n)}{\phi(n)}P\left(\frac{x}{n}\right) + 0(\rho^{-3/2}x^{-1/2} + \rho)\Bigg\}$$
$$- \left\{\frac{Ax^2}{2} - \frac{x}{2} - x\sum_{n\le\rho x}\frac{\mu^2(n)}{\phi(n)}P\left(\frac{x}{n}\right) + 0\left(\rho^{-3/2}x^{1/2} + \rho x\right)\right\}$$
$$- \frac{A}{2}(x^2 + x) + \frac{1}{2}(x\log x - x + 0(\log x))$$
$$= \frac{1}{2}\left(A - \gamma - \log 2\pi - \sum_p\frac{\log p}{p(p-1)}\right)x + 0(\rho^{-3/2}x^{1/2} + \rho x + \log x).$$

On setting $\rho = [x^{1/5}]^{-1}$ in the above, we obtain (1.8),

To prove (1.9), we proceed as in the proof of (1.7) and use (1.8) instead of (1.6).

Finally, we note that, in proving (1.6) and (1.8), we supposed that x is an integer but passage from integral x to real x could be easily affected.

REMARK 4.1. From (1.2) and (1.9), we obtain

$$\sum_{n\le x}(x-n)\frac{n}{\phi(n)} = \int_1^x\left(\sum_{n\le t}\frac{n}{\phi(n)}\right)dt$$
$$= \frac{A}{2}x^2 - \frac{1}{2}x\log x + \left(\frac{1-D}{2}\right)x + 0(x^{4/5}).$$

We conjecture that the order of the error function on the right of above is $O_\varepsilon(x^{1/4+\varepsilon})$ for each $\varepsilon > 0$.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TOLEDO, TOLEDO, OHIO 43606