ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 3, Summer 1980

EXPRESSING GROUP ELEMENTS AS COMMUTATORS

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Let G be a group. If $x, y \in G$, the commutator of x and y, $[x, y]$, is defined to be the element $xyx^{-1}y^{-1}$. These elements generate a characteristic subgroup, called the commutator subgroup of G, and is denoted G'. It is not true in general that G' consists only of commutators. Thus we define $\lambda(G)$ to be the smallest integer n such that every element on G' is a product of n commutators.

Perhaps the first example of a group G with $\lambda(G) > 1$ was given by Fite [1]. He discovered a group G of order 256 with G' elementary abelian of order 16 containing only 15 commutators. In [2] a group G with $|G| = 240$, G' cyclic of order 60, and $\lambda(G) = 2$ was exhibited. This was later shown to be the smallest such example in the case G' is cyclic (cf. [3]).

Then the question naturally arises: what are the groups G with $|G|$ or $|G'|$ minimal satisfying $\lambda(G) > 1$? In [3] the author has partially answered this question by proving the following theorem.

THEOREM 1. *If (i) G' is abelian and $|G| < 128$ or $|G'| < 16$, or* (ii) *G' is nonabelian and $|G| < 96$ or $|G'| < 24$, then $\lambda(G) = 1$.*

In this paper, we construct groups to show that the bounds in Theorem 1 can not be improved. The next two lemmas will be useful in determining which elements are commutators.

LEMMA 1. *If G is a group and $x, y, z \in G$, then*
(i) $[x, yz] = [x, y]\, y\, [x, z]\, y^{-1}$
(ii) $[xy, z] = x\, [y, z]\, x^{-1}\, [x, z]$.
(iii) *If y and $[x, y]$ commute, then $[x, y^e] = [x, y]^e$.*
(iv) *If x and $[x, y]$ commute, then $[x^e, y] = [x, y]^e$.*

PROOF. (i) and (ii) follow by writing out the elements. (iii) and (iv) follow by a straightforward induction. For details see [6].

LEMMA 2. *Suppose G is a group with a subgroup H such that $H \supseteq G'$ and $G = \langle H, x\rangle$. If w is a commutator in G, then $w = [ax^e, b]$ for some $a, b \in H$ and $e \in \mathbf{Z}$.*

PROOF. We first note that if $g \in G$, then $g = hx^m$ for some $h \in H$ and $m \in \mathbf{Z}$. Also, if $h_1, h_2 \in H$, then

Received by the editors on February 13, 1978.

$$(h_1x^s)(h_2x^t)^r \equiv h_1h_2^r x_1{}^{s+rt} \pmod{G'}$$

since G/G' is abelian. Thus since $H \supseteq G'$,

$$(h_1x^s)(h_2x^t)^r = h_3x^{s+rt} \text{ for some } h_3 \in H.$$

Now suppose $w = [g_1, g_2]$ is a commutator in G. Then $g_1 = h_1x^s$ and $g_2 = h_2x^t$, where $h_1, h_2 \in H$. If $t = 0$, we are done. Suppose $s = 0$. Then by Lemma 1 we have

$$\begin{aligned} w &= [h_1, h_2x^t] = [h_1(h_2x^t), h_2x^t] \\ &= [h_1h_2x^t, h_2x^t(h_1h_2x^t)^{-1}] = [h_1h_2x^t, h_1^{-1}]. \end{aligned}$$

The proof now proceeds by induction on $k = \min\{|s|, |t|\}$. If $k = 0$, we are done as above. Suppose $|s| \geqq |t|$. By the division algorithm, $s = qt + r$, where $0 \leqq r < |t|$. Hence as above,

$$\begin{aligned} w &= [h_1x^s, h_2x^t] = [(h_1x^s)(h_2x^t)^{-q}, h_2x^t] \\ &= [h_3x^{s-qt}, h_2x^t] = [h_3x^r, h_2x^t]. \end{aligned}$$

Since $r < |t| \leqq s$, we are finished by induction. A similar argument works if $|s| < |t|$.

Let $G = \langle u, v, w, x, y\rangle$, where $u^2 = v^2 = w^2 = x^8 = y^2 = 1, uvu = vw$, $uxu = x^{-1}$, $vxv = x^5y$, and w, $y \in ZG$. Then $G' = \langle w, x^2, x^4y\rangle = \langle w, x^2, y\rangle \cong C_4 \times K_4$ (where C_n is the cyclic group of order n and K_4 is the Klein group of order 4).

CLAIM. *y is not a commutator in G.*

PROOF OF CLAIM. Suppose y is a commutator in G. Applying Lemma 2 with $H = \langle u, w, x, y\rangle$, we have

$$y = [x^ey^fw^gu^hv^k, x^qy^rw^su^t].$$

Using Lemma 1 repeatedly, we obtain

$$\begin{aligned} y &= [x^eu^hv^k, x^qu^t] \\ &= [x, u^t]^e [u^h, x]^q [v, x]^{kq} [v, u]^{kt} \\ &= [x, u^t]^e [u^h, x]^q x^{4kq}y^{kq} w^{kt}. \end{aligned}$$

Since the only term involving y is y^{kq}, we must have that $k \equiv q \equiv 1 \pmod 2$. Hence $w^{kt} = 1$ and so $t \equiv 0 \pmod 2$. Thus we have

$$y = [u^h, x]^q x^4 y.$$

If h is even, then $u^h = 1$, and $y = x^4y$, a contradiction. If h is odd, then $[u^h, x] = x^{-2}$ and since q is odd, $y = x^2y$ or x^6y, a contradiction. Hence y is not a commutator in G.

Note that $G = \langle x, y \rangle \langle u, v \rangle$, where $\langle x, y \rangle \cap \langle u, v \rangle = 1$ and $\langle x, y \rangle$ is normal in G. Hence G is isomorphic to a semi-direct product of $\langle x, y \rangle \cong C_8 \times C_2$ and $\langle u, v \rangle \cong D_4$ (where D_n is the dihedral group of order $2n$). Thus G is a group of order 128 with G' abelian of order 16 and $\lambda(G) = 2$. This shows that (i) of Theorem 1 can not be improved. It can be shown [**3**] that if $|G'| = 16$ and $\lambda(G) > 1$, then either $G' \cong C_4 \times K_4$ or $G' \cong K_4 \times K_4$.

We now construct a class of examples. Let $G_1 = \langle a, b, x \rangle$, where $a^4 = b^4 = x^3 = 1$, $xax^{-1} = b$, $xbx^{-1} = ab$, $a^2 = b^2$, and $aba^{-1} = b^{-1}$($G_1 \cong SL_2(3)$, the group of 2×2 matrices of determinant 1 over the field of 3 elements). Then $G_1 = \langle H_1, x \rangle$, where $H_1 = G_1' = \langle a, b \rangle \cong Q_8$ (where Q_8 is the quaternion group of order 8).

CLAIM. *If* $u, v \in H_1$ *and* $[ux^e, v] = a^2$, *then* $e \equiv 0 \pmod 3$.

PROOF OF CLAIM. Suppose not. Note that since $3 \nmid e$, $[x^e, v]$ has order 4, unless $v \in \langle a^2 \rangle$. If $[x^e, v]$ has order 4, then $[ux^e, v] = [u, v]u[x^e, v]u^{-1}$ has order 4, a contradiction. Hence $v \in \langle a^2 \rangle$ and so $[ux^e, v] = 1$, also a contradiction.

Choose G_2 to be any nonabelian group with a normal abelian subgroup H_2 of index 3. Then there exists $y \in G_2$ such that $G_2 = \langle H_2, y \rangle$ and $y^3 \in H_2$. Let G be the subgroup of $G_1 \times G_2$ generated by $H_1 \times H_2$ and the element (x, y). Note $G' = G_1' \times G_2'$.

PROPOSITION. $\lambda(G) > 1$.

PROOF. Choose $1 \neq c \in G_2'$. We will show that (a^2, c) is not a commutator in G. By Lemma 2 with $H = H_1 \times H_2$, if (a^2, c) is a commutator, then

$$(a^2, c) = [(h_1, h_2)(x^e, y^e), (k_1, k_2)],$$

where $(h_1, h_2), (k_1, k_2) \in H_1 \times H_2$. Hence $a^2 = [h_1x^e, k_1]$ and $c = [h_2y^e, k_2]$. By the previous claim, $e \equiv 0 \pmod 3$, and thus $y^e \in H_2$. Hence $c \in H_2' = \{1\}$, a contradiction, yielding the result.

The group G constructed above has order $24\,|H_2|$. If we take G_2 to be A_4, the alternating group on 4 letters, then $H_2 = A_4' \cong K_4$. In this case $|G| = 96$ and $G' \cong Q_8 \times K_4$. In fact there are exactly 2 groups of order 96 with $\lambda(G) > 1$(cf. [**3**]). We can also take G_2 to be any nonabelian group of order 27. Then $|H_2| = 9$ and $G_2' \cong C_3$. Hence $G' \cong Q_8 \times C_3$, and so G' is nonabelian of order 24. These two examples show that (ii) of Theorem 1 can not be improved.

For other examples of groups G in which some element of G' is not a product of a fixed number of commutators, one can consult [**3**], [**4**], [**5**], and [**7**]. An open question in this field is whether G simple implies $\lambda(G) = 1$.

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