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INFINITE GROUPS WITH A SUBNORMALITY CONDITION ON INFINITE SUBGROUPS

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1. **Introduction and notation.** If $\mathfrak{X}$ is a subgroup theoretic property, let $(\mathfrak{X})$ denote the class of groups all of whose subgroups are $\mathfrak{X}$-subgroups, and let $I(\mathfrak{X})$ denote the class of all infinite groups, all of whose infinite subgroups are $\mathfrak{X}$-subgroups. In [4] and [5] Černikov studies the structure of groups in three classes (we do not require a trivial group in a class) of the form $I(\mathfrak{X}) - (\mathfrak{X})$, for $\mathfrak{X}$ denoting normal, ascendant, and complemented, respectively. In [14], R. Phillips studies a class of this form for $\mathfrak{X}$ denoting serial; this class is the same as for $\mathfrak{X}$ denoting ascendant. In the present paper we study the structure of $I\mathfrak{P} - \mathfrak{P}$, where $\mathfrak{P}$ is the class of all groups, all of whose subgroups are subnormal of bounded defects, and where $I\mathfrak{P}$ is the class of all infinite groups, all of whose infinite subgroups are subnormal of bounded defects.

Our major result is that locally nilpotent groups in $I\mathfrak{P} - \mathfrak{P}$ are Černikov groups and we obtain a structure theorem for them in § 2. By studying certain automorphisms of divisible abelian p-groups of finite rank in § 3, we further characterize locally nilpotent groups in $I\mathfrak{P} - \mathfrak{P}$ in terms of direct limits of p-groups of maximal class in § 4. We explain why we restrict our attention to locally nilpotent groups following Theorem 2.4.

$\mathfrak{N}$, $\mathfrak{N}_c$, $L\mathfrak{N}$, Min, Z, ZA, and ZD denote the classes of nilpotent groups, nilpotent groups of class at most c, locally nilpotent groups, groups satisfying the minimal condition or descending chain condition on subgroups, groups having a central series, hypercentral groups, and groups with a descending central series, respectively. A group of type p^∞ is a group with generators $x_1, x_2, \cdots$ and defining relations $px_1 = 0$, $px_{n+1} = x_n$.

A divisible abelian group of finite rank is a direct sum of finitely many groups each of which is the full rational group or a group of type p^∞ for various primes p; if such a group is a p-group the rank is the number of summands. A Černikov (or extremal) group is a finite extension of an abelian group in Min. A Černikov group G possesses a characteristic divisible abelian group $D(G)$ of finite rank and finite index. $H \leqq G$ and $H < G$ denote that H is a subgroup of G and H is a proper subgroup of G, respectively. $H \operatorname{sn} G$, $H \operatorname{sn}_r G$, $H \operatorname{ser} G$, and

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H char G denote that H is subnormal, subnormal of defect at most r, serial, or characteristic in G, respectively. H^G denotes the normal closure of H in G. We define $H^{G,n}$ inductively by $H^{G,0} = G$ and $H^{G,n+1} = H^{(H^{G,n})}$. For X and Y subsets of a group G, $[X, Y]$ denotes the group generated by commutators $[x, y]$ where $x \in X$, $y \in Y$. If $X_1, X_2, \cdots$ are subsets of a group we define more general commutator subgroups by $[X_1] = \langle X_1 \rangle$, $[X_1, \cdots, X_{n+1}] = [[X_1, \cdots, X_n], X_{n+1}]$. The derived group of G is $G' = [G, G]$.

The center of a group G is denoted by $Z(G)$. Higher centers $Z_\alpha(G)$ for α any ordinal number are defined inductively by $Z_0(G) = 1$,

$$\frac{Z_{\alpha+1}(G)}{Z_\alpha(G)} = Z\left(\frac{G}{Z_\alpha(G)}\right),$$

and $Z_\lambda(G) = \bigcup_{\alpha<\lambda} Z_\alpha(G)$ for α an ordinal and λ a limit ordinal.

2. **Structure of $L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$.** We will make considerable use of Roseblade's Theorem that there is a function f with domain and range the positive integers, such that if every subgroup of a group G is subnormal with subnormal defect at most s, then G is nilpotent of class not exceeding $f(s)$. (See [19; Theorem 1] or [18; Theorem 7.42 and Corollary].)

LEMMA 2.1. *If $G \in L\mathfrak{N} \cap I\mathfrak{P}$ and M is a finite normal subgroup of G, with $G/M \in \mathfrak{N}$, then $G \in \mathfrak{N}$.*

PROOF. Let F be a finite subgroup of G and let r be the class of G/M. Then $MF \in \mathfrak{N}$ and hence

$$F \operatorname{sn}_{|M|} MF \operatorname{sn}_r G.$$

Hence, by Roseblade's Theorem, $G \in \mathfrak{P} = \mathfrak{N}$.

LEMMA 2.2. *Let $G \in L\mathfrak{N} \cap I\mathfrak{P}$. If G has any nonempty collection $\mathbf{H} = \{H_\gamma \mid \gamma \in \Gamma\}$ of infinite normal subgroups such that $\bigcap \mathbf{H}$ is finite, then $G \in \mathfrak{N}$.*

PROOF. First we suppose that $\bigcap \mathbf{H} = 1$. Let b be the bound for subnormal defects of infinite subgroups of G. Then for all γ, $G/H_\gamma \in \mathfrak{N}_{f(b)}$. Since $\bigcap \{H_\gamma \mid \gamma \in \Gamma\} = 1$, we have G isomorphically contained in the direct product of the G/H_γ, which is nilpotent of class at most $f(b)$. Thus $G \in \mathfrak{N}_{f(b)}$. The desired conclusion for the general case, $\bigcap \mathbf{H} \neq 1$, now follows from 2.1.

The following theorem obtains $(L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}) \leqq$ Min and shows that we may limit our attention to p-groups in $L\mathfrak{N} \cap I\dot{\mathfrak{P}} - \mathfrak{P}$.

THEOREM 2.3. *$G \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$ if and only if $G = P \times K$, where P is a Sylow p-subgroup of G, P is an infinite Černikov group, $P \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$, and K is a finite nilpotent group.*

PROOF. Necessity. In [14; § VI] it is shown that if $G \in I\mathfrak{P} - \mathfrak{P}$, then G is periodic. Let $G \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$. Let

$$A = \bigcap \{H \lhd G \mid H \text{ is infinite}\}.$$

By 2.2, A is infinite. Since $A \in L\mathfrak{N}$ and A is periodic, we may write $A = \Sigma\{A_i \mid i \in I\}$, each A_i a Sylow p_i-subgroup of A. Notice that $A_i \operatorname{char} A$; thus $A_i \lhd G$ for all $i \in I$. If $|I| > 1$, A_i is finite for all $i \in I$ since A can have no proper infinite subgroup which is normal in G. Hence, $|I| > 1$ implies that I is infinite since A is infinite; but then A has a proper infinite subgroup which is normal in G. Hence $|I| = 1$; i.e., A is a p-group. Let P be a Sylow p-subgroup of G containing A. Since $G \in L\mathfrak{N}$ and G is periodic, write $G = P \times K$, where K is a Sylow p'-subgroup of G. If K is infinite, then by definition of A and P we have $A \leqq P \cap K = 1$. Hence K is finite and thus also nilpotent since G is locally nilpotent. Clearly $P \notin \mathfrak{N}$ since $G = P \times K \notin \mathfrak{N}$. However, $P \in L\mathfrak{N} \cap I\mathfrak{P}$, and hence it remains only to show that P is a Černikov p-group. To this end we claim first that A' is finite. Suppose A' is infinite. Since $A' \operatorname{char} A \operatorname{char} G$, $A' \lhd G$ and hence $A' = A$. Since $P \in L\mathfrak{N} \leqq \mathfrak{Z}$, we have for all x, $1 \neq x \in A$, $[P, x^P] < x^P$. Hence $x^P < A$. Since A has no proper infinite subgroups which are normal in P, x^P is finite. Thus $P/C_P(x^P)$ is finite and hence $A \leqq C_P(x^P)$. Since $x \in A$ was arbitrary we have $A' = 1$, a contradiction. Thus A' is a finite normal subgroup of P.

Next we claim that A is a hypercentral Černikov group. Let

$$\frac{B}{A'} = \frac{A}{A'}(p)$$

be the maximal elementary abelian subgroup of A/A'. Suppose that B/A' is infinite. Then $A \leqq B$; i.e., $A = B$ and A/A' is infinite elementary abelian. Let $F \leqq P$ be finite. $P/A \in \mathfrak{N}_s$, for some s. Thus

$$[P, \underbrace{F, \cdots, F}_{s}] \leqq [P, \underbrace{AF, \cdots, AF}_{s}] \leqq A. \tag{1}$$

Now AF/A' is abelian by finite and is not a Černikov group. Thus by a theorem of Černikov ([3; Theorem 3] or [18; Lemma 10.21]), its center is infinite. Hence $Z(AF/A') \cap (A/A')$ is also infinite, and an elementary abelian p-group; i.e., it is an infinite direct sum of cyclic groups of order p and contained in the center of AF/A'. Thus $AF/A' \in L\mathfrak{N} \cap I\mathfrak{P}$

(bound b) and has an infinite descending chain of normal subgroups with trivial intersection. By 2.2, $AF/A' \in \mathfrak{N}_c$, for some c. Thus

$$[P, \underbrace{F, \cdots, F}_{s+c}] \leqq [A, \underbrace{AF, \cdots, AF}_{c}] \leqq A'$$

where we have used (1). Hence every subgroup of P/A' is subnormal with defect at most $\max\{b, s + c\}$. Thus $P/A' \in \mathfrak{N}$. By 2.1, $P \in \mathfrak{N}$, a contradiction. Thus B/A' is finite; i.e., A/A' has Min and so $A \in$ Min. Since $A \in L\mathfrak{N}$, A is a hypercentral Černikov p-group. (See [**17**; Theorem 5.27, Corollary 2].)

Next we claim that $P \in ZA$. Since $A \lhd P$ and $A \in$ Min we may choose $1 \neq B \leqq A$, B minimal with respect to $B \lhd P$. By a result of McLain, $B \leqq Z(P)$. Hence $Z(P) \neq 1$. Now $P/Z(P) \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$ and so by induction

$$Z_n(P) < Z_{n+1}(P),\ n = 0,\ 1,\ 2,\ \cdots$$

whence $Z_\omega(P) = \bigcup_{n<\omega} Z_n(P)$ is infinite. Thus $P/Z_\omega(P) \in \mathfrak{N}$ by Roseblade's Theorem and we have $P \in ZA$.

Next note that $Z_n(P)$ is finite for all $n < \omega$ since otherwise $P/Z_n(P)$ is nilpotent for some n, implying that P is nilpotent. Thus by a result of Muhammedžan ([**13**; Theorem 8] or [**18**; Theorem 10.23, Corollary 1]), P is a Černikov group. This completes the necessity.

Sufficiency. Let $G = P \times K$, P an infinite p-group, $P \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$ and K a finite nilpotent p'-group. Then $G \in L\mathfrak{N} - \mathfrak{N}$ and we claim furthermore that $G \in I\mathfrak{P}$. Let r be the bound on subnormal defects of infinite subgroups of P and $K \in \mathfrak{N}_c$. Let H be an infinite subgroup of G. Since $H \in L\mathfrak{N}$, write $H = H_p \times H_{p'}$, where H_p is the Sylow p-subgroup of H and $H_{p'}$ is the Sylow p'-subgroup of H. Then $H_p \leqq P$ and $H_{p'} \leqq K$; hence H_p is infinite. But then we have

$$H = H_p \times H_{p'} \operatorname{sn}_c H_p \times K \operatorname{sn}_r P \times K,$$

whence every infinite subgroup of G is subnormal of defect no more than $c + r$, as desired.

Now we restrict our attention to p-groups in the class $L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$. The next theorem gives a version of their structure which is further pursued, together with examples, in § 4.

THEOREM 2.4. *Let P be an infinite p-group. Then $P \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$ if and only if P is a Černikov p-group satisfying the following: Let $D = D(P)$ and $C = C_P(D)$.* (i) *$C < P$ with $C \in \mathfrak{N}$, and* (ii) *$x \in P - C$ implies that x does not normalize any infinite proper subgroup of D.*

PROOF. Necessity. P is Černikov by 2.3 and D is a divisible abelian p-group of finite index in P. Thus $D \leqq Z(C)$ and hence $C/Z(C)$ is a finite p-group and thus nilpotent. Thus $C \in \mathfrak{N}$ and we conclude that $C < P$. Now let $x \in P$ normalize an infinite proper subgroup H of D; using [17; Lemma 3.29.1] we have $D = C_D(x)[D, \langle x \rangle]$. If $C_D(x)$ is finite, then $[D, \langle x \rangle]$ has finite index in D. Since D has no proper subgroups of finite index, we have for all $n \geqq 1$,

$$[D, \underbrace{\langle x \rangle, \langle x \rangle, \cdots, \langle x \rangle}_{n}] = D.$$

But $H \lhd D\langle x \rangle$ and H is infinite. Thus by hypothesis and using Roseblade's Theorem

$$\frac{D\langle x \rangle}{H} \in \mathfrak{N}.$$

Hence there exists an r such that

$$[D, \underbrace{\langle x \rangle, \langle x \rangle, \cdots, \langle x \rangle}_{r}] \leqq H < D,$$

a contradiction. Thus we have $C_D(x)$ infinite. Thus $\langle x \rangle \lhd C_P(x) \operatorname{sn}_s P$, where s is the bound on subnormal defects of infinite subgroups of P, and by [17; Lemma 3.13] we have $[D, x] = 1$; i.e., $x \in C = C_P(D)$.

Sufficiency. Let H be an infinite subgroup of P. We claim that either $D \leqq H$ or $H \leqq C$. For suppose otherwise. Then $H \cap D$ is a proper infinite subgroup of D and there is some $x \in H - C$. By condition (ii) we have $x \notin N_P(H \cap D)$. But $H \cap D \lhd H$ and $x \in H$, a contradiction, which establishes the claim. But now clearly $P \in I\mathfrak{P}$ because $D \leqq H$ implies $H \, sn_d \, P$ where $P/D \in \mathfrak{N}_d$, (P/D being a finite p-group) and $H \leqq C$ implies $H \operatorname{sn}_c C \lhd P$, where $C \in \mathfrak{N}_c$. Hence $H \operatorname{sn}_s P$, where $s = \max\{d, c + 1\}$. Now let $x \in P - C$ (we use condition (i)). By [16; Lemma 2.1 (iii)] we have $\langle x \rangle$ is not descendant in P. Thus $P \notin \mathfrak{N}$. A Černikov p-group is locally finite and hence in $L\mathfrak{N}$ and so $P \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$.

Why we have restricted our attention to $L\mathfrak{N}$ groups deserves some discussion. Let $\mathfrak{X}$ be any subgroup theoretic property such that for all groups G, G is an $\mathfrak{X}$-subgroup of G. If there exists an infinite non-abelian group S, all of whose proper subgroups are finite, then $S \in I(\mathfrak{X})$. Similarly $S \in I\mathfrak{P}$. Whether such a group S exists is an unsolved problem posed by Schmidt (see [6]; also [17; § 3.4]). Thus in studying groups of the types $I(\mathfrak{X})$ and $I\mathfrak{P}$, one must either solve Schmidt's problem or impose additional restrictions to avoid the problem.

In [4] and [14] the additional restriction of local finiteness avoids the problem because of a theorem discovered independently by Kargapolov [10] and by P. Hall and Kulatilaka [9] (see also [17; Theorem 3.43]), which says that an infinite locally finite group always possesses an infinite abelian subgroup. Some groups in the class $L\mathfrak{F} \cap I\mathfrak{P} - \mathfrak{P}$ are also in $L\mathfrak{F} \cap I(\text{ser}) - (\text{ser})$, which are studied in [14]. It can be shown (see the author's dissertation [15; Lemma 3.16]) that $L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P} = (L\mathfrak{F} \cap I\mathfrak{P} - \mathfrak{P}) - (L\mathfrak{F} \cap I(\text{ser}) - (\text{ser}))$.

3. **Strongly irreducible automorphisms.** Throughout this section p will denote a fixed prime and D a divisible abelian p-group of finite rank, r.

Let $\alpha \in \operatorname{Aut} D$. Following [14], we call α a *strongly irreducible automorphism* (abbreviated *$S-I$ automorphism*) if no proper infinite subgroup of D is α-invariant; i.e., if for all $H < D$, H infinite, we have

$$H^{\langle\alpha\rangle} = D.$$

Furthermore, we call a group A of automorphisms of D a group of $S-I$ automorphisms if every nontrivial element of A is an $S-I$ automorphism. Note that the identity mapping on D is an $S-I$ automorphism if D has rank 1, but not if D has rank at least 2. Note also that an automorphism α of D is an $S-I$ automorphism if and only if for every proper nontrivial divisible subgroup D_1 of D we have $D_1{}^{\alpha} \neq D_1$. In the notation of 2.4, P/C is a finite group of $S-I$ automorphisms of D.

Since the endomorphism ring of a group of type p^{∞} is the ring R_p of p-adic integers (see [7; Theorem 55.1] or [11; pp. 154–157]) we may take $\operatorname{Aut} D$ to be $\mathrm{GL}(r, R_p)$, the ring of all $r \times r$ matrices over R_p with determinant a unit of R_p.

THEOREM 3.1. *Let α be an automorphism of D of order p. Then α is an $S-I$ automorphism if and only if $r = p - 1$.*

PROOF. Necessity. Let α be an $S-I$ automorphism of D of order p. By an argument due to Černikov in [4; Theorem 3.2] we have $r \leqq p - 1$. View α as an $r \times r$ matrix over F_p, the quotient field of the PID R_p. Let f be its minimal polynomial. Since α has order p,

$$f \mid (X^p - 1).$$

But $X^p - 1 = (X - 1)\,\Phi_p(X)$, where $\Phi_p(X)$ is the cyclotomic polynomial of degree $p - 1$. Since $\Phi_p(X + 1)$ is irreducible over F_p using Eisenstein's criterion, so is $\Phi_p(X)$. Thus $f \in \{X - 1,\ \Phi_p(X),\ (X - 1)\Phi_p(X)\}$; i.e., $\deg f \in \{1, p - 1, p\}$. Now let g be the characteristic polynomial of α. Since $g \in F_p[X]$ we have $f|g$. Now $\deg g = r$. Thus if $r < p - 1$ we have $f = X - 1$; i.e., α is trivial, a contradiction. Hence $r \geqq p - 1$ and the necessity is proved.

Sufficiency. Let α be an automorphism of order p and let $r = p - 1$. Choose D_1 to be the maximal divisible subgroup of D on which α acts trivially. Since D_1 is a direct summand of D, it is easily verified that if α also acts trivially on D/D_1, then α acts trivially on D, contradicting its having order p. Thus α has order p as it acts on D/D_1. Now let D_2/D_1 be the minimal nontrivial divisible subgroup of D/D_1 such that $(D_2/D_1)^\alpha = D_2/D_1$. As above, if α is trivial on D_2/D_1 then it is trivial on D_2; by the maximality of D_1 we would then have $D_2 = D_1$, contrary to the choice of D_2. Hence α is nontrivial on D_2/D_1, and by the minimality of D_2/D_1 we conclude α is an $S-I$ automorphism of D_2/D_1 of order p. By the necessity just proved,

$$\operatorname{rank} \frac{D_2}{D_1} = p - 1 = \operatorname{rank} D = r.$$

Hence $D_1 = 0$ and $D_2 = D$, yielding the desired result.

It can be shown that a p-group of $S-I$ automorphisms of D has exponent p; in fact, by using 4.2 it has order p.

4. **Relationship to direct limits of p-groups of maximal class.** There are precisely two nontrivial direct limits of p-groups of maximal class and presentations for them are known. Both are Černikov p-groups G, satisfying

$$|G : Z_\omega(G)| = |Z_{n+1}(G) : Z_n(G)| = p$$

and the rank of $D(G)$ is $p - 1$. (See [**2**; § 5] and [**1**].) In this section we use some characterizations of Blackburn for such groups to show a relationship with the groups we studied in § 2 and to provide examples of groups in $L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$.

THEOREM 4.1. [**2**; *Theorem* 5.1]. *Let P be a Černikov p-group for which $D(P)$ has rank $r \leqq p - 1$. Then either $G/Z(G)$ is finite or G has a finite normal subgroup N such that G/N is a direct limit of p-groups of maximal class.*

THEOREM 4.2. *Let P be an infinite p-group. Then the following are equivalent*:

(1) $P \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$
(2) *P is a Černikov group with $D(P)$ of rank $p - 1$ and $|P : C_P(D)| = p$.*
(3) *There is a finite normal subgroup N of P such that P/N is a direct limit of p-groups of maximal class.*

PROOF. (1) implies (3). By 2.4, P is a Černikov group, $D = D(P)$ has finite index in P and has finite rank, and P/C (where $C = C_P(D)$) is isomorphic to a nontrivial p-group of $S-I$ automorphisms of D. By 3.1 $\operatorname{rank} D = p - 1$. Now $Z(P)$ is finite since $P \in I\mathfrak{P} - \mathfrak{P}$ so that by 4.1 there exists a finite normal subgroup $N \lhd P$ such that P/N is a direct limit of p-groups of maximal class.

(3) implies (2). Let $N \lhd P$, N finite, P/N a direct limit of p-groups of maximal class. Since P/N is Černikov and N is finite, P is a Černikov p-group. Let $D = D(P)$ and let $C = C_P(D)$. Now DN/N is a normal divisible subgroup of P/N and

$$\left| \frac{P}{N} : \frac{DN}{N} \right| \leqq |P : D| < \infty.$$

Thus $DN/N = D(P/N) = Z_\omega(P/N)$. Hence

$$\operatorname{rank} D = \operatorname{rank}\left(\frac{D}{D \cap N} \right) = \operatorname{rank}\left(\frac{DN}{N} \right) = p - 1$$

and $|P : DN| = p$. Let $Z = Z(P)$. Notice that

$$\frac{Z}{Z \cap N} \cong \frac{ZN}{N} \leqq Z\left(\frac{P}{N} \right).$$

Since P/N has a finite center, Z is finite. Thus $C < P$. But by [17; Lemma 3.13], $[D, N] = 1$; i.e., $DN \leqq C$. Hence $p \leqq |P : C| \leqq |P : DN| = p$. Thus $|P : C| = p$.

(2) implies (1). Let P be Černikov with $D = D(P)$ of rank $p - 1$ and with $|P : C| = p$ where $C = C_P(D)$. Then $x \in P - C$ implies that x acts as an $S-I$ automorphism of D by 3.1. Hence P satisfies condition 2.4(ii). Now $D \leqq Z(C)$ and so $C/Z(C)$ is finite. Thus $C \in \mathfrak{B}$. Hence 2.4(i) is also satisfied, yielding the desired result.

We complete our discussion by providing two examples of groups in $L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$, corresponding to the two direct limits of p-groups of maximal class.

EXAMPLE 4.3. Let D be a direct sum of $p - 1$ groups of type p^∞. The companion matrix of the cyclotomic polynomial $\Phi_p(X)$ (of degree $p - 1$) is an element of order p in $GL(p - 1, R_p)$. Thus D has an automorphism α of order p. By 2.4 and 3.1, the split extension $D]\langle\alpha\rangle \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$.

EXAMPLE 4.4. Let D be a direct sum of $p - 1$ groups of type p^∞. Let $\langle\beta\rangle$ be a cyclic group of order p^2 operating on D so that $C_{\langle\beta\rangle}(D) =$

$\langle\beta^p\rangle$. Form the split extension $H = D]\langle\beta\rangle$ and let z be an element of order p in $D \cap Z(H)$. Then $\langle z^{-1}\beta^p\rangle \leqq Z(H)$. By 2.4 and 3.1,

$$\frac{H}{\langle z^{-1}\beta^p\rangle} \in L\mathfrak{N} \cap I\mathfrak{P} - \mathfrak{P}$$

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