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OSCILLATION AND EVEN ORDER LINEAR DIFFERENTIAL EQUATIONS

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Let q be a continuous function from $[0, \infty)$ to $(0, \infty)$, and let n be a positive integer. We shall obtain herein oscillation and nonoscillation criteria for

$$(1) \qquad u^{(2n)} + qu = 0$$

on $[0, \infty)$. It follows from results of J. G. Mikusiński [7] and I. T. Kiguradze [5] (see also G. V. Anan'eva and V. I. Balaganskiĭ [1], H. C. Howard [4], V. A. Kondrat'ev [6], and C. A. Swanson [8, Theorem 4.59, p. 173]) that if $0 < \alpha < 2n - 1$ and $\int_0^\infty t^\alpha q(t)\, dt = \infty$, then every solution of (1) is oscillatory (i.e., every solution of (1) has an unbounded set of zeros). *Thus, we shall assume throughout that if* $0 < \alpha < 2n - 1$, *then* $\int_0^\infty t^\alpha q(t)\, dt < \infty$. Also, since our primary results compare (1) to certain second order equations, we assume $n \geqq 2$. R. Grimmer [2] has recently done some related work comparing higher order equations to second order equations.

THEOREM 1. *Suppose the second order equation*

$$(2) \qquad w''(t) + \left((1/(2n-3)!) \int_t^\infty (s-t)^{2n-3} q(s)\, ds \right) w(t) = 0$$

is oscillatory. Then every solution of (1) *is oscillatory.*

THEOREM 2. *Suppose the second order equation*

$$(3) \qquad w''(t) + (t^{2n-2}/(2n-2)!) q(t) w(t) = 0$$

is nonoscillatory. Then there exists a nonoscillatory solution of (1).

COROLLARY 1. *Suppose*

$$\limsup_{t \to \infty} t \int_t^\infty (s-t)^{2n-2} q(s)\, ds > (2n-2)!.$$

Then every solution of (1) *is oscillatory.*

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COROLLARY 2. *Suppose*

$$\limsup_{t\to\infty} t \int_t^\infty s^{2n-2}q(s)\,ds < (2n-2)!/4.$$

Then there exists a nonoscillatory solution of (1).

Note that Corollaries 1 and 2 are partial extensions of more detailed results obtained by E. Hille [3] in the second order case. In fact, Corollaries 1 and 2 are immediate consequences of Theorems 1 and 2 and of [3]. (See [8, Theorem 2.1, p. 45] for a summary of those results of Hille which are relevant here). Since there is a voluminous literature on oscillation and nonoscillation in second order equation (see, for example, [8, Chapter 2] and D. Willett [9]), Theorems 1 and 2 permit the drawing of a great many additional conclusions regarding (1). With the exceptions of Corollaries 1 and 2, we leave this to the reader.

PROOF OF THEOREM 1. We shall assume the existence of a nonoscillatory solution of (1), and show that this implies that (2) is nonoscillatory. Suppose u is a nonoscillatory solution of (1). If u is eventually negative, we may replace u by $-u$, so we assume u is eventually positive. Find $a \geqq 0$ such that $u(t) > 0$ if $t \geqq a$. Now $u^{(2n)} < 0$ on $[a, \infty)$, so $u^{(2n-1)}$ is eventually one-signed. Since $u^{(2n-1)}$ is eventually one-signed, $u^{(2n-2)}$ is eventually one-signed. Continuing this, we see that there is $c \geqq a$ such that none of $u, u', \cdots, u^{(2n-1)}$ has any zeros in $[c, \infty)$. Let j be the largest integer such that $u^{(i)} > 0$ on $[c, \infty)$ if $i \leqq j$ (where we write $u = u^{(0)}$). Now j is odd. (Note that, thus far, our argument is essentially due to Kiguradze [5]). Suppose $1 < j < 2n-1$. Now,

$$-u^{(j+1)}(t) = (1/(2n-j-2)!) \int_t^\infty (s-t)^{2n-j-2}q(s)u(s)\,ds$$

whenever $t \geqq c$, and

$$u(s) \geqq (1/(j-2)!) \int_c^s (s-\xi)^{j-2}u^{(j-1)}(\xi)\,d\xi$$

if $s \geqq c$, so

$$-u^{(j+1)}(t) \geqq (1/(2n-j-2)!(j-2)!)$$

$$\int_t^\infty (s-t)^{2n-j-2}q(s)\left(\int_c^s (s-\xi)^{j-2}u^{(j-1)}(\xi)\,d\xi\right)ds$$

$$\geqq (1/(2n-j-2)!(j-2)!)$$

$$\int_t^\infty (s-t)^{2n-j-2}q(s)\left(\int_t^s (s-\xi)^{j-2}u^{(j-1)}(\xi)\,d\xi\right) ds$$

if $t \geqq c$. Since $u^{(j)} > 0$ on $[c, \infty)$, $u^{(j-1)}$ is increasing on $[c, \infty)$, and $(2n-j-2)!(j-1)! \leqq (2n-3)!$, so

$$-u^{(j+1)}(t) \geqq u^{(j-1)}(t)(1/(2n-3)!) \int_t^\infty (s-t)^{2n-3}q(s)\,ds,$$

and

$$u^{(j+1)}(t)/u^{(j-1)}(t) \leqq -(1/(2n-3)!) \int_t^\infty (s-t)^{2n-3}q(s)\,ds \tag{4}$$

if $t \geqq c$. If $j = 1$, then

$$-u''(t) = (1/(2n-3)!) \int_t^\infty (s-t)^{2n-3}q(s)u(s)\,ds$$

$$\geqq u(t)(1/(2n-3)!) \int_t^\infty (s-t)^{2n-3}q(s)\,ds$$

whenever $t \geqq c$, so we see that (4) holds if $j < 2n-1$. Let v be given on $[c, \infty)$ by $v(t) = u^{(j)}(t)/u^{(j-1)}(t)$, and note that $v(t) > 0$ if $t \geqq c$. Now

$$v'(t) = u^{(j+1)}(t)/u^{(j-1)}(t) - v(t)^2,$$

and (4) says that

$$v'(t) + v(t)^2 \leqq -(1/(2n-3)!) \int_t^\infty (s-t)^{2n-3}q(s)\,ds \tag{5}$$

if $t \geqq c$. A classical result of A. Wintner [**10**] (see also [**8**, Theorem 2.15, p. 63]) says that (5) implies nonoscillation for (2), so the proof is complete in the case $j < 2n-1$.

Finally, suppose $j = 2n-1$. Now,

$$u^{(2n-1)}(t) = u^{(2n-1)}(\tau) + \int_t^\tau q(s)u(s)\,ds$$

$$\geqq \int_t^\tau q(s)u(s)\,ds$$

if $\tau \geqq t \geqq c$, so

$$u^{(2n-1)}(t) \geqq \int_t^\infty q(s)u(s)\,ds$$

if $t \geqq c$. Thus

$$u^{(2n-1)}(t) \geqq (1/(2n-3)!)$$
$$\int_t^\infty q(s)\left(\int_c^s (s-\xi)^{2n-3}u^{(2n-2)}(\xi)\,d\xi\right) ds$$
$$\geqq (1/(2n-3)!)$$
$$\int_t^\infty q(s)\left(\int_t^s (s-\xi)^{2n-3}u^{(2n-2)}(\xi)\,d\xi\right) ds$$

if $t \geqq c$. Let $\{y_m\}_{m+1}^\infty$ be a sequence, each value of which is a continuous function from $[c, \infty)$ to $[u^{(2n-2)}(c), \infty)$ such that $y_1 = u^{(2n-2)}$, $y_m(c) = u^{(2n-2)}(c)$, if $m \geqq 1$, and

$$y'_{m+1}(t) = (1/2n-3)!)\int_t^\infty q(s)\left(\int_t^s (s-\xi)^{2n-3}y_m(\xi)\,d\xi\right) ds$$

if $t \geqq c$ and $m \geqq 1$. Clearly $\{y_m\}_{m=1}^\infty$ is equicontinuous and locally bounded, so z, given by $z(t) = \lim_{m\to\infty} y_m(t)$, exists and is continuous. Also, since $y_m(t) \leqq u^{(2n-2)}(t)$ for all $m \geqq 1$, $t \geqq c$, and

$$\lim_{m\to\infty}\int_t^\infty q(s)\left(\int_t^s (s-\xi)^{2n-3}y_m(\xi)\,d\xi\right) ds$$

exists, locally uniformly in t, we see that z is differentiable and

$$z'(t) = (1/(2n-3)!)\int_t^\infty q(s)\left(\int_t^s (s-\xi)^{2n-3}z(\xi)\,d\xi\right) ds$$

whenever $t \geqq c$. Now

$$z''(t) = -(1/(2n-3)!)\int_t^\infty (s-t)^{2n-3}q(s)z(t)\,ds$$

and

$$z''(t)/z(t) = -(1/(2n-3)!)\int_t^\infty (s-t)^{2n-3}q(s)\,ds$$

whenever $t \geqq c$. If we now let v be given on $[c, \infty)$ by $v(t) = z'(t)/z(t)$, the remainder of the proof follows as before, and we are through.

PROOF OF THEOREM 2. Suppose (3) is nonoscillatory, and let w be an eventually positive solution of (3). Find $a \geqq 0$ such that $w(t) > 0$ if $t \geqq a$. Now $w' > 0$ on $[a, \infty)$, so

$$w'(t) = w'(\tau) + (1/(2n-2)!) \int_t^\tau s^{2n-2} q(s) w(s)\, ds$$

$$\geqq (1/(2n-2)!) \int_t^\tau s^{2n-2} q(s) w(s)\, ds$$

whenever $\tau \geqq t \geqq a$, and

$$w'(t) \geqq (1/(2n-2)!) \int_t^\infty s^{2n-2} q(s) w(s)\, ds$$

$$\geqq (1/2n-2)!) \int_t^\infty (s-t)^{2n-2} q(s) w(s)\, ds$$

whenever $t \geqq a$. As in the last part of the proof of Theorem 1, this says that there is a differentiable function u from $[a, \infty)$ to $[w(a), \infty)$ such that $u(a) = w(a)$, $u(t) \leqq w(t)$ whenever $t \geqq a$, and

$$u'(t) = (1/(2n-2)!) \int_t^\infty (s-t)^{2n-2} q(s) u(s)\, ds$$

whenever $t > a$. Differentiating this last equation $2n-2$ times yields

$$u^{(2n-1)}(t) = \int_t^\infty q(s) u(s)\, ds,$$

and then

$$u^{(2n)}(t) = -q(t) u(t);$$

so we see that u satisfies (1) on $[a, \infty)$. Clearly u can be extended to a solution of (1) on $[0, \infty)$, and since u has no zeros in $[a, \infty)$, this solution is nonoscillatory. The proof is complete.

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