DIFFEOMORPHISMS OF TORI, THEIR LIFTS AND SUSPENSIONS

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ABSTRACT. Every element $f \in \text{Diff}^r(T^n)$ is shown to have a lift to \mathbb{R}^n of the form $F(x) = Ax + \pi(x)$ where $\pi(x)$ is a map of \mathbb{R}^n periodic of period 1 in each component of x and A is a matrix with integer entries of determinant ± 1 . Diff $^r(T^n)$ is then shown to be decomposed into a disjoint union of open sets $\bigcup_{A \in UM(z)} \text{Diff}^r(T^n)(A)$, where UM(z) are matrices as described above. For the particular case Diff $^r(T^2)$, Diff $^r(\mathbb{R}^2)$ (A) is shown to be pathwise connected in Diff $^r(T^2)$ for each $A \in UM(z)$ yielding an arithmetic classification of isotopy classes of Diff $^r(T^2)$. As a corollary one obtains an arithmetic classification of the manifolds of suspension of elements of Diff $^r(T^2)$.

0. Introduction. Consider a homeomorphism of the circle $f: S^1 \rightarrow S^1 \cdot S^1$ may be viewed as the real line R where points are identified according to whether their coordinates differ by an element of Z, the integers, i.e., $R/Z = S^1$. For any choice of a function $F: R \rightarrow R$ which projects onto f under the quotient map, $p: R \rightarrow R/Z$, if f is orientation preserving, the "lift" F may be shown to satisfy the following conditions:

(i) F is continuous,

(ii) F is strictly increasing, and

(iii) F(x + 1) = F(x) + 1, for all $x \in R$.

From (iii) it is easy to see that $F(x) = x + \pi(x)$, where $\pi(x)$ is a periodic function of period 1. Topological properties of f may almost be completely studied by examining arithmetic properties of F. The function, $f: S^1 \to S^1$, may arise as the map induced on a cross-section of a flow on a two-dimensional torus T^2 by that flow. Topological properties of the flow can then be studied almost completely by studying topological properties of f. The situation just described has been extensively studied, initially by A. Denjoy [1] (or see [2]). Denjoy considered $\lim_{n\to\infty} (F^n(x)/n)$, where F^n means the *n*-th iterate of F, and shows that this limit exists and is independent of x. The limit is called the rotation number. It is then shown that if the rotation number α is

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irrational and F is twice continuously differentiable the set of points $\{f^n(x) : n = 0, \pm 1, \pm 2, \cdots\}$ is dense on S^1 for all x and if α is rational, then f^k has a fixed point for some k. If f is the map induced on a cross-section of a flow on T^2 by that flow, continuity properties of differential equations can then be used to show that if α is rational, the flow has a periodic orbit, and if α is irrational and the flow is C^2 , every path of the flow is dense on T^2 .

The novice then seeking to explore extensions of these techniques to *n*-dimensional tori T^n may study homeomorphisms $f: T^n \to T^n$. Viewing T^n as $\mathbb{R}^n/\mathbb{Z}^n$, the natural assumption may possibly be made (in view of (iii)) that any lift $F: \mathbb{R}^n \to \mathbb{R}^n$ of an orientation preserving homeomorphism $f: T^n \to T^n$ satisfies $F(x_1, \dots, x_i + 1, \dots, x_n) = F(x_1, \dots, x_n) + (0, \dots, 0, 1, 0, \dots, 0)$, where F is an *n*-vector; i.e., $F(x) = \mathrm{Id}(x) + \pi(x)$, where $\pi(x)$ is a map of period 1 in each component of $x = (x_1, \dots, x_n)$. Everything proceeds smoothly (or not) until the discovery is made that

$$F(\mathbf{x}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

projects onto an orientation preserving homeomorphism of T^2 .

One purpose of this paper is to put on a precise footing the relationships between homemorphisms (not necessarily orientation preserving) of a torus (denoted $\text{Diff}^{(0)}(T^n)$) and their lifts. To this end, $\text{Diff}^{(0)}(T^n)$ is decomposed into a disjoint union of subsets (called lift classes) (\S 1); in particular it is shown that any element of $\text{Diff}^0(T^n)$ has an almost unique lift of the form $F(x) = Ax + \pi(x)$, where A is a matrix with integer entries whose determinant is $= \pm 1$, and $\pi(x)$ is a map of period 1 in each coordinate of x. In $\S 2$, an arithmetic for lift classes is discussed and applied to situations of topological conjugacy. In §3, a topology is put on Diff^r (T^n) (the set of diffeomorphisms of class C^r of $T^n, r \ge 0$, and it is shown that lift classes are open. In § 4, it is proven that lift classes are connected in case $T^n = T^2$ (theorem 4.22). Although many of the theorems in §4 are known, most are only implicitly contained in the literature. They are proven here, for the most part, by means of elementary methods. § 4 also contains extensive discussions of suspensions of diffeomorphisms of T^2 and shows that the manifolds of suspension of elements of the same lift class are diffeomorphic (theorem 4.23). The results in § 4 may also be of interest in so far as theorem 4.22 is essentially an arithmetic classification of the isotopy classes of diffeomorphisms of T^2 .

The mathematics in the first half of this paper is not particularly difficult and may be read by many students. The presentation is particularly detailed in order that this be possible, since it is a purpose of this paper to foster a better intuitive feeling for diffeomorphisms of tori. Questions appear at the ends of several of the sections. The author does not know whether they are easy or difficult, but it is felt that their answers will yield a complete arithmetic and qualitative understanding of lifts and hence may bear some scrutiny.

Once this study has been made, one is then in a position to return to the original problem of extending of the techniques of Denjoy to higher dimensional manifolds. An appropriate generalization of the notion of rotation number must be found, and following that, it would be hoped that results could be obtained pertaining to the topological properties of flows, not just on T^n , but on any manifold admitting toral cross-sections (see theorem 4.23 and succeeding remarks).

1. Lift classes. Before proceeding, we will need a couple of facts about covering spaces.

DEFINITION 1.1. Let X be a topological space. A covering space of X is a pair consisting of a space \tilde{X} and a continuous map $p: \tilde{X} \to X$ such that each $x \in X$ has an arcwise connected open neighborhood U such that each arc component of $p^{-1}(U)$ is mapped homeomorphically onto U by $p \cdot p$ is called the projection.

DEFINITION 1.2. Let X and Y be topological spaces. Let (\tilde{X}, p) be a covering of X. A map $f: Y \to X$ can be lifted iff there exists a map $\tilde{f}: Y \to \tilde{X}$ such that the following diagram commutes:



f is called a lifting of f.

DEFINITION 1.3. Let X and Y be topological spaces. Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be coverings of X and Y respectively. Let $f: Y \to X$. A function $F : \tilde{Y} \to \tilde{X}$ will be called a lift of f iff the following diagram commutes:



Note the difference between the words lifting and lift in the previous two definitions. In fact, F of definition 1.3 is a lifting of the map $f \circ q$.

The following two theorems (see [5], for instance) will be necessary.

THEOREM 1.4. Let (\tilde{X}, p) be a covering space of X, and let Y be a connected locally connected space. Given any two continuous maps $f_0, f_1: Y \to \tilde{X}$ such that $p \circ f_0 = p \circ f_1$, the set $E = \{y \in Y : f_0(y) = f_1(y)\}$ is either empty or all of Y.

THEOREM 1.5. Let (\tilde{X}, p) be a covering space of X, and let Y be a connected and locally arcwise connected space. Let $y_0 \in Y, \tilde{x}_0 \in \tilde{X}$, and $x_0 = p(\tilde{x}_0)$. Given a continuous map $f: (Y, y_0) \to (X, x_0)$ (i.e., $f(y_0) = x_0$), there exists a lifting $\tilde{f}: (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ iff $f_*\pi_1(Y, y_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$. Here, $\pi_1(X, x_0)$ denotes the fundamental group of (X, x_0) , and f_* denotes the homomorphism between fundamental groups induced by f.

PROPOSITION 1.6. Let $f: X \to X$ be continuous, and let $p: \tilde{X} \to X$ be a covering of X, where \tilde{X} is simply connected. Then there always exists a map $F: \tilde{X} \to \tilde{X}$ such that the following diagram commutes:



i.e., f always has a lift.

PROOF. This is a direct consequence of theorem 1.5, since $\pi_1(\tilde{X}, \tilde{x}_0) = 1$ so that $(f \circ p)_* \pi_1(\tilde{X}, \tilde{x}_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

Let $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (*n*-times) and $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ the integral lattice points of \mathbb{R}^n . Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$, and let $p: \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the natural projection of \mathbb{R}^n onto the quotient $\mathbb{R}^n/\mathbb{Z}^n$. We then see that (\mathbb{R}^n, p) is a covering space for $\mathbb{R}^n/\mathbb{Z}^n$, and, further, that p may be used to induce a differentiable structure on T^n making T^n a differentiable manifold. We call T^n an *n*-dimensional torus. We fix $p: \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ once and for all as the covering of T^n we will use. An immediate consequence of proposition 1.6 is

PROPOSITION 1.7. Let $f: T^n \to T^n$ be continuous. Then f has a lift $F: \mathbb{R}^n \to \mathbb{R}^n$.

PROOF. \mathbb{R}^n is simply connected.

DEFINITION 1.8. Let (\tilde{X}, p) be a covering of a space X. The fibre over a point $x \in X$ is the set $p^{-1}(x) \in \tilde{X}$.

For the particular covering of T^n that we have, two points $x = (x_1, x_2, \dots, x_n)$ and $x_0 = (x_{01}, \dots, x_{0n})$ in \mathbb{R}^n are in the same fibre if and only if $x_i - x_{0i} \in \mathbb{Z}$ for all $i = 1, \dots, n$. This implies immediately that if $f: T^n \to T^n$, and $x - x_0 \in \mathbb{Z}^n$, then $F(x) - F(x_0) \in \mathbb{Z}^n$, for any lift F of f.

PROPOSITION 1.9. Let $f: T^n \to T^n$ be continuous. Let F be any lift of f. Then for each $(m_1, \dots, m_n) \in Z^n$, there exists a unique $(p_1, \dots, p_n) \in Z^n$ such that

 $F(x_1 + m_1, \cdots, x_n + m_n) = F(x_1, \cdots, x_n) + (p_1, \cdots, p_n).$

Moreover (p_1, \dots, p_n) is determined by the set $\{(a_{i_1}, \dots, a_{i_n})\} \subset \mathbb{Z}^n$

such that $F(0, \dots, 1, \dots, 0) = F(0, \dots, 0) + (a_{i_1}, \dots, a_{i_n})$, where the 1 occurs in the *i*-th place.

PROOF. Suppose $F(x_1 + m_1, \dots, x_n + m_n) = F(x_1, \dots, x_n) + (p_1, \dots, p_n)$ for some $(x_1, \dots, x_n) \in \mathbb{R}^n$. Suppose $(p_1, \dots, p_n) \in \mathbb{Z}^n$ varies with $x \in \mathbb{R}^n$. Then $F(y_1 + m_1, \dots, y_n + m_n) = F(y_1, \dots, y_n) + (p_1(y), \dots, p_n(y))$. But F is continuous (as may easily be shown, since f is), hence, $F(y_1 + m_1, \dots, y_n + m_n) - F(x_1 + m_1, \dots, x_n + m_n) = (p_1(y), \dots, p_n(y)) - (p_1, \dots, p_n)$ is continuous. But $(p_1(y), \dots, p_n(y)) - (p_1, \dots, p_n)$ is a continuous map into a discrete set, and hence is constant.

It is easy to see that given $(m_1 \cdots, m_n) \in \mathbb{Z}^n$, $F(x_1 + m_1, \cdots, x_n + m_n) = F(x_1, \cdots, x_n) + \sum_{i=1}^n m_i(a_{i1}, \cdots, a_{in})$, by using the first part of the theorem. In other words,

$$p_j = \sum_{j=1}^n m_i a_{ij}.$$

PROPOSITION 1.10. Suppose $f: T^n \to T^n$ is continuous, and $F: \mathbb{R}^n \to \mathbb{R}^n$ is a lift of f. Then F is uniquely determined up to an element of \mathbb{Z}^n ; i.e., if \underline{F} is any other lift of f, then $F = \underline{F} + (q_1, \dots, q_n)$ for some $(q_1, \dots, q_n) \in \mathbb{Z}^n$.

PROOF. We prove that the statement of the proposition is equivalent to the following statement \mathcal{S} and prove \mathcal{S} . \mathcal{S} : If $x_0 = (x_{01}, \dots, x_{0n}) \in \mathbb{R}^n$ with $p(x_0) = f \circ p(0)$, then the lift F such that $F(0) = x_0$ is uniquely determined.

Necessity. If \underline{F} is any other lift of F, then $F = \underline{F} + (q_1, \dots, q_n)$. But $F(0) = \underline{F}(0) = x_0$ implies $0 = \underline{F}(0) - F(0) = (q_1, \dots, q_n)$.

Sufficiency. Suppose \overline{F} is a lift of f such that $F(0) = x_0$ and \overline{F} is uniquely determined. Let \underline{F} be any other lift of f. If $F \neq \underline{F}$, there exists $x \in \mathbb{R}^n$ such that $F(x) \neq \underline{F}(x)$. We may suppose that this x = 0. Consider $F(0) - \underline{F}(0) = x_0 - x_0^{-1}$. Then, since $p \circ F(x) = p \circ \underline{F}(x) =$ $f \circ p(x), F(x)$ and $\underline{F}(x)$ must be in the same fibre over $f \circ p(x)$; i.e., $F(x) - \underline{F}(x) \in \mathbb{Z}^n$. Hence continuity of F(x) and $\underline{F}(x)$ implies that $F(x) - \underline{F}(x)$ is constant for all $x \in \mathbb{R}^n$.

Statement \mathcal{S} follows directly from theorem 1.4, replacing Y by \mathbb{R}^n and f_0 and f_1 by F and \underline{F} .

PROPOSITION 1.11. If F and <u>F</u> are lifts of continuous maps f, $f : T^n \to T^n$, then f = f if and only if $F = \underline{F} + q$ for some $q \in Z^n$.

PROOF. Sufficiency is the preceding proposition. Necessity is easily proven as follows. Let $F = \underline{F} + q$. Let $t \in T^n$. Then t = p(x) for some $x \in \mathbb{R}^n$, and $f(t) = f \circ p(x) = p \circ F(x) = p \circ \underline{F}(x) = f \circ p(x) = f(t)$.

Proposition 1.7 and 1.10 guarantee that any continuous function $f: T^n \to T^n$ has a lift unique up to translation by elements of Z^n . It is clear that any continuous function $F: R^n \to R^n$ such that $F(x + m) = F(x) + p, m, p \in Z^n$, projects onto a continuous function of T^n , and proposition 1.11 guarantees that any such function projects onto a unique function $f: T^n \to T^n$.

In view of the preceding paragraph we will decompose $C(T^n)$, the continuous maps of a torus into itself, into "lift class" as follows:

DEFINITION 1.12. Let $A = (a_{ij})$ be an $n \times n$ matrix with integer entries; denote the set of such matrices by M(Z). Define $C(T^n)$ (A) to be $C(T^n)$ (A) = { $f \in C(T^n) : f$ has a lift F satisfying $F(1, 0, \dots, 0) =$ $F(0, \dots, 0) + (a_{11}, \dots, a_{1n}); F(0, 1, 0, \dots, 0) = F(0, \dots, 0) + (a_{21}, \dots, a_{2n}); \dots; F(0, \dots, 0, 1) = F(0, \dots, 0) + (a_{n1}, \dots, a_{nn})$ }.

Proposition 1.11 guarantees that $C(T^n)$ $(A) \cap C(T^n)$ $(B) = \emptyset$ if $A \neq B$, and proposition 1.7 guarantees that $C(T^n) = \bigcup_{A \in M(\mathbb{Z})} C(T^n)$ (A). We are now in a position to give the precise structures of lifts $F: \mathbb{R}^n \to \mathbb{R}^n$ of maps $f: T^n \to T^n$ in $C(T^n)$.

PROPOSITION 1.13. Suppose $f \in C(T^n)$ (A). Then any lift F of f is of the form $F(x) = A^t x + \pi(x)$, where A^t denotes the transpose of A, and $\pi(x)$ is a continuous map, $\pi : \mathbb{R}^n \to \mathbb{R}^n$, of period 1 in each component of $x = (x_1, \dots, x_n)$.

PROOF. Consider

$$F(x_1 + m_1, \cdots, x_n + m_n) - A^t \begin{pmatrix} x_1 + m_1 \\ \vdots \\ \vdots \\ x_n + m_n \end{pmatrix}$$

$$= F(x_1, \cdots, x_n) + A^t \begin{pmatrix} m_1 \\ \vdots \\ \vdots \\ m_n \end{pmatrix} - A^t \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} - A^t \begin{pmatrix} m_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

$$= F(x_1, \cdots, x_n) - A^t \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}, \text{ where } (m_1, \cdots, m_n) \in \mathbb{Z}_n.$$

The second equality follows from the proof of proposition 1.9. The

equality between the first and last expressions above immediately yields the fact that

$$F(x_1, \cdots, x_n) - A^t \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \pi(x_1, \cdots, x_n)$$

is a map of period 1 in each component of *x*.

Up to now, everything that has been done has been for continuous maps $f: T^n \to T^n$. Henceforth, we shall assume that f is a homeomorphism. For such f, we can give a very precise description of its lift. This is done in the next three propositions. We define $\text{Diff}^0(T^n)(A) = \{f: f \text{ is a homeomorphism and } f \in C(T^n)(A)\}$. The sets $\text{Diff}^0(T^n)(A)$ are called lift classes.

We will need the following theorem:

THEOREM 1.14. (BROUWER INVARIANCE OF DOMAIN) (see [10]). Suppose that U_1 and U_2 are homeomorphic subsets of S^n . Then, if U_1 is open, U_2 is also open.

PROPOSITION 1.15. If $f \in \text{Diff}^{0}(T^{n})$, then any lift F of f is a homeomorphism of \mathbb{R}^{n} with itself.

PROOF. F is onto. For suppose not, let $s \in \partial$ Im $F \subset \mathbb{R}^n$. Then $p(s) \in T^n$, and there is a neighborhood U of p(s) and a neighborhood η of s such that $p: \eta \to U$ is a homeomorphism. Let $s_0 \in F^{-1}(\sigma)$, where $\sigma = p^{-1} \circ p(s)$. $F^{-1}(\sigma) \neq \emptyset$ since $F^{-1}(\sigma) = p^{-1} \circ f^{-1} \circ p(s)$. There exists a neighborhood U_0 of $p(s_0)$ and a neighborhood η_0 of s_0 such that $p: \eta_0 \to U_0$ is a homeomorphism. Consider $N = p^{-1} \circ f^{-1}$ $(f(U_0) \cap U)$. N is open in \mathbb{R}^n and $s_0 \in N$. Furthermore, $F(N) = p^{-1} \circ f^{-1} \circ f \circ p(N)$ is a homeomorphic image of N in \mathbb{R}^n and $s \in F(N)$. Since F(N) is open in \mathbb{R}^n , by theorem 1.14, s is an interior point of F(N), contradicting the fact that $s \in \partial$ Im F.

F is $1 - \widecheck{1}$, for suppose $F(x_0) = F(x_1) = s$. There are two cases: (i) x_0 and x_1 are not in the same fibre over T^n , and (ii) x_0 and x_1 are in the same fibre over T^n . In the first case, $p(x_0) \neq p(x_1)$, but $f \circ p(x_0) =$ $f \circ p(x_1) = p(s)$, contradicting the fact that f is a homeomorphism. In the second case, let γ be any non-self intersecting curve joining x_0 and x_1 not passing through the same fibre twice. Then $p(\gamma)$ is a closed curve on T^n which is not null-homotopic, hence $f \circ p(\gamma)$ is a non-nullhomotopic closed curve on T^n . But $F(\gamma)$ is null homotopic in \mathbb{R}^n , so that $p \circ F(\gamma)$ is null-homotopic on T^n , contradicting the fact that



commutes.

With a careful selection of open neighborhoods it is easy to see locally that $F = p^{-1} \circ f \circ p$ and $F^{-1} = p^{-1} \circ f^{-1} \circ p$ are functions that are continuous. Hence F and F^{-1} are continuous.

PROPOSITION 1.16. If $f \in \text{Diff}^{0}(T^{n})(A)$, then A is non-singular.

PROOF. Suppose A is singular. Let N(A) be the null-space of A. Since the entries in A are all integers, it is not difficult to show that the null space contains points of the form $h = (h_1, \dots, h_n) \in \mathbb{Z}^n$ $(h_1, \dots, h_n) \neq 0$. Let $\& \subset N(A)$ be a line joining the origin with h. Then $F(0) = \pi(0) = \pi(h) = A^th + \pi(h) = F(h)$. The second equality follows from the periodicity of π . Hence F is not a homeomorphism contradicting proposition 1.15.

Note that the converse of this theorem is not true. For instance, on T^2 take

$$F(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \sin \pi x_2 \\ 2 \sin \pi x_1 \end{pmatrix} .$$

PROPOSITION 1.17. If $f \in \text{Diff}^{0}(T^{n})(A)$, then $|\det A| = 1$.

PROOF. Let $F = A^t x + \pi(x)$ be a lift of f. Since A^t is non-singular, $\overline{F}(x) = A^t x$ maps the rectangular parallelepiped R with vertices $(1, 0, \dots, 0)$, $(0, 1, 0, \dots,), \dots, (0, \dots, 0, 1)$ into a non-degenerate parallelepiped with vertices in Z^n . Also F(x) maps the parallelepiped R onto a curvilinear parallelepiped with the same vertices because of the periodicity of $\pi(x)$. Since F is the lift of a diffeomorphism of the torus, the image of R must cover the torus since R does. This implies that F(R) must have volume one since R does. Now F(R) may be gotten from $\overline{F}(R)$ by taking congruent perturbations of opposite (n-1)dimensional faces; the perturbations given by $\pi(x)$. Hence the volume of F(R) must be the same as that of $\overline{F}(R)$; i.e., the volume of $\overline{F}(R)$ is 1. The only way the volume of $\{y: y = A^tx, x \in R\}$ can be 1 is if $|\det A^t| = 1$.

The following diagram may help in visualizing the proof of the last proposition.



Diagram l.l

The solid lines form the boundary of $\overline{F}(R)$ and the dotted lines form the boundary of F(R).

The results of this first section may be summarized in the following theorem.

THEOREM 1.18. Let Diff⁰(T^n) (A) be as defined above. Then the set of homeomorphisms of T^n may be written as the disjoint union $\text{Diff}^0(T^n) = \bigcup_{A \in UM\{Z\}} \text{Diff}^0(T^n)$ (A), where UM(Z) is the set of all $n \times n$ matrics with integer entries whose determinant is ± 1 .

QUESTION. When talking about a lift class we may also define $\pi(A) = \{\pi(x) \text{ periodic of period } 1 \text{ in each component of } x : A^t x + \pi(x) \text{ is the lift of a homeomorphism of } T^n \}$. It is not difficult to see that $A \neq B$ often implies $\pi(A) \neq \pi(B)$. Can some definitive information be given as to the functions which may belong to a given $\pi(A)$?

A "good" answer to this question would make an element of $\text{Diff}^{0}(T^{n})$ particularly susceptible to a complete arithmetic analysis of its qualitative characteristics.

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2. Arithmetic of Lift Classes. In this section, we are interested in the lift class that is obtained by taking compositions and inverses of elements of $\text{Diff}^{0}(T^{n})$ that are contained in arbitrary lift classes. The section also contains some consequences of the arithmetic relating to topological conjugacy.

PROPOSITION 2.1. If F is a lift of $f \in \text{Diff}^0(T^n)$, and G is a lift of $g \in \text{Diff}^0(T^n)$, then $G \circ F$ is a lift of $g \circ f$.

PROOF. Immediate from the commutativity of the following diagram:



PROPOSITION 2.2. If $f \in \text{Diff}^{0}(T^{n})$ (A) and $g \in \text{Diff}^{0}(T^{n})$ (B), then $g \circ f \in \text{Diff}^{0}(T^{n}(AB))$.

PROOF. f has a lift $F(x) = A^t x + \pi_1(x)$, and g has a lift $G = B^t x + \pi_2(x)$.

$$G \circ F(x) = B^{t}(A^{t}x + \pi_{1}(x)) + \pi_{2}(A^{t}x + \pi_{1}(x))$$

= $B^{t}A^{t}x + B^{t}\pi_{1}(x) + \pi_{2}(A^{t}x + \pi_{1}(x))$
= $(AB)^{t}x + B^{t}\pi_{1}(x) + \pi_{2}(A^{t}x + \pi_{1}(x))$

is then a lift of $g \circ f$. $g \circ f \in \text{Diff}^{0}(T^{n})$ (AB), since $B^{t}\pi_{1}(x) + \pi_{2}(A^{t}x + \pi_{1}(x))$ is a function of period 1 in each component of x, since π_{1} and π_{2} are, and $AB \in UM(Z)$, since A and B are.

PROPOSITION 2.3. If $f \in \text{Diff}^{0}(T^{n})(A)$, then $f^{-1} \in \text{Diff}^{0}(T^{n})(A^{-1})$.

PROOF. First note that $A^{-1} \in UM(Z)$, so that $\text{Diff}^{0}(T^{n})$ (A^{-1}) is a properly defined subset of $\text{Diff}^{0}(T^{n})$. Let G be a lift of f^{-1} . Then $G \circ F$ = id within a translation by elements of Z^{n} . Hence if $f^{-1} \in \text{Diff}^{0}(T^{n})(B)$, BA = I; i.e., $B = A^{-1}$.

It should be noted that proposition 2.2 is also valid for elements $f \in C(T^n)$. At this point we shift our perspective to Diff^r (T^n) .

DEFINITION 2.4. Diff^{*r*} $(T^n) = \{f \in \text{Diff}^0(T^n) : \text{any lift } F \text{ of } f \text{ and } G \text{ of } f^{-1} \text{ are } r \text{ times continuously differentiable maps of } R^n \}.$

Note: Definition 2.4 is equivalent to saying that f and f^{-1} are r times continuously differentiable since $p: \mathbb{R}^n \to T^n$ induces a C^{∞} differentiable structure on T^n . Note also that $\text{Diff}^r(T^n) \subset \text{Diff}^0(T^n), r \ge 0$.

PROPOSITION 2.5. If $f \in \text{Diff}^r(T^n)$ (A), then there exists a $g \in \text{Diff}^r(T^n)$ (Id) $\exists f = pA \circ g$, where $p : R^n \to R^n/Z^n$ is the covering projection.

PROOF. Let $F = Ax + \pi(x)$ be a lift of f. There exists a $g \in$ Diff T^n (Id) such that $F = A \circ G$, where $G = \text{Id } x + \pi_1(x)$ is a lift of g. Take G to be $G = A^{-1}F = \text{Id } x + A^{-1}\pi(x)$. Then $g = p \circ G$.

DEFINITION 2.6. f and $g \in \text{Diff}^r(T^n)$ are called topologically conjugate if and only if there is an $h \in \text{Diff}^0(T^n)$ such that $f \circ h = h \circ g$.

It is an immediate consequence of propositions 2.2 and 2.3

PROPOSITION 2.7. If $f \in \text{Diff}^r(T^n)$ (A) and $g \in \text{Diff}^r(T^n)$ (B) are topologically conjugate, then the eigenvalues of A are the same as those of B.

PROOF. $B = C^{-1}AC$ for some matrix C in UM(Z).

The converse of this proposition is not, in general, true. If $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}$, A and B have the same eigenvalues and A and B are in UM(Z); however, there is no matrix C in UM(Z) such that $B = C^{-1}AC$.

This observation leads us to state a proposition and pose a question.

PROPOSITION 2.8. If $f \in \text{Diff}^r(T^n)$ (A) is topologically conjugate to $g \in \text{Diff}^r(T^n)$ (B), then for each $f_1 \in \text{Diff}^r(T^n)$ (A), there is a $g_1 \in \text{Diff}^r(T^n)$ (B) which is topologically conjugate to f_1 .

PROOF. Let $F = Ax + \pi_A(x)$ be a lift of f. Since $g = h^{-1} \circ f \circ h$ for some $h, G = H^{-1} \circ F \circ H$ is a lift of g, where $H = Cx + \pi_c(c)$ is a lift of h. Now, if $H^{-1} = C^{-1}x + \psi(x)$, we get

$$G(x) = Bx + \pi_B(x) = H^{-1} \circ F \circ H(x)$$

= $C^{-1}ACx + \bar{\pi}(x)$, for some $\bar{\pi}$,

i.e. $B = C^{-1}AC$.

Let $F_1 = Ax + \pi_1(x)$ be a lift of f_1 . Then, $G_1 = C^{-1} \circ F_1 \circ C(x) = C^{-1} \circ A \circ Cx + C^{-1}\pi_1 \circ C(x) = Bx + \overline{\pi}(x)$ is the lift of some element g_1 of Diff^r (T^n) (B), and the topological conjugacy in R^n of F_1 and G_1 projects down to a topological conjugacy in T^n between f_1 and g_1 .

It is clear that a topological conjugacy of an element in $\text{Diff}^r(T^n)$ (Id) is again an element of $\text{Diff}^r(T^n)$ (Id).

This last proposition indicates that we can identify element-wise certain lift classes between which topological conjugacy occurs. However, the preceding example leaves serious doubt in our minds as to which they are. Hence,

Question: Characterize those lift classes between which topological conjugacy can hold.

Topological conjugacy will be pursued a bit further in § 4.

3. Lift Classes are Open. Before we can talk about openness in Diff⁷(T^n) (A), we must put a topology on Diff⁷(T^n). At this point, we ought perhaps to make precise a fact that we have stated loosely before; namely that $p: R^n \to R^n/Z^n = T^n$ makes T^n into a differentiable manifold. In particular, given any point $m \in T^n$, there is a point x^m in R^n and a neighborhood N_m of x^m such that p is a homeomorphism of N_m with a neighborhood of p in T^n (see definition 1.1). Take neighborhoods $\{p(N_m)\}_{m\in T^n}$ and homeomorphisms $\{(p \mid_{N_m})^{-1}\}_{m\in T^n}$ to be an atlas for a C^r differentiable structure of T^n . Now if $f \in Diff^r(T^n)$, in terms of local coordinates, the derivative of f at m is $Df(m) = Df(x^m)$, where F is any lift of f. If x and $y \in R^n$, we define $\|x-y\| = \max_i |x_i - y_i|$. Then we can define a base for a topology of Diff^r(T^n) by defining $\rho(f_1g) = \max_{0 \le i \le r} \sup_{m \in T^n} \|D^if(m) - D^ig(m)\|$, where $D^if(m)$ is the *i*-th derivative and taking as basic open sets the sets $\{g \in Diff^r(T^n): \rho(f, g) < \epsilon$ for any $\epsilon > 0$ and each $f \in Diff^r(T^n)$.

PROPOSITION 3.1. Diff^r (T^n) (A) is open in Diff^r (T^n) for $r \ge 0$, $A \in UM(Z)$.

REMARK. This proposition is, in a sense, a stability property; i.e., if $f \in \text{Diff}^r(T^n)$ (A), then any element $g \in \text{Diff}^r(T^n)$ sufficiently close to f will also be in $\text{Diff}^r(T^n)$ (A).

PROOF. Suppose $f \in \text{Diff}(T^n)$ (A) and $g \in \text{Diff}(T^n)$ (B) with $\rho(f,g) < \epsilon$. Then $||f - g|| < \epsilon$. Let $F = A^t x + \pi_1(x)$ and $G = B^t x + \pi_2(x)$ be lifts of f and g respectively. We may take $\pi_1(0) = \pi_2(0) = 0$ by adding appropriate a and $b \in \mathbb{R}^n$ to F and G, i.e.; $F(x) = A^t x + \pi_1(x) + a$ and $G(x) = B^t x + \pi_2(x) + b$. The components $a_i - b_i$ of a - b may be taken such that $0 < |a_i - b_i| < 1$ because of the periodic nature of the covering. This eliminates "inessential differences" between the lifts and allows us to say that $||f - g|| < \epsilon$ implies $||F - G|| = ||(A^t - B^t)x + \pi_1(x) - \pi_2(x) + (a - b)|| < \epsilon$, or $||(A^t - B^t)x + (a - b)|| < \epsilon + ||\pi_1(x) - \pi_2(x)||$, for all x such that $0 \le x_i \le 1$, i = 1, \cdots , n.

Since $\pi_1(0) = \pi_2(0) = 0$, $||a - b|| < \epsilon$.

Then, $||(A^t - B^t)x|| < 2\epsilon + ||\pi_1(x) - \pi_2(x)||$. But $||(A^t - B^t)x||$ is largest at one of the vertices v of $\{x : 0 \le x_i \le 1, i = 1, \dots, n\}$, and at this point $\pi_1(v) = \pi_2(v) = 0$. Hence, $||(A^t - B^t)x|| \le ||(A^t - B^t)v|| < 2\epsilon$. If $\epsilon < 1/2$, $||(A^t - B^t)x|| < 1$. But it is possible to choose x so as to yield $(A^t - B^t)x =$ any column of $A^t - B^t$. Hence the absolute value of the maximum of the entries of $A^t - B^t$ in any column is less than $2\epsilon < 1$. Since the entries are integers, the difference of them, being less than 1, must be 0, i.e., $A^t - B^t = 0$. Hence, f and g are in the same lift class.

4. Suspensions, Flows and Lift Classes.

In this section, we apply the previous work to T^2 to show that the manifolds of suspensions of diffeomorphisms of tori are diffeomorphic if the diffeomorphisms are in the same lift class. As a corollary, we will see that lift classes are pathwise connected. This last result yields an arithmetic classification of isotopy classes of diffeomorphisms on T^n . From a certain viewpoint we will also see that the lift class of the identity map of T^n is particularly important.

DEFINITION 4.1. f and g in $\text{Diff}^r(T^n)$ are called C^r isotopic if and only if there is a C^r homotopy $h(m, t) : T^n \times I \to T^n$ such that (i) h(m, o) = f(m), (ii) h(m, 1) = g(m), and (iii) $h(m, t) \in \text{Diff}^r(T^n)$ for $0 \le t \le 1$. Here I = [0, 1].

DEFINITION 4.2. A C^r flow on a manifold M is a function $\phi(m, t)$: $M \times R \to M$ such that (i) ϕ is of class C^r , (ii) $\phi(m, 0) = m$, (iii) $\phi(\phi(m, t)s) = \phi(m, t + s)$, and (iv) $\phi(m, t)$ is a diffeomorphism for each t. The orbit through m is defined to be the set { $\phi(m, t) : t \in R$ }.

Note: A flow itself defines an isotopy between the diffeomorphism $\phi(m, 1) : M \to M$ and the identity.

DEFINITION 4.3.a. A cross-section of a flow ϕ on a compact manifold M is defined to be a compact submanifold N of codimension 1 such that (i) every orbit of ϕ intersects N, (ii) the intersection of each orbit with N is transverse, (iii) if $n \in N$, there is a t > 0 such that $\phi(n, t) \in N$, and there is a t < 0 such that $\phi(n, t) \in N$.

b. If a flow on M has a cross-section N, we can define an associated diffeomorphism of N by the formula $f(n) = \phi(n, t_n)$, where t_n is the smallest positive t satisfying $\phi(n, t_n) \in N$.

An extremely simple example of the above concepts is given by the solution of the following differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix},$$

where a is any real number. The projection $p: \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ gives a flow on a torus. The projection of the x_1 -axis gives a cross-section of the flow which is a circle. The map $p \circ F \circ p^{-1}: p(x_1\text{-axis}) \to p(x_1\text{-axis})$, where F = a + x, is the associated diffeomorphism. Despite its simplicity, this is an important example and has been extensively studied (see [1] or [2] for instance).

DEFINITION 4.3.b has a converse definition associated with it, that of suspension.

DEFINITION 4.4. Suppose N is a compact manifold and $f \in \text{Diff}^{r}(N)$. Define a diffeomorphism $\tau : N \times R \to N \times R$ by $\tau(n, t) = (f^{-1}(n), t+1)$. The space $N \times R/(n, t) \sim \tau(n, t)$ is a manifold, say M_0 . Define a flow $\psi : N \times R \times R \to N \times R$ by $\psi(n, u, t) = (n, u + t)$. This induces a flow on M_0 by projection which we call $\phi_0(m, t) : M_0 \times R \to M_0$. The pair (M_0, ϕ_0) is called the suspension of f; we call M_0 the manifold of the suspension. The flow ϕ_0 has as cross-section $N_0 = q(N \times 0) \subset M_0$, where $q : N \times R \to N \times R/Z = M_0$ is the quotient map. Note that M_0 is diffeomorphic to $N \times I/(n, 0) \sim \tau(n, 0)$.

One may study the diffeomorphism $f = p \circ F \circ p^{-1}$, where F(x) = a + x is from the previous example, and see that the suspension of f is the flow on the torus given in that example.

The notion of suspension was first introduced by S. Smale in [9] and extensively studied by G. Ikegami in [3] and [4].

We now proceed with the exposition as outlined at the beginning of this section, but it will be necessary to have a couple of preliminary definitions and theorems at our disposal.

DEFINITION 4.5. Let $f, g \in \text{Diff}^r(M) \cdot f$ and g are said to be pseudoisotopic if and only if there is a C^r diffeomorphism $G: M \times I \to M \times I$ such that G(x, 0) = (f(x), 0) and G(x, 1) = (g(x), 1).

It is clear that if f and g are isotopic they are pseudoisotopic. Whether the converse is true is not known. It is also easy to show that pseudo-isotopy is an equivalence relation.

THEOREM 4.6. (See [6]). Let M and N be compact manifolds with boundary. Let f_0 be a fixed C^r-diffeomorphism of ∂M onto ∂N . Then for any C^r-diffeomorphism $f: \partial M \to \partial N$, let $\tilde{f} = f_0^{-1} \circ f \in$ Diff^r(∂M). Then $M \cup_f N$ depends only on the pseudo-isotopy class of \tilde{f} , up to diffeomorphism. Here $M \cup_f N$ denotes $M \cup N/m \sim f(m)$. **LEMMA** 4.7. Suppose $f, g \in \text{Diff}^r(M)$, and f and g are pseudoisotopic. Then $M \times I/(x, 0) \sim (f(x), 1)$ and $M \times I/(x, 0) \sim (g(x), 1)$ are diffeomorphic.

PROOF. We apply theorem 4.6. Let $f_0: \partial(M \times I) \rightarrow \partial(M \times I)$ be defined by $f_0(m, 0) = (m, 1)$ and $f_0(m, 1) = (m, 0)$. Also, let $g_0: \partial(M \times I) \rightarrow \partial(M \times I)$ be defined by $g_0(m, 0) = (m, 1)$ $g_0(m, 1) = (m, 0)$.

We compute with f, similar results hold when we compute with g. $f_0^{-1}(m, 0) = (m, 1)$, and $f_0^{-1}(m, 1) = (m, 0)$.

Let $f: \partial(M \times I) \rightarrow \partial(M \times I)$ be defined by

(*)
$$f(m, 1) = (m, 0) \text{ and } f(m, 0) = (f(m), 1).$$

Here we have confused $f: M \to M$ with $f: \partial(M \times I) \to \partial(M \times I)$. It is clear from the number of independent variables in the parentheses which we mean.

Then,

$$f(m, 0) = f_0^{-1} \circ f(m, 0) = f_0^{-1}(f(m), 1) = f(m, 0)$$
, and
 $f(m, 1) = f_0^{-1} \circ f(m, 1) = f_0^{-1}(m, 0) = (m, 1)$.

Similar computations yield $\tilde{g}(m, 0) = (g(m), 0)$ and $\tilde{g}(m, 1) = (m, 1)$.

Since f(m) is pseudo-isotopic to g(m), say by G(m, t), \tilde{f} is pseudo-isotopic to g by a map $G: \partial(M \times I) \times I \rightarrow \partial(M \times I) \times I$;

$$\tilde{G}(x, t) = \begin{cases} G(m, t), 0, & \text{for } x = (m, 0) \\ (m, t), & \text{for } x = (m, 1). \end{cases}$$

Hence, $M \times I \cup_f M \times I$, where f is defined in (*), is diffeomorphic to $M \times I \cup_g M \times I$, for the analogous g. But these identifications are equivalent to $M \times I/(m, 0) \sim (f(m), 1)$ and $M \times I/(m, 0) \sim (g(m), 1)$ respectively.

It is clear, now, where we are headed. We wish to prove that the manifolds of suspension of diffeomorphism f and g in Diff^r(T^{<math>n}) (A) are diffeomorphic. To do this, all we must do is prove that f and g are pseudo-isotopic and apply theorem 4.7. We carry this out only for the case n = 2 and first only for Diff^r(T²) (Id).

The problem is formulated as follows.

Consider $T^2 \times I$. $\partial(T^2 \times I) = T^2 \times \{0\} \cup T^2 \times \{1\}$. Let $f \in \text{Diff}^r(\partial(T^2 \times I))$. Define $f \in \text{Diff}^r(\partial(T^2 \times I))$ by

$$f(m, 0) = (m, 0)$$
 and
 $f(m, 1) = (f(m), 1).$

Note that the notational difficulty here is the same as in the proof of theorem 4.7. If we can prove that $f \in \text{Diff}^r(\partial(T^2 \times I))$ can be extended to a diffeomorphism $\phi \in \text{Diff}^r(T^2 \times I)$, we will have proven the existence of a pseudo-isotopy between f and id.

The tool that we will use is the following important theorem.

THEOREM 4.8. (see [7]). Let M and N be combinatorially equivalent n-manifolds. Let ∂M be the disjoint union of M_0 and M_1 , where each is a union of components of ∂M ; similarly, let $\partial N = N_0 \cup N_1$. Let $f: M_0 \to N_0$ be a diffeomorphism which is extendible to a combinatorial equivalence $\mathfrak{P}: M \to N$. The obstructions to extending f to a diffeomorphism of M onto N are elements of $H_m(M, M_1, \Gamma^{n-m})$; if the obstructions vanish, f may be so extended.

In this theorem, H_m is homology based on infinite chains, m = 0, $1, \dots, n$.

 Γ^{n-m} is a group defined in [8], and Γ^k is known to be 0 for $k \leq 4$.

In view of the last statement, the homology groups above of a 3dimensional manifold are 0, and since the dimensions of the suspension manifolds of elements of $\text{Diff}^r(T^2)$ are 3-dimensional, we may disregard the last requirement of the theorem.

We must still take note of some of the other terminology in the theorem.

DEFINITION 4.9. If v_0, \dots, v_m are independent points of \mathbb{R}^n , called vertices, the simplex $\sigma = v_0, \dots, v_m$ they span is the set of points $x \in \mathbb{R}^n$ such that $x = \sum_i b_i v_i$, where $b_i \ge 0$ and $\sum_i b_i = 1$. A face of a simplex σ is a simplex spanned by a subset of the vertices of σ .

DEFINITION 4.10. A simplicial complex K is a collection of simplices in \mathbb{R}^n such that

(i) every face of a simplex of K is in K,

(ii) the intersection of two simplices of K is a single face of each of them, and

(iii) each point of |K| has a neighborhood intersecting only finitely many simplices of K. Here |K| denotes the union of the points of K and is called the polytope of K.

DEFINITION 4.11. Let K be a complex, M a manifold, possibly with boundary. The map $f: |K| \to M$ is differentiable of class C^r relative to K if $f|_{\sigma}$ is a class C^r for each simplex σ of K.

DEFINITION 4.12. Let $f: \sigma \to M$ be a C^r map. Given the point b of σ , define the map $df_b: \sigma \to \mathbb{R}^n$ by the formula $df_b(x) = Df(b) \cdot (x - b)$, where Df(b) is the derivative of f at b.

DEFINITION 4.13. If x is a point of |K|, the star of x in K, denoted st(x, K), is the union of the interiors of all simplices σ such that x lies on σ .

DEFINITION 4.14. If $f: K \to M$ is a C^r map we have maps $df_b: \sigma \to R^n$ for each σ in $\overline{st}(b, K)$, and these maps agree on the intersection of any two simplices in $\overline{st}(b, K)$ (since either (i) one is a face of the other or (ii) their intersection is a union of rays emanating from b). Also a single coordinate neighborhood of M can be found such that $f(\overline{st}(b, K))$ is contained in that coordinate neighborhood. Hence df_b is well-defined and called the differential of f. f is called an immersion if $df_b: st(b, k) \to R^n$ is one-to-one for each b. If f is also a homeomorphism, it is called a triangulation of M.

DEFINITION 4.15. A mapping $f: M \to N$ is a C^r-combinatorial equivalence between M and N if and only if there are C^r-triangulations $h: K \to M$ and $k: L \to N$, K and L complexes, and a linear isomorphism $\ell: K \to L$ such that $f = k \circ \ell \circ h^{-1}$. ℓ is a linear isomorphism between K and L if and only if ℓ maps $|K| \to |L|$ homemorphically and maps simplices of K linearly onto simplices of L.

We will have cause to use the following theorems (these can be found fully discussed in [6]).

THEOREM 4.16. If M is a C^r-manifold, possibly with boundary, and $f: K \to M$ and $g: L \to M$ are C^r-triangulations of M, there are subdivisions of K and L which are linearly isomorphic.

THEOREM 4.17.a. If M is a C-manifold without boundary, M has a C^r -triangulation.

b. If M is a C^r-manifold with boundary, any C^r-triangulation of the boundary may be extended to a C^r-triangulation of M.

The terminology in the above two theorems are covered in the following two definitions.

DEFINITION 4.18. If $f: J \to \partial M$ is a C^r -triangulation of ∂M , an extension of f is a C^r -triangulation $g: L \to M$ such that $g^{-1} \circ f$ is a linear isomorphism of J with a subcomplex of L, where a subcomplex of L is a subset which is itself a complex.

DEFINITION 4.19. A subdivision K' of K is a complex such that |K'| = |K| and each simplex of K' is contained in a simplex of K.

We now attack the problem as outlined preceding theorem 4.8.

PROPOSITION 4.20. Let $f \in \text{Diff}^r(T^2)$ (Id). Let $f: \partial(T^2 \times I) \rightarrow \partial(T^2 \times I)$ be defined by

$$f(m, 0) = (m, 0)$$
 and
 $f(m, 1) = (f(m), 1).$

Then, there is a combinatorial equivalence \mathfrak{P} between $T^2 \times I$ and itself such that $\mathfrak{P}|_{\partial(T^2 \times I)} = f$.

PROOF. We perform the appropriate constructions in $R^2 \times I/\sim$ and view $T^2 \times I$ as $R^2 \times I/\sim$, where \sim represents the relation $(x, y, t) \sim (\bar{x}, \bar{y}, \bar{\tau})$ if and only if $(x - \bar{x}, y - \bar{y}) \in Z^2$ and $t = \bar{t}$.

We can without loss of generality assume that the lift F of f is of the form

$$F(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \pi_1(x_1, x_2) \\ \pi_2(x_1, x_2) \end{pmatrix},$$

where $\pi_i(0, 0) = 0$.

Let $R = \{(x, y, t) : 0 \le x, y, t \le 1\}$. The recipe for the proof is as follows:

(1) Let K be a triangulation of R that is extendable throughout $R^2 \times I$ by periodicity.

(2) Construct a volume V diffeomorphic to R having as part of its bounding surface the surfaces $\{(x, y, 0) : 0 \le x, y \le 1\}$ and $\{(F(x, y), 1) : 0 \le x, y \le 1\}$. Moreover V must be constructed in such a way as to be periodically extendible throughout $R^2 \times I$.

(3) Let J_0 and J_1 be the subcomplexes of K that triangulate the faces of R that lie in the planes t = 0 and t = 1, respectively. Triangulate the faces of V that lie in the planes t = 0 and t = 1 by J_0 and $F(J_1)$ respectively.

(4) Extend this partial triangulation of ∂V to all of ∂V , and extend again to all of V in such a way that the triangulation is extendible throughout $R^2 \times I$ by periodicity.

(5) Subdivide the triangulations of R and V in such a way that the projection $R^2 \times I \rightarrow R^2 \times I / \sim = T^2 \times I$ induces triangulations of $T^2 \times I$.

What we have so far are two triangulations of $T^2 \times I$, say, $h: K \rightarrow T^2 \times I$ induced from R, and $K: L \rightarrow T^2 \times I$ induced from V, such that the second triangulation of the boundary is the image under $f: \partial(T^2 \times I) \rightarrow \partial(T^2 \times I)$ of the first.

(6) Subdivide K and L so that they become linearly isomorphic. It is clear that the proof is now finished. We justify steps (1)-(6). Step 1 is clear.

Step 2 is somewhat more difficult. The situation we have is pictured below (Diagram 4.1). The map (id, F) on $(R \cap \{t = 0\}) \cup (R \cap \{t = 1\})$ maps the shaded area in (a) onto the shaded area in (b).



Diagram 4.1.

Label the curves bounding (F(x, y), 1) by $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$, as in the diagram. Υ_1, Υ_3 are translations of each other in the plane t = 1 as are Υ_2 and Υ_4 , hence congruent; this follows from the periodicity of π . Furthermore they do not intersect each other. Consider the following diagram representing their projections onto the plane t = 0 (Diagram 4.2). The union of the rectangle $S = \{(x, y) : 0 \leq x, y \leq 1\}$ and portions of the eight adjoining rectangles can be divided up periodically into four simply connected closed sets S_1, S_2, S_3 and S_4 such that S_1 contains the projection of Υ_1 and the line segment $C_1 = \{0 \leq x \leq 1, y = 0\}$, etc., and only the endpoints of these curves lie on the boundaries of the respective sets in which they are contained.

Since Υ_1 is a C^r -smooth curve homotopic to the line segment C_1 relative to the endpoints, and S_1 is simply connected, there is a C^r -smooth homotopy $H_1(x, y, t)$ between Υ_1 and C_1 keeping the endpoints fixed and otherwise deforming Υ_1 into C_1 , through points of the interior of S_1 . Let σ_1 be the surface defined by $\{(H_1(x, y, t), t) : 0 \leq t \leq 1\}$. Likewise construct σ_2 via a homotopy $H_2(x, y, t)$. Let σ_3 and σ_4 be surfaces obtained from the homotopies $H_1 + (0, 1)$ and $H_2 + (1, 0)$, respectively. These four surfaces are of class C^r , do not intersect each other, and bound a region in $R^2 \times I$ that projects onto $T^2 \times I$. Extend this picture periodically throughout $R^2 \times I$.



Diagram 4.2.

Step 3 is self evident.

Step 4 again is a bit more complicated. Triangulate the four lines $L_1 = \{(0, 0, t) : 0 \leq t \leq 1\}, L_2 = \{(1, 0, t) : 0 \leq t \leq 1\}, L_3 = \{(1, 1, t) : 0 \leq t \leq 1\}, \text{ and } L_4 = \{(0, 1, t) : 0 \leq t \leq 1\} \text{ in the same way. The triangulations of } \{(F(x, y), 0) : 0 \leq x, y \leq 1\} \text{ and } \{(F(x, y), 1) : 0 \leq x, y \leq 1\} \text{ and } \{(F(x, y), 1) : 0 \leq x, y \leq 1\} \text{ induce triangulations of } C_1, C_2, C_3, C_4 \text{ and } Y_1, Y_2, Y_3, Y_4 \text{ such that } C_1 \text{ and } C_3 \text{ have the same triangulation as do } C_2 \text{ and } C_4, Y_1 \text{ and } Y_3 \text{ as } Y_2 \text{ and } Y_4.$ This follows from the fact that R was triangulated in such a way as to be extendible throughout $R^2 \times I$ by periodicity. Hence the boundaries of $\sigma_1, \sigma_2, \sigma_3$ and σ_4 are triangulated in such a way that $\partial \sigma_1$ and $\partial \sigma_3$ have the same triangulations to $\sigma_1, \sigma_2, \sigma_3$ and σ_4 in such a way that σ_1 and σ_3 (σ_2 and σ_4) have the same triangulation to all of V (again, by theorem 4.18b). Now extend this triangulation throughout $R^2 \times I$ by periodicity. This completes step 4.

The triangulations constructed in $\mathbb{R}^2 \times I$ in steps 1 and 4 may not project to a triangulation of $T^2 \times I$, since upon projection two simplices may meet in more than one face; hence the necessity of performing the operation in step 5.

Step 6 follows from theorem 4.17.

REMARK. The proof of this theorem in case $\pi(0) \neq 0$ is now clear.

Proposition 4.20 shows that $f \in \text{Diff}^r(T^2)(\text{Id})$ is pseudo-isotopic to the identity map on T^2 .

PROPOSITION 4.21. The suspension manifold of $f \in \text{Diff}^r(T^2)(\text{Id})$ is a torus.

PROOF. The suspension manifold of the identity is a torus. Since f is pseudo-isotopic to the identity, apply theorem 4.7.

THEOREM 4.22. If f and g are elements of $\text{Diff}^r(T^2)(A)$, then f and g are C^r-isotopic.

PROOF. In the case where A is the identity, the suspension manifold is a torus (proposition 4.20), and $\phi_1(t)$, the flow of the suspension, provides an isotopy between f and the identity since $\phi_0(1) = f$.

Suppose $A \neq Id$. Then $f = p \circ A(Id x + \pi(x))$ for some lift $Id x + \pi(x)$ of a map $g \in Diff'(T^2)(Id)$ (by proposition 2.5). Let ϕ_0 be the isotopy of g with the identity guaranteed above. Define ϕ by $\phi(t) = p \circ A \circ p^{-1} \circ \phi_0(t)$, where $p^{-1}(m)$ is in a single simply connected subset of $p^{-1}(T^2)$ for all $m \in T^2$. $\phi(0) = p \circ A$, and $\phi(1) = f$. Since ϕ_0 is an isotopy, so is ϕ , so that every element of Diff' $(T^2)(A)$ is isotopic to $p \circ A$. The proof that f and g in Diff' $(T^2)(A)$ are isotopic is now immediate.

Another way of stating this theorem is

THEOREM 4.22A. Diff^r (T^2) may be decomposed into a disjoint union of open connected subsets, $\bigcup_{A \in UM(Z)} \text{Diff}^r(T^2)(A)$.

Since isotopies are pseudo-isotopies proposition 4.21 has a natural generalization.

THEOREM 4.23. The manifolds of suspension of f and g, elements of Diff⁷ $(T^2)(A)$, are diffeomorphic.

PROOF. Since f and g are pseudo-isotopic apply theorem 4.7.

A few final remarks are in order. Of particular interest to one studying differential equations on T^3 is the following.

PROPOSITION 4.25. $f \in \text{Diff}^r(T^2)(\text{Id})$ if and only if f is the associated diffeomorphism of a flow on T^3 .

PROOF. Necessity. If f is the associated diffeomorphism of a flow ϕ on T^3 , then ϕ is an isotopy of f with the identity. Hence by theorem 4.22, $f \in \text{diff}^r(T^2)(\text{Id})$.

Sufficiency. If $f \in \text{diff}^{r}(T^{2})(\text{Id})$, the manifold of its suspension is T^{3} .

This shows that the naive approach to induced diffeomorphisms of cross-section of flows on T^3 was not without merit.

Finally, how good is theorem 4.23? Are there as many manifolds of suspension as lift classes? The following theorem of S. Smale ([9]) says no.

THEOREM 4.24. Let (M_0, ϕ_0) and (M_1, ϕ_1) be suspensions of f_0 and f_1 in Diff⁷ (T^2) . If f_0 and f_1 are topologically conjugate, then (M_0, ϕ_0) and (M_1, ϕ_1) are topologically conjugate.

DEFINITION 4.27. Two dynamical systems (M_0, ϕ_0) and (M_1, ϕ_1) are said to be topologically conjugate, if there is a homeomorphism $h: M_0 \rightarrow M_1$ which maps sensed trajectories of ϕ_0 onto sensed trajectories of ϕ_1 .

Hence, theorem 4.24 implies that the manifolds of suspension of diffeomorphisms in lift classes between which a topological conjugacy exist (see proposition 2.8) are homeomorphic.

It is clear that there is more than one such manifold, since if there is only one, it is that of the lift class $\text{Diff}^r(T^2)(\text{Id})$, i.e., a torus, implying that every diffeomorphism is isotopic to id, a contradition of theorem 4.22A.

Question: Classify manifolds of suspension of elements of $diff(T^2)$ up to diffeomorphism.

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