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QUOTIENT CATEGORIES AND RINGS OF QUOTIENTS[1,2]

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A Serre class in an abelian category $\mathcal{A}$ is a nonempty subclass $\mathcal{S}$ of $\mathcal{A}$ closed under subobjects, quotient objects, and extensions. The importance of such classes stems from the fact that it is for such classes $\mathcal{S}$ that the quotient category $\mathcal{A}/\mathcal{S}$ is defined [4]. Quotient categories provide the natural categorical setting for certain considerations. (See, for example, [4] and [16].)

In studying the categories $\mathcal{R}/\mathcal{S}$, with $\mathcal{R}$ the category of (left) R modules and $\mathcal{S}$ a Serre class of $\mathcal{R}$, the opposite ring $R_{\mathcal{S}}$ of the endomorphism ring of R as an object of $\mathcal{R}/\mathcal{S}$ plays a fundamental role. The ring $R_{\mathcal{S}}$ has many properties reminiscent of "rings of quotients," and $R_{\mathcal{S}}$ is examined from this point of view in §2. In particular, each of the various generalizations of rings of quotients known to the authors is an $R_{\mathcal{S}}$ for a suitable $\mathcal{S}$, and several examples are given. The modules $M_{\mathcal{S}}$ are also discussed, and it is indicated how these objects provide a reasonable generalization of rings and modules of quotients. This point of view unifies previous generalizations and places them in a natural categorical setting. §2 concludes with an examination of $R_{\mathcal{S}}$ in the case where R is a commutative Noetherian ring and $\mathcal{S}$ is the Serre class of modules with essential socles.

In [3], Gabriel has investigated the quotient categories $\mathcal{R}/\mathcal{S}$, where $\mathcal{R}$ is the category of all (left) modules over a ring R, and $\mathcal{S}$ is a Serre class closed under arbitrary infinite direct sums (a strongly complete Serre class). There is a canonical functor, called the localization functor, from $\mathcal{R}$ to $\mathcal{R}_{\mathcal{S}}$, the category of all left modules over $R_{\mathcal{S}}$. Every $R_{\mathcal{S}}$ module is an R module via a natural ring homomorphism $\phi : R \rightarrow R_{\mathcal{S}}$, $\mathcal{R}_{\mathcal{S}} \cap \mathcal{S}$ (the class of $R_{\mathcal{S}}$ modules which belong to $\mathcal{S}$ when considered as R modules via ϕ) is a Serre class of $\mathcal{R}_{\mathcal{S}}$, and in [3], a

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natural equivalence $\mathcal{R}/\mathcal{S} \to \mathcal{R}_{\mathcal{S}}/(\mathcal{R}_{\mathcal{S}} \cap \mathcal{S})$ is established. In particular, if $\mathcal{R}_{\mathcal{S}} \cap \mathcal{S} = \{0\}$, $\mathcal{R}/\mathcal{S}$ is equivalent to $\mathcal{R}_{\mathcal{S}}$. A sufficient condition for this to happen is that $\mathcal{F}(\mathcal{S}) = \{I \mid R/I \in \mathcal{S}\}$ have a cofinal set of finitely generated left ideals and that the localization functor be exact [3, Corollaire 2]. In §3, the converse is established (Theorem 3.2), and several other equivalent conditions given. It is further shown that $\mathcal{R}/\mathcal{S}$ is equivalent to a full subcategory of $\mathcal{R}_{\mathcal{S}}$, and that this subcategory is an exact subcategory if and only if the localization functor is exact (Theorem 3.13).

If R is a commutative ring, a Serre class $\mathcal{B}$ of $\mathcal{R}$ is bounded complete if it satisfies the condition: $R/(0 : B) \in \mathcal{B}$ if and only if $B \in \mathcal{B}$. Quotient categories $\mathcal{R}/\mathcal{B}$ for bounded complete Serre classes $\mathcal{B}$ provide a natural setting for the study of quasi-isomorphisms of abelian groups [16]. Two additional categories are associated with $\mathcal{R}/\mathcal{B}$, the category $\mathcal{H}_{\mathcal{B}}$ with objects the objects of $\mathcal{R}$ and with $\mathrm{Hom}_{\mathcal{H}_{\mathcal{B}}}(A, B) = \mathrm{Hom}_{\mathcal{R}/\mathcal{B}}(R, \mathrm{Hom}_R(A, B))$, and the category $\mathcal{T}_{\mathcal{B}}$ with objects the objects of $\mathcal{R}$ and with $\mathrm{Hom}_{\mathcal{T}_{\mathcal{B}}}(A, B) = R_{\mathcal{B}} \otimes_R \mathrm{Hom}_R(A, B)$. A natural equivalence $\mathcal{H}_{\mathcal{B}} \to \mathcal{R}/\mathcal{B}$ is established in 3.15. There is a canonical functor $\mathcal{T}_{\mathcal{B}} \to \mathcal{R}/\mathcal{B}$, and 3.16 together with 3.2 give several equivalent necessary and sufficient conditions for this functor to be an equivalence. An immediate corollary of these results is Theorem 3.1 in [16].

Serre classes $\mathcal{S}$ of $\mathcal{R}$ that are closed under arbitrary direct sums are in natural one-one correspondence with certain filters of left ideals of R, as has been pointed out in [3]. In §1, this phenomenon is further investigated, and natural one-one correspondences between several kinds of classes of $\mathcal{R}$ and filters of left ideals of R are established. The basic reason for these correspondences is that certain kinds of classes of R-modules are uniquely determined by the cyclic modules they contain, hence uniquely determined by the set of left ideals I of R such that R/I belongs to that class. The results of §1 are of special value in §5, which studies concordant and harmonic functors. A concordant functor is a left-exact subfunctor of the identity functor of an abelian category $\mathcal{A}$, and a harmonic functor is a right-exact quotient functor of the identity functor of $\mathcal{A}$. Such functors are in natural one-one correspondence with certain subclasses of $\mathcal{A}$ (5.10 and 5.14). In particular, if $\mathcal{A} = \mathcal{R}$ for some ring R, concordant functors are in natural one-one correspondence with filters of left ideals of R (5.11), and harmonic functors are in natural one-one correspondence with the two-sided ideals of R (5.15). These functors are of interest because of the relative homological algebras to which they give rise (5.6, 5.7).

§4 is devoted to the generalization of some of the results of [9]. If $\mathcal{S}$ is a Serre class of $\mathcal{R}$, a module $M \in \mathcal{R}$ is $\mathcal{S}$-injective if $\mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to 0$ is exact whenever $0 \to A \to B$ is exact and $B/A \in \mathcal{S}$. Various concepts in [9] such as "quasi-injective" and "closed," and theorems in [9] about these concepts are extended to their natural generalizations of quasi-$\mathcal{S}$-injective and $\mathcal{S}$-closed. Results in [9] are obtained from those in §4 by setting $\mathcal{S} = \mathcal{R}$. If P is a set of primes, a P-group is an abelian group, every element of which has order a product of powers of primes in P. If $\mathcal{P}$ is the class of all P-groups, then letting R be the ring of integers and $\mathcal{S} = \mathcal{P}$, the results of §4 also yield the principal results in [5].

1. **Some classes of modules.** Let R be a ring with identity and $\mathcal{R}$ the category of all unitary left R-modules. This section is concerned with the characterization of certain important subclasses of $\mathcal{R}$.

1.1. Definition. A *Serre class* of $\mathcal{R}$ is a nonempty subclass $\mathcal{S}$ of $\mathcal{R}$ such that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of R modules, then $B \in \mathcal{S}$ if and only if A and C are in $\mathcal{S}$. Equivalently, $\mathcal{S}$ is a nonempty subclass of $\mathcal{R}$ closed under submodules, homomorphic images and extensions.

The importance of Serre classes stems from the fact that it is for such classes $\mathcal{S}$ that the quotient category $\mathcal{R}/\mathcal{S}$ is defined [4]. It is a hopeless task to attempt to characterize all Serre classes for an arbitrary ring R. However, there are intimate relations between certain types of Serre classes and sets of ideals of R, as Gabriel has pointed out in [3]. In investigating this phenomenon further, it is profitable to begin by examining classes more general than Serre classes.

1.2. Definition. An *additive class* of $\mathcal{R}$ is a nonempty subclass $\mathcal{C}$ of $\mathcal{R}$ that is closed under submodules, homomorphic images and finite direct sums. An additive class $\mathcal{C}$ is *complete* if for any $C \in \mathcal{C}$ and any index set I, if $C_i \approx C$ for all $i \in I$, then $\sum_{i \in I} C_i \in \mathcal{C}$. (This sum will be written as $\sum_I C$.) An additive class $\mathcal{C}$ is *strongly complete* if $\mathcal{C}$ is closed under arbitrary infinite direct sums.

If R is commutative, it is easy to see that an additive class $\mathcal{C}$ is complete if and only if $A \otimes_R C \in \mathcal{C}$ for all $A \in \mathcal{R}$ and $C \in \mathcal{C}$. Another type of complete class which is of interest is the following.

1.3. Definition. A complete additive class is *bounded* if it is contained in every complete additive class with the same cyclic modules.

This terminology is motivated by the fact that nontrivial bounded complete classes of abelian groups are just those complete classes whose members are bounded groups. Note that a bounded complete additive class is uniquely determined by the cyclic modules it contains. Further, the following lemma asserts that the set of cyclic modules of any additive class is the set of cyclic modules of some bounded complete additive class.

1.4. LEMMA. *Let $\mathcal{C}$ be an additive class of $\mathcal{R}$ and let $\mathcal{C}_b$ be the class consisting of those modules that are a submodule of a homomorphic image of a finite sum $\sum_{i=1}^{n} A_i$, where each $A_i = \sum_{I_i} C_i$ for some cyclic module $C_i \in \mathcal{C}$, with I_i arbitrary index sets. Then $\mathcal{C}$ and $\mathcal{C}_b$ contain the same cyclic modules and $\mathcal{C}_b$ is the smallest complete additive class containing all the cyclic modules in $\mathcal{C}$.*

PROOF. It is clear that $\mathcal{C}_b$ and $\mathcal{C}$ contain the same cyclic modules, and it remains only to show that $\mathcal{C}_b$ is a complete additive class. Since a homomorphic image of a submodule of any module M is a submodule of a homomorphic image of M, $\mathcal{C}_b$ is closed under submodules and homomorphic images. If $B \in \mathcal{C}_b$, then there is an exact sequence

$$S = \sum_{i=1}^{n} A_i \to A \to 0$$

with $B \subset A$ and with $A_i = \sum_{I_i} C_i$ where C_i is a cyclic module in $\mathcal{C}$. For any set I,

$$\sum_I S \to \sum_I A \to 0$$

is exact and $\sum_I B \subset \sum_I A$. Now $\sum_I S$ is a module of the form described in the lemma, so that $\sum_I B \in \mathcal{C}_b$. Similarly, if $B_1, B_2 \in \mathcal{C}_b$ then $B_1 \oplus B_2 \in \mathcal{C}_b$.

1.5. LEMMA. *Let $\mathcal{C}$ be an additive class of $\mathcal{R}$, and let $\mathcal{C}_s$ be the class consisting of those modules that are homomorphic images of modules of the form $\sum_{\lambda \in \Lambda} C_\lambda$, with C_λ a cyclic module in $\mathcal{C}$. Then $\mathcal{C}_s$ and $\mathcal{C}$ have the same cyclic modules and $\mathcal{C}_s$ is a strongly complete additive class.*

PROOF. Clearly $\mathcal{C}_s$ and $\mathcal{C}$ have the same cyclic modules and $\mathcal{C}_s$ is closed under arbitrary direct sums and homomorphic images. Since every module is a homomorphic image of the (external) direct sum of the cyclic modules it contains, $\mathcal{C}_s$ is closed under submodules.

By 1.5, every additive class determines a strongly complete additive class with the same cyclic modules. But a strongly complete additive class is uniquely determined by the cyclic modules it contains. This proves

1.6. PROPOSITION. *$\mathcal{C} \to \mathcal{C}_s$ is a natural one-one correspondence between the bounded complete additive classes and the strongly complete additive classes of $\mathcal{R}$.*

Because of the significance of the set of cyclic modules which an additive class $\mathcal{C}$ contains (in particular, there being a unique bounded complete and a unique strongly complete additive class with the same set of cyclics as $\mathcal{C}$), it is natural to ask which sets of cyclic modules are the cyclic modules of an additive class. But every cyclic module of $\mathcal{R}$ is of the form R/I for some left ideal I of R. Thus the question is which sets $\mathcal{I}$ of left ideals of R are such that $\{R/I \mid I \in \mathcal{I}\}$ is the set of cyclic modules for some additive class $\mathcal{C}$.

1.7. LEMMA. *Let $\mathcal{C}$ be an additive class of $\mathcal{R}$ and let $\mathcal{I}(\mathcal{C}) = \{I \mid R/I \in \mathcal{C}\}$. Then*

(a) *Every left ideal of R containing an ideal in $\mathcal{I}(\mathcal{C})$ is in $\mathcal{I}(\mathcal{C})$.*

(b) *$I, J \in \mathcal{I}(\mathcal{C})$ imply $I \cap J \in \mathcal{I}(\mathcal{C})$.*

(c) *$I \in \mathcal{I}(\mathcal{C})$, $r \in R$ imply $I : r \in \mathcal{I}(\mathcal{C})$ (where $I : r = \{x \in R \mid xr \in I\}$).*

PROOF. Since $\mathcal{C}$ is closed under homomorphic images, (a) holds. The map $R/(I \cap J) \to (R/I) \oplus (R/J) : x + (I \cap J) \to (x + I,\ x + J)$ is a monomorphism, and (b) follows. The map $R/(I : r) \to (Rr + I)/I$ given by $x + (I : r) \to xr + I$ is an isomorphism and (c) follows.

1.8. DEFINITION. A set $\mathcal{I}$ of left ideals of R satisfying (a), (b), and (c) above is a *filter* of left ideals of R.

Thus every additive class yields a filter of ideals of R, and every filter yields a set of cyclic modules. Such a set of cyclic modules is the set of cyclic modules of some additive class, as shown by the following lemma. The proof is immediate and thus omitted.

1.9. LEMMA. *Let $\mathcal{I}$ be a filter of left ideals of R, and let $\mathcal{S}(\mathcal{I}) = \{M \in \mathcal{R} \mid 0 : m \in \mathcal{I}$ for each $m \in M\}$. Then $\mathcal{S}(\mathcal{I})$ is a strongly complete additive class and $\{R/I \mid I \in \mathcal{I}\}$ is the set of cyclic modules of $\mathcal{S}(\mathcal{I})$.*

The following theorem is announced by Gabriel in [3]. From the previous lemmas its proof is also immediate.

1.10. THEOREM (GABRIEL). *$\mathcal{S} \to \mathcal{I}(\mathcal{S})$ is a natural one-one correspondence between the strongly complete additive classes of $\mathcal{R}$ and the filters of left ideals of R. The inverse correspondence is $\mathcal{I} \to \mathcal{S}(\mathcal{I})$.*

Because of 1.10 and the one-one correspondence between strongly complete additive classes and bounded complete additive classes (1.6), there is a one-one correspondence between bounded complete additive classes and filters of left ideals of R. To establish this correspondence directly (that is, to get a correspondence for bounded complete additive classes similar to the correspondence $\mathcal{S} \to \mathcal{F}(\mathcal{S})$ for strongly complete additive classes), it is convenient to introduce the concept *weak annihilator*.

1.11. DEFINITION. Let $M \in \mathcal{R}$ and let I be a left ideal of R. An element $m \in M$ is *weakly annihilated* by I if there exist $r_1, \cdots, r_n \in R$ such that $(\bigcap_{i=1}^n (I : r_i))m = 0$. If every element of M is weakly annihilated by I, then M is *weakly annihilated* by I and I is a *weak annihilator* of M.

1.12. LEMMA. *Let $\mathcal{F}$ be a filter of left ideals of R and let $\mathcal{B}(\mathcal{F}) = \{M \in \mathcal{R} \mid M$ is weakly annihilated by some $I \in \mathcal{F}\}$. Then $\mathcal{B}(\mathcal{F})$ is a bounded complete additive class of $\mathcal{R}$ whose cyclic modules are the modules R/I with $I \in \mathcal{F}$.*

PROOF. Clearly $\mathcal{B}(\mathcal{F})$ is closed under submodules and homomorphic images. For $B_1, B_2 \in \mathcal{B}(\mathcal{F})$, let I_i be a weak annihilator of B_i, $I_i \in \mathcal{F}$. For $x \in B_1$, $y \in B_2$ let $r_1, \cdots, r_m$ and $s_1, \cdots, s_n \in R$ be such that $(\bigcap_{i=1}^m (I_1 : r_i))x = 0$ and $(\bigcap_{i=1}^n (I_2 : s_i))y = 0$. It may be assumed that $m = n$ and $r_i = s_i$. Since $(I_1 : r_i) \cap (I_2 : r_i) = (I_1 \cap I_2):r_i$, then $(\bigcap_{i=1}^n (I_1 : r_i)) \cap (\bigcap_{i=1}^n (I_2 : r_i)) = \bigcap_{i=1}^n ((I_1 \cap I_2) : r_i)$, which annihilates $x + y$. Hence $I_1 \cap I_2$ weakly annihilates $B_1 \oplus B_2$. If $B \in \mathcal{B}(\mathcal{F})$, a similar argument shows that if I weakly annihilates B then I weakly annihilates $\sum_\Lambda B$ for any index set Λ.

Now let $\mathcal{C}$ be a complete additive class containing the cyclic modules of $\mathcal{B}(\mathcal{F})$, and let $B \in \mathcal{B}(\mathcal{F})$. There is an $I \in \mathcal{F}$ which weakly annihilates B. Now $R/I \in \mathcal{B}(\mathcal{F})$ since $(I : r)(r + I) = 0$ for each $r \in R$. For $r_1, \cdots, r_n \in R$ the maps

$$R\Big/\left(\bigcap_{i=1}^n (I : r_i)\right) \to \sum_{i=1}^n (R/(I : r_i))$$

$$\text{with } r + \bigcap_{i=1}^n (I : r_i) \to \{r + (I : r_i)\}_i$$

and

$$R/(I : r_i) \to R/I \quad \text{with } r + (I : r_i) \to rr_i + I$$

are monomorphisms. For any set Λ, the sum $\sum_\Lambda (R/I)$ belongs to $\mathcal{C}$

since $R/I \in \mathcal{B}(\mathcal{F})$ (and hence is in $\mathcal{C}$). For $b \in B$, choose $r_1, \cdots, r_n$ such that $(\bigcap_{i=1}^{n} (I : r_i))b = 0$ and let $I_b = \bigcap_{i=1}^{n} (I : r_i)$. Then the above remarks show that $\sum_{b \in B} (R/I_b)$ is in $\mathcal{C}$. But there is an epimorphism $\sum_{b \in B} (R/I_b) \to B$ since $I_b b = 0$. Hence $B \in \mathcal{C}$, so that $\mathcal{B}(\mathcal{F}) \subset \mathcal{C}$.

Finally, if $R/J \in \mathcal{B}(\mathcal{F})$ then there are $K \in \mathcal{F}$ and $s_1, \cdots, s_m \in R$ such that $(\bigcap_{i=1}^{m} (K : s_i))(1 + J) = 0$, so that $\bigcap_{i=1}^{m} (K : s_i) \subset J$. Since $\bigcap_{i=1}^{m} (K : s_i) \in \mathcal{F}$ then $J \in \mathcal{F}$. This completes the proof.

This lemma and the fact that bounded complete additive classes are uniquely determined by their cyclic modules yields

1.13. Theorem. *$\mathcal{F} \to \mathcal{B}(\mathcal{F})$ is a natural one-one correspondence between the filters of left ideals of R and the bounded complete additive classes of $\mathcal{R}$. The inverse correspondence is $\mathcal{B} \to \mathcal{F}(\mathcal{B})$.*

Note that in the notation of 1.4 and 1.5, $(\mathcal{B}(\mathcal{F}))_s = \mathcal{S}(\mathcal{F})$ and $(\mathcal{S}(\mathcal{F}))_b = \mathcal{B}(\mathcal{F})$.

If R is commutative the concept of weakly annihilate is equivalent to that of annihilate, so that in this case $\mathcal{B}(\mathcal{F}) = \{M \in \mathcal{R} \mid IM = 0$ for some $I \in \mathcal{F}\}$.

There is a third type of additive class which is of interest, namely those additive classes that are the smallest additive class containing the cyclics in them. There is a one-one correspondence between filters and them too, and given a filter $\mathcal{F}$, the class of modules that are submodules of homomorphic images of modules of the form $\sum_{k=1}^{n} (R/I_k)$, $I_k \in \mathcal{F}$, is the smallest additive class containing the cyclics R/I with $I \in \mathcal{F}$. There does not seem to be a way to describe these classes purely in terms of annihilators. In fact the example of the additive class of finite abelian groups defies description in this way.

Let $\mathcal{F}$ be a filter of left ideals of R. In [3] Gabriel has stated necessary and sufficient conditions on $\mathcal{F}$ such that $\mathcal{S}(\mathcal{F})$ is a Serre class. Here different (but equivalent, and perhaps simpler) such conditions will be given, and the corresponding problem for $\mathcal{B}(\mathcal{F})$ will be considered.

A filter $\mathcal{F}$ is *multiplicative* if $I, J \in \mathcal{F}$ implies $IJ \in \mathcal{F}$ (where IJ is the left ideal of $\mathcal{R}$ generated by the products xy with $x \in I$ and $y \in J$).

1.14. Lemma. *If $\mathcal{C}$ is a complete Serre class of $\mathcal{R}$, then $\mathcal{F}(\mathcal{C})$ is a multiplicative filter.*

Proof. Let $I, J \in \mathcal{F}(\mathcal{C})$. The exact sequence

$$0 \to J/IJ \to R/IJ \to R/J \to 0$$

shows that $IJ \in \mathcal{F}(\mathcal{C})$ if $J/IJ \in \mathcal{C}$. To get $J/IJ \in \mathcal{C}$, define, for each $j \in J$,

$$f_j : R \to J/IJ : r \to rj + IJ.$$

Then $\operatorname{Ker} f_j \supset I$ so f_j induces an epimorphism $f_j : R/I \to (R_j + IJ)/IJ$. Now $R/I \in \mathcal{C}$ so $\sum_{j \in J} (R/I) \in \mathcal{C}$, and the (external) direct sum $\sum_{j \in J} (Rj + IJ)/IJ$ is in $\mathcal{C}$. But there is a natural epimorphism $\sum_{j \in J} (Rj + IJ)/IJ \to J/IJ$, whence $J/IJ \in \mathcal{C}$.

1.15. LEMMA. *Let $\mathcal{F}$ be a multiplicative filter and suppose R is commutative. Then $\mathcal{B}(\mathcal{F})$ is a bounded complete Serre class.*

PROOF. By 1.12 it suffices to show that if

$$0 \to A \to B \to C \to 0$$

is exact with A, $C \in \mathcal{B}(\mathcal{F})$ then $B \in \mathcal{B}(\mathcal{F})$. Assuming A, $C \in \mathcal{B}(\mathcal{F})$, there are ideals I, $J \in \mathcal{F}$ such that $IA = 0$ and $JC = 0$, R being commutative. But then $(IJ)B = 0$, whence $B \in \mathcal{B}(\mathcal{F})$.

From 1.13, 1.14 and 1.15 results

1.16. THEOREM. *Let R be commutative. Then $\mathcal{F} \to \mathcal{B}(\mathcal{F})$ is a natural one-one correspondence between the multiplicative filters of left ideals of R and the bounded complete Serre classes of $\mathcal{R}$. The inverse of this correspondence is $\mathcal{B} \to \mathcal{F}(\mathcal{B})$.*

For noncommutative rings R, necessary and sufficient conditions on $\mathcal{F}$ are not known such that $\mathcal{F} \to \mathcal{B}(\mathcal{F})$ is one-one between those filters and bounded complete Serre classes of $\mathcal{R}$. The following proposition gives a necessary condition, however.

1.17. PROPOSITION. *If $\mathcal{C}$ is a complete Serre class, then $I \in \mathcal{F}(\mathcal{C})$ if and only if there is a $J \in \mathcal{F}(\mathcal{C})$ such that $(I + J)/I$ is weakly annihilated by an ideal $K \in \mathcal{F}(\mathcal{C})$.*

PROOF. Since $\mathcal{B}(\mathcal{F}(\mathcal{C}))$ is the smallest complete additive class containing the cyclics R/I with $I \in \mathcal{F}(\mathcal{C})$, $\mathcal{B}(\mathcal{F}(\mathcal{C})) \subset \mathcal{C}$. If $(I + J)/I$ is weakly annihilated by $K \in \mathcal{F}(\mathcal{C})$ then $(I + J)/I \in \mathcal{B}(\mathcal{F}(\mathcal{C}))$ and hence in $\mathcal{C}$. Also $R/(I + J) \in \mathcal{C}$ since $J \in \mathcal{F}(\mathcal{C})$, so the exact sequence

$$0 \to (I + J)/I \to R/I \to R/(I + J) \to 0$$

yields $R/I \in \mathcal{C}$, and $I \in \mathcal{F}(\mathcal{C})$.

If $I \in \mathcal{F}(\mathcal{C})$ let $J = I$, $K = R$ and the condition is satisfied.

Call a filter $\mathcal{F}$ *weakly complete* if $I \in \mathcal{F}$ whenever there is a $J \in \mathcal{F}$ such that $(I + J)/I$ is weakly annihilated by some $K \in \mathcal{F}$.

Then it is easy to see that $\mathcal{F}$ weakly complete implies $\mathcal{F}$ is multiplicative. Furthermore, if R is commutative, it follows from 1.15 and 1.17 that $\mathcal{F}$ is weakly complete if and only if $\mathcal{F}$ is multiplicative. Also 1.17 says that if $\mathcal{C}$ is a complete additive class then $\mathcal{F}(\mathcal{C})$ is weakly complete.

Call a filter $\mathcal{F}$ *strongly complete* if $I \in \mathcal{F}$ whenever there is a $J \in \mathcal{F}$ such that $I:j \in \mathcal{F}$ for all $j \in J$.

1.18. LEMMA. *A filter $\mathcal{F}$ is strongly complete if and only if $\mathcal{S}(\mathcal{F})$ is a strongly complete Serre class.*

PROOF. Suppose $\mathcal{F}$ is strongly complete, and

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact with $A, C \in \mathcal{S}(\mathcal{F})$. For $b \in B$, there is a $J \in \mathcal{F}$ such that $Jg(b) = 0 = g(Jb)$. Thus $Jb \subset f(A)$. Let I be the annihilator of b. Then $I:j$ is the annihilator of jb for $j \in J$, and since $R/(I:j) \simeq Rjb \in \mathcal{S}(\mathcal{F})$, then $I:j \in \mathcal{F}(\mathcal{S}(\mathcal{F})) = \mathcal{F}$. Now $I:j \in \mathcal{F}$ for all $j \in J$ implies $I \in \mathcal{F}$. It follows that $B \in \mathcal{S}(\mathcal{F})$ and that $\mathcal{S}(\mathcal{F})$ is a strongly complete Serre class.

Suppose $\mathcal{S}(\mathcal{F})$ is a strongly complete Serre class, that $J \in \mathcal{F}$ and that $I:j \in \mathcal{F}$ for all $j \in J$. Now $R/(I:j) \simeq (Rj + I)/I$ via $x + I:j \to xj + I$, so that $(Rj + I)/I \in \mathcal{S}(\mathcal{F})$ for all $j \in J$. Thus the external direct sum $\sum_{j\in J}((Rj + I)/I) \in \mathcal{S}(\mathcal{F})$. An epimorphic image of this sum is $(J + I)/I$, whence $(J + I)/I \in \mathcal{S}(\mathcal{F})$. The exact sequence

$$0 \to (J + I)/I \to R/I \to R/(J + I) \to 0$$

yields $R/I \in \mathcal{S}(\mathcal{F})$ and $I \in \mathcal{F}$.

From 1.10 and 1.18 follows

1.19. THEOREM (GABRIEL). *$\mathcal{F} \to \mathcal{S}(\mathcal{F})$ is a natural one-one correspondence between the strongly complete filters of R and the strongly complete Serre classes of $\mathcal{R}$.*

When R is a commutative ring, the filters which arise in the following way will be of special interest.

1.20. PROPOSITION. *Let R be a commutative ring and P a prime ideal of R. Then $\mathcal{F}_P = \{I \subset R \mid I \not\subset P\}$ is a strongly complete and multiplicative filter.*

PROOF. $\mathcal{F}_P$ is clearly a filter, and is multiplicative. Suppose $I:j \in \mathcal{F}_P$ for all $j \in J$, with $J \in \mathcal{F}_P$. Let $j \in J$ with $j \notin P$ and $x \in I:j$ with $x \notin P$. Then $xj \in I$ but $xj \notin P$, so that $I \not\subset P$. Thus $I \in \mathcal{F}_P$, and $\mathcal{F}_P$ is strongly complete.

Attention is now turned briefly to the case where R is Noetherian.

1.21. LEMMA. *Let R be a ring and $\mathcal{S}$ a Serre class of $\mathcal{R}$. If $\mathcal{I}(\mathcal{S})$ contains a cofinal set of finitely generated ideals, then $\mathcal{I}(\mathcal{S})$ is a multiplicative filter.*

PROOF. Let $I, J \in \mathcal{I}(\mathcal{S})$. There is a finitely generated $K \in \mathcal{I}(\mathcal{S})$ with $K \subset J$. Let $j_1, \cdots, j_n$ generate K, and let $L = K + IJ$. Then $L \in \mathcal{I}(\mathcal{S})$, $\{j_i + IJ\}_i$ generates L/IJ and the maps

$$f_i : R/I \to (Rj_i + IJ)/IJ \quad \text{with } x + I \to xj_i + IJ$$

are epimorphisms. Hence there is an epimorphism $\sum_{i=1}^{n} (R/I) \to L/IJ$. But $\sum_{i=1}^{n} (R/I) \in \mathcal{S}$, whence $L/IJ \in \mathcal{S}$. The exact sequence

$$0 \to L/IJ \to R/IJ \to R/L \to 0$$

shows that $IJ \in \mathcal{I}(\mathcal{S})$.

1.22. PROPOSITION. *If R is a commutative Noetherian ring then a filter $\mathcal{I}$ of R is multiplicative if and only if it is strongly complete.*

PROOF. If $\mathcal{I}$ is strongly complete, $\mathcal{I} = \mathcal{I}(\mathcal{S}(\mathcal{I}))$ is multiplicative by 1.21.

Suppose $\mathcal{I}$ is multiplicative and

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact with $A, C \in \mathcal{S}(\mathcal{I})$. For $b \in B$, there is a $J \in \mathcal{I}$ such that $Jg(b) = 0 = g(Jb)$. Thus $Jb \subset f(A)$. Let $j_1b, \cdots, j_nb$ generate Jb. There exist $I_1, \cdots, I_n \in \mathcal{I}$ such that $I_ij_ib = 0$. Thus $(\bigcap_{i=1}^{n} I_i)Jb = 0$. Now $I = \bigcap_{i=1}^{n} I_i \in \mathcal{I}$ so $IJ \in \mathcal{I}$, whence $B \in \mathcal{S}(\mathcal{I})$ and $\mathcal{S}(\mathcal{I})$ is a strongly complete Serre class. Since $\mathcal{I}(\mathcal{S}(\mathcal{I})) = \mathcal{I}$, $\mathcal{I}$ is strongly complete by 1.18.

Of course one can give an easy direct (purely ring theoretic) proof of 1.22.

1.23. COROLLARY. *Let R be a commutative Noetherian ring. Then for any Serre class $\mathcal{S}$ of $\mathcal{R}$, there is a bounded complete and a strongly complete Serre class with the same cyclic modules as $\mathcal{S}$. Further, a set S of cyclic modules is the set of cyclic modules of a Serre class of $\mathcal{R}$ if and only if $\{I \mid R/I \in S\}$ is a multiplicative filter of R.*

1.24. PROPOSITION. *Let R be a commutative Noetherian ring and $\mathcal{I}$ a strongly complete filter of ideals of R. Then*

$$\mathcal{I} = \bigcap_{P \text{ prime};\, P \notin \mathcal{I}} \mathcal{I}_P \quad \text{and} \quad \mathcal{S}(\mathcal{I}) = \bigcap_{P \text{ prime};\, P \notin \mathcal{I}} \mathcal{S}(\mathcal{I}_P)$$

where $\mathcal{I}_P = \{I \subset R \mid I \not\subset P\}$.

Proof. Let $\mathcal{P} = \{P \mid P$ is a prime ideal of R and $P \notin \mathcal{I}\}$. Clearly $\mathcal{I} \subset \bigcap_{P \in \mathcal{P}} \mathcal{I}_P$. Let $I \in \bigcap_{P \in \mathcal{P}} \mathcal{I}_P$ and let $I = Q_1 \cap \cdots \cap Q_n$ be a representation of I as the intersection of primary ideals. Now $I \subset P_i = \sqrt{Q_i}$ so $P_i \in \mathcal{I}$, $i = 1, \cdots, n$, but for some integer k, $P_1{}^k \cap \cdots \cap P_n{}^k \subset I$ so $I \in \mathcal{I}$ since $P_i{}^k \in \mathcal{I}$ by 1.22. The equality $\mathcal{S}(\mathcal{I}) = \bigcap_{P \in \mathcal{P}} \mathcal{S}(\mathcal{I}_P)$ follows easily since both sides are strongly complete Serre classes, so are completely determined by the cyclics in them by 1.19.

Of the filters of ideals in a ring, the maximal ones, namely the ultrafilters, should be of some special significance. A characterization of such filters would seem to be worthwhile, but it is probably very difficult. For commutative Noetherian rings a characterization of the maximal multiplicative (equivalently, maximal strongly complete) filters follows.

1.25. Lemma. *Let $\mathcal{I}$ be a maximal multiplicative filter of a commutative ring R. Then $I \notin \mathcal{I}$ if and only if there is a $J \in \mathcal{I}$ and an integer $n > 0$ such that $I^nJ = 0$.*

Proof. If for every $n > 0$ and $J \in \mathcal{I}$, $I^nJ \neq 0$, then the ideals containing an ideal of the form I^nJ form a multiplicative filter, include $I^1R = I$ and the ideals of $\mathcal{I}$, and do not include 0. The other implication is obvious.

1.26. Lemma. *Let $\mathcal{I}$ be a maximal multiplicative filter of the commutative ring R, and suppose P is a prime ideal of R such that $P \notin \mathcal{I}$. Then*

(a) *P is a minimal prime ideal of R.*
(b) *$\mathcal{I} = \{I \mid I \not\subset P\}$.*

Proof. Suppose $Q \subset P$, Q prime. Then there is a $J \in \mathcal{I}$ and an integer $n > 0$ such that $P^nJ = 0 \subset Q$. Since $J \not\subset Q$ and Q is prime, $P \subset Q$. Hence $P = Q$ and P is a minimal prime, proving (a).

Note that $\mathcal{I} \subset \{I \mid I \not\subset P\}$ since $P \notin \mathcal{I}$. But I, $J \not\subset P$ implies $IJ \not\subset P$. It is easily seen that $\{I \mid I \not\subset P\}$ is a multiplicative filter, and (b) follows.

1.27. Lemma. *Let P be a prime ideal of the commutative ring R. Suppose there is an ideal $I \not\subset P$ and an integer $n > 0$ such that $P^nI = 0$. Then $\mathcal{I} = \{J \mid J \not\subset P\}$ is a maximal multiplicative filter.*

Proof. Clearly $\mathcal{I}$ is a proper multiplicative filter. Any bigger multiplicative filter would contain P and hence would contain $P^nI = 0$. Thus $\mathcal{I}$ is as asserted.

1.28. LEMMA. *Let R be a commutative Noetherian ring and P a minimal prime ideal of R. Then there is an ideal $I \not\subset P$ and an integer $n > 0$ such that $P^nI = 0$.*

PROOF. The ideal P is an isolated prime of the 0 ideal. If P is the only minimal prime, then P is nilpotent and one may take $I = R$. Otherwise, let $P = P_1, P_2, \cdots, P_n$ be the associated prime ideals of 0. Now $0 = Q_1 \cap \cdots \cap Q_n$ with $P_i = \sqrt{Q_i}$. There is an $m > 0$ such that $P_i^m \subset Q_i$ for all i. Hence $P_1^mP_2^m \cdots P_n^m \subset P_1^m \cap P_2^m \cap \cdots \cap P_n^m = 0$. Now if $I = P_2^m \cdots P_n^m \subset P_1$ then $P_i \subset P_1 = P$ for some $i > 1$. Thus $I \not\subset P$ and $P^mI = 0$.

1.29. THEOREM. *Let R be a commutative Noetherian ring, and let $P_1, \cdots, P_n$ be the minimal prime ideals of R. Then $P_i \to \mathcal{F}_i = \{I \mid I \not\subset P_i\}$ is a one-one correspondence between the minimal primes of R and the maximal multiplicative filters of R.*

PROOF. If $\mathcal{F}$ is a maximal multiplicative filter of R and contains all the prime ideals of R then $0 = P_1^m \cdots P_n^m$ for some m implies $0 \in \mathcal{F}$ which is impossible. Thus there is a prime ideal $P \notin \mathcal{F}$, it is minimal by 1.26(a), and 1.26(b) says that $\mathcal{F} = \{I \mid I \not\subset P\}$. Now 1.27 and 1.28 complete the proof.

2. **Rings and modules of quotients.** Rings of quotients, modules of quotients and the generalizations of these which have appeared nearly all have been or can be interpreted as direct limits of certain groups of homomorphisms. Most of these direct limits can be identified with endomorphism rings or homomorphism groups in an appropriate quotient category. Several examples will be given to illustrate this connection after a brief discussion of quotient categories and relevant maps.

Let $\mathcal{R}$ denote the category of left R modules, and let $\mathcal{S}$ be a Serre class of $\mathcal{R}$. The ring of endomorphisms of R in the quotient category $\mathcal{R}/\mathcal{S}$ is

$$\varinjlim_{R/I, S \in \mathcal{S}} \operatorname{Hom}_R(I, R/S).$$

Let $R_{\mathcal{S}}$ denote the opposite ring of this ring of endomorphisms. There is a natural ring homomorphism $\operatorname{Hom}_R(R, R) \to \operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, R)$, inducing for the opposite rings a ring homomorphism $R \to R_{\mathcal{S}}$, which will be denoted throughout this paper by ϕ.

For $M, N \in \mathcal{R}$,

$$\operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(N, M) = \varinjlim_{N/N', M' \in \mathcal{S}} \operatorname{Hom}_R(N', M/M').$$

The elements of this group are equivalence classes of maps, and the equivalence class of a map f will be denoted by $[f]$. The natural bilinear map

$$\mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(R, R) \times \mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(R, M) \to \mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(R, M)$$

makes the group $\mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(R, M)$ into a right module over $\mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(R, R)$, and hence into a left module over $R_{\mathcal{S}}$. This left $R_{\mathcal{S}}$-module will be denoted by $M_{\mathcal{S}}$. Via the ring homomorphism $\phi : R \to R_{\mathcal{S}}$, $M_{\mathcal{S}}$ is also a left R-module. (In the case $\mathcal{S}$ is strongly complete, $M_{\mathcal{S}}$ coincides with $M_{\mathcal{G}}$ for $\mathcal{G} = \{I \subset R \mid R/I \in \mathcal{S}\}$, as defined in [1, p. 159].)

Homomorphisms $M \to M_{\mathcal{S}}$ are obtained as follows. For $M \in \mathcal{R}$, each $m \in M$ determines an R-homomorphism $\bar{m} : R \to M : r \to rm$, and hence the element $[\bar{m}]$ of $M_{\mathcal{S}}$. The map $\phi_M : M \to M_{\mathcal{S}} : m \to [\bar{m}]$ is an R-homomorphism, and for $M = R$, $\phi_R = \phi$ as defined above.

Let S be a multiplicative subset of the center of a ring R. Then $\mathcal{G}(S) = \{I \subset R \mid I$ is a left ideal of $\mathcal{R}$ and $Rs \subset I$ for some $s \in S\}$ is a strongly complete filter. Let $\mathcal{S} = \mathcal{S}(\mathcal{G}(S))$. The ring of quotients R_S can be identified with the ring $R_{\mathcal{S}}$ under the map $R_S \to R_{\mathcal{S}} : r/x \to [f]$, where $f : Rx \to R/(0 : x)r$ by the rule $f(yx) = yr + (0 : x)r$. Under this identification, the usual map $R \to R_S$ becomes $\phi : R \to R_{\mathcal{S}}$.

The extended centralizer of a ring R over a module M as defined by R. E. Johnson in [6] can be identified with

$$\varinjlim_{M' \in \mathcal{M}^*} \mathrm{Hom}_R(M', M)$$

where $\mathcal{M}^*$ is the set of essential submodules of M. In particular, if R has zero left singular ideal the left regular quotient ring of R is

$$\varinjlim_{I \in \mathcal{R}^*} \mathrm{Hom}_R(I, R)$$

where $\mathcal{R}^*$ denotes the set of essential left ideals of R. Assume no nonzero elements of R or M are annihilated by an essential left ideal. Then $\mathcal{R}^*$ is a strongly complete filter and the extended centralizer of R over M is the ring of endomorphisms of M in the quotient category $\mathcal{R}/\mathcal{S}(\mathcal{R}^*)$. In particular, the left regular quotient ring of R is $R_{\mathcal{S}(\mathcal{R}^*)}$.

The left maximal quotient ring of a ring R given by Y. Utumi [14] can be identified with

$$\varinjlim_{I \in R^\bullet} \mathrm{Hom}_R(I, R)$$

where $R^{\blacktriangle}$ is the set of left ideals I of R such that $\mathrm{Hom}_R(R/I, \bar{R}) = 0$ (with $\bar{R}$ denoting an injective envelope of R).

G. D. Findlay and J. Lambek [2] discuss rational extensions of modules. A rational extension of a module M can be identified with a unique submodule of

$$\varinjlim_{I \in M(R)} \mathrm{Hom}_R(I, M)$$

where $M(R)$ denotes the set of left ideals I of R for which $\mathrm{Hom}_R(R/I, \bar{M}) = 0$ (with $\bar{M}$ denoting an injective envelope of M). The direct limit itself yields a maximal rational extension of M. Now $M(R)$ and $R^{\blacktriangle} = R(R)$ are strongly complete filters of left ideals, $R_{\mathcal{S}(R^{\blacktriangle})}$ is the left maximal quotient ring of Y. Utumi [14], and $M_{\mathcal{S}(M(R))}$ is the maximal rational extension of M of G. D. Findlay and J. Lambek [2].

For any commutative ring R, a prime ideal P determines a strongly complete filter $\mathcal{I}_P = \{I \mid I \not\subset P\}$. This is the same filter as that determined by the multiplicative set $R \backslash P$. F. Richman [13] defines a ring of quotients of an integral domain R to be any subring S of the field of quotients Q of R which contains R and which is flat as an R module. He has shown that such rings are necessarily of the form $S = \bigcap_P R_P$ for some set of prime ideals P of R. That such rings S are of the form $R_{\mathcal{S}}$ for suitable $\mathcal{S}$ is a consequence of the fact that $R_P = R_{\mathcal{S}(\mathcal{I}_P)}$ and 2.1(b) below.

2.1. THEOREM. *Let R be an integral domain with field of quotients Q. Let $\mathcal{S}_\lambda (\lambda \in \Lambda)$ be strongly complete Serre classes of $\mathcal{R}$, and set $\mathcal{S} = \bigcap_{\lambda \in \Lambda} \mathcal{S}_\lambda$. Then*

(a) $R_{\mathcal{S}} = \{q \in Q \mid Iq \subset R$ *for some* $I \in \mathcal{I}(\mathcal{S})\}$,

(b) $R_{\mathcal{S}} = \bigcap_{\lambda \in \Lambda} R_{\mathcal{S}_\lambda}$.

PROOF. Since $\mathcal{S}$ is strongly complete, (a) follows from the facts that $R_{\mathcal{S}}/R \in \mathcal{S}$ and $Q/R_{\mathcal{S}}$ has no nonzero submodules in $\mathcal{S}$.

Since $\mathcal{S} \subset \mathcal{S}_\lambda$ for each λ, $R_{\mathcal{S}} \subset R_{\mathcal{S}_\lambda}$ and thus $R_{\mathcal{S}} \subset \bigcap_{\lambda \in \Lambda} R_{\mathcal{S}_\lambda}$. If $q \in \bigcap_{\lambda \in \Lambda} R_{\mathcal{S}_\lambda}$, then for each λ there is an $I_\lambda \in \mathcal{I}(\mathcal{S}_\lambda)$ such that $I_\lambda q \subset R$. But $\sum_{\lambda \in \Lambda} (I_\lambda q) = (\sum_{\lambda \in \Lambda} I_\lambda) q \subset R$ and $\sum_{\lambda \in \Lambda} I_\lambda \in \mathcal{S}$. Thus $q \in R_{\mathcal{S}}$, and (b) follows.

Now if R is an integral domain and $R \subset S \subset Q$, S is a generalized quotient ring of R in the sense of Richman if and only if $S = R_{\mathcal{S}(\mathcal{I})}$ where $\mathcal{I} = \bigcap_{P \text{ prime};\, SP \neq S} \mathcal{I}_P$ and $R_{\mathcal{S}(\mathcal{I})}\phi(P) = R_{\mathcal{S}(\mathcal{I})}$ for all primes $P \in \mathcal{I}$ (see [13, Theorem 1]). (The Serre classes $\mathcal{S}$ for which $R_{\mathcal{S}}\phi(I) = R_{\mathcal{S}}$ for all $I \in \mathcal{I}$ are studied further in §3.)

Let $\mathcal{I} = \mathcal{I}(\mathcal{S}) = \{I \in R \mid R/I \in \mathcal{S}\}$ be the filter of left ideals associated with a Serre class $\mathcal{S}$, and for $M \in \mathcal{R}$ let $\mathcal{I}M = \{x \in M \mid$

$0 : x \in \mathcal{I}\}$. It is easy to show that $\mathcal{I}M$ is a submodule of M, $\mathcal{I}M$ is generated by the set of submodules of M that belong to $\mathcal{S}$, and $\mathcal{I}M = \{x \in M \mid Rx \in \mathcal{S}\} = \operatorname{Ker} \phi_M$. (Some of this is included in [1, Exercise 19b, p. 159], in the case $\mathcal{S}$ is strongly complete.) Note that, if $\mathcal{S}$ is not strongly complete, $\mathcal{I}M$ is not necessarily in $\mathcal{S}$. Every finitely generated submodule of $\mathcal{I}M$ does, however, belong to $\mathcal{S}$. An R module M will be called $\mathcal{S}$-free if $\mathcal{I}M = 0$, or equivalently, if ϕ_M is a monomorphism.

The kernel of ϕ_M is described above. The following two propositions concern the image and cokernel of ϕ_M.

2.2. Proposition. *If* $\operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, \mathcal{I}M) = 0$, *then the image of* ϕ_M *is an essential* R *submodule of* $M_{\mathcal{S}}$.

Proof. Suppose $\operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, \mathcal{I}M) = 0$. Let $f : I \to M/M'$ represent a nonzero element of $M_{\mathcal{S}}$. From the homomorphism $\operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, \mathcal{I}M) \to \operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, M) = M_{\mathcal{S}}$, it follows that $f(I) \not\subset \mathcal{I}M/M'$. Thus there is an $i \in I$ with $f(i) = m + M'$, $m \notin \mathcal{I}M = \operatorname{Ker} \phi_M$. Now $i[f] = \phi(i)[f] = [f \circ \bar{i}]$, and $f \circ \bar{i}(x) = f(xi) = xm + M' = \bar{m}(x) + M'$ for $x \in R$. Thus $i[f] = [\bar{m}] = \phi_M(m) \neq 0$, and it follows that $\phi_M(M)$ is essential in $M_{\mathcal{S}}$.

The hypothesis of 2.2 is satisfied for all $M \in \mathcal{R}$ in many important cases. It is satisfied, for example, if $\mathcal{S}$ is strongly complete or if $\mathcal{I}(\mathcal{S})$ has a cofinal set of finitely generated ideals.

2.3. Proposition. *For each* $M \in \mathcal{R}$, $\mathcal{I}(\operatorname{Coker} \phi_M) = \operatorname{Coker} \phi_M$.

Proof. Let $[f] \in M_{\mathcal{S}}$, $f : I \to M/N$ with $I \in \mathcal{I} = \mathcal{I}(\mathcal{S})$ and $N \in \mathcal{S}$. Let $i \in I$ and $f(i) = m + N$. For $r \in R$, $f(\bar{i}(r)) = f(ri) = rf(i) = rm + N = \bar{m}(r) + N$, and since $(\bar{i}f)(r) = f(\bar{i}(r))$, $i[f] = [\bar{i}]\,[f] = [f\bar{i}] = [\bar{m}] = \phi_M(m) \in \phi_M(M)$. Thus $I[f] \subset \phi_M(M)$, and $I([f] + \phi_M(M)) = 0$ with $I \in \mathcal{I}$ implies $[f] + \phi_M(M) \in \mathcal{I}(M_{\mathcal{S}}/\phi_M(M)) = \mathcal{I}(\operatorname{Coker} \phi_M)$. The proposition follows.

Note that if $\mathcal{S}$ is strongly complete then $\operatorname{Ker} \phi_M$ and $\operatorname{Coker} \phi_M$ are in $\mathcal{S}$, whence ϕ_M induces an isomorphism between M and $M_{\mathcal{S}}$ in the category $\mathcal{R}/\mathcal{S}$.

When $\mathcal{S}$ is strongly complete, $M_{\mathcal{S}}$ can be obtained by a different method. Let E be an injective envelope of $M' = M/\mathcal{I}M$. By 2.2 it may be assumed that $M_{\mathcal{S}} \subset E$. Then $M_{\mathcal{S}}/M' = \mathcal{I}(E/M')$ [3, Proposition 4, p. 413]. Note that $\mathcal{I}(E/M_{\mathcal{S}}) = 0$ in this case.

Let M and $A \subset B$ be R modules. The set of left ideals $M(R) = \{I \subset R \mid \operatorname{Hom}_R(R/I, \bar{M}) = 0\}$ is a strongly complete filter, and $S \in \mathcal{S}(M(R))$ if and only if $\operatorname{Hom}_R(S, \bar{M}) = 0$. The symbol $A \leqq B(M)$ of [2] translates to $B/A \in \mathcal{S}(M(R))$. G. D. Findlay and J. Lambek call M *rationally complete* if every homomorphism $A \to M$ can be extended to a homomorphism $B \to M$, whenever $B/A \in \mathcal{S} = \mathcal{S}(M(R))$. They show this to be equivalent to the property that M has no proper rational extension [2, Theorem 4.1], i.e., $\mathrm{Ext}_R{}^1(S, M) = 0$ for all $S \in \mathcal{S}$. In light of the characterization of rational extensions mentioned earlier, this is equivalent to $\phi_M : M \to M_{\mathcal{S}(M(R))}$ being an isomorphism. The following proposition shows that the modules $M_{\mathcal{S}}$ satisfy similar completeness properties when $\mathcal{S}$ is any strongly complete Serre class, assuming that $\mathrm{Ker}\,\phi_M = 0$. Note that in the setting above, $\phi_M : M \to M_{\mathcal{S}(M(R))}$ always has zero kernel.

2.4. PROPOSITION. *If $\mathcal{S}$ is a strongly complete Serre class in $\mathcal{R}$, and M an R module, then*

(a) (i)–(iv) *are equivalent.*

(i) $\mathrm{Ext}_R{}^1(S, M) = 0$ *for all* $S \in \mathcal{S}$.

(ii) $\mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to 0$ *is exact whenever* $A \subset B$ *and* $B/A \in \mathcal{S}$.

(iii) $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M) \to 0$ *is exact for all* $I \in \mathcal{I}(\mathcal{S})$.

(iv) $\mathcal{I}(\bar{M}/M) = 0$, *where $\bar{M}$ is an injective envelope of M.*

(b) *If $M = A_{\mathcal{S}}$ for some R module A then $\phi_M : M \to M_{\mathcal{S}}$ is an isomorphism and the conditions* (i)–(iv) *above hold for M.*

(c) *If* $\mathrm{Ker}\,\phi_M = 0$ *and any of the conditions* (i)–(iv) *above hold, then ϕ_M is an isomorphism.*

PROOF. That (i) implies (ii) follows from the exact sequence

$$\mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to \mathrm{Ext}^1(B/A, M).$$

Condition (iii) is a special case of (ii). Assume (iii) holds and let $x \in \bar{M}$ such that $x + M \in \mathcal{I}(\bar{M}/M)$. Then $I = \{r \in R \mid rx \in M\}$ belongs to $\mathcal{I}$, and the homomorphism $f : I \to M : i \to ix$ can be extended to a homomorphism $\bar{f} : R \to M$. Let $m = \bar{f}(1)$. Then $I(x - m) = 0$, and $R(x - m) \cap M = I(x - m) \cap M = 0$, implying $x - m = 0$ since M is essential in $\bar{M}$. Thus $x \in M$ and $\mathcal{I}(\bar{M}/M) = 0$.

Assume (iv) holds. The exact sequence

$$0 \to M \to \bar{M} \to \bar{M}/M \to 0$$

yields the exact sequence

$$\mathrm{Hom}(S, \bar{M}/M) = 0 \to \mathrm{Ext}^1(S, M) \to \mathrm{Ext}^1(S, \bar{M}) = 0$$

so $\mathrm{Ext}^1(S, M) = 0$ for $S \in \mathcal{S}$.

Assume $M = A_{\mathcal{S}}$ with $\mathcal{S}$ strongly complete. Then $\mathcal{I}(A/\mathcal{I}A) = 0$ and $A/\mathcal{I}A$ an essential submodule of $A_{\mathcal{S}}$ imply $\mathcal{I}A_{\mathcal{S}} = 0$. Thus $\mathrm{Ker}\,\phi_M = 0$. Let E be an injective envelope of $A/\mathcal{I}A$. Then

$A/\mathcal{T}A \subset A_{\mathcal{S}} \subset E$ so E is also an injective envelope of $A_{\mathcal{S}}$, i.e., $E = \bar{M}$. Thus $\mathcal{T}(\bar{M}/M) = \mathcal{T}(E/A_{\mathcal{S}}) = 0$, and $\operatorname{Cok}\phi_M \subset \mathcal{T}(\bar{M}/M)$ implies $\operatorname{Cok}\phi_M = 0$. Thus ϕ_M is an isomorphism, and since $\mathcal{T}(\bar{M}/M) = 0$, (i)–(iv) hold for M.

Assume $\operatorname{Ker}\phi_M = 0$ and (ii) holds for M. Since $\operatorname{Cok}\phi_M \in \mathcal{S}$, the sequence

$$\operatorname{Hom}(M_{\mathcal{S}}, M) \to \operatorname{Hom}(M, M) \to 0$$

must be exact, implying $\operatorname{Im}\phi_M$ is a summand of $M_{\mathcal{S}}$. Since it is also an essential submodule of $M_{\mathcal{S}}$, $\operatorname{Im}\phi_M$ must be equal to $M_{\mathcal{S}}$, hence ϕ_M is an isomorphism.

If $\mathcal{S}$ is not a strongly complete Serre class the modules $M_{\mathcal{S}}$ may not be complete in the sense described above. However certain homomorphisms can always be extended.

2.5. Proposition. *Let $\mathcal{S}$ be any Serre class of $\mathcal{R}$. If $I \in \mathcal{T}(\mathcal{S})$ and $M \in \mathcal{R}$ with $f : I \to M$ an R-homomorphism, then there exists an R-homomorphism $g : R \to M_{\mathcal{S}}$ such that the diagram*

$$\begin{array}{ccc} I & \xrightarrow{f} & M \\ \downarrow & & \downarrow{\scriptstyle \phi_M} \\ R & \xrightarrow{g} & M_{\mathcal{S}} \end{array}$$

commutes (with $I \to R$ the inclusion map).

Proof. The map $g : R \to M_{\mathcal{S}} : r \to r[f]$ is an R-homomorphism, and for $i \in I$, $g(i) = \phi_M(f(i))$.

The following proposition describes the group of extensions of a module A by a cyclic $R/I \in \mathcal{S}$ as a subgroup of the cokernel of ϕ_A. For an R module M and a left ideal I, let $M[I]$ denote the subgroup of M annihilated by I, i.e., $M[I] = \{x \in M \mid Ix = 0\}$.

2.6. Proposition. *Let $\mathcal{S}$ be a Serre class of $\mathcal{R}$ and $A \in \mathcal{R}$ with $\mathcal{T}A = 0$. Then for each $I \in \mathcal{T}(\mathcal{S})$,*

$$\operatorname{Ext}_R{}^1(R/I, A) \simeq (A_{\mathcal{S}}/\phi_A(A))[I] = (\operatorname{Coker}\phi_A)[I].$$

Proof. In the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Hom}_R(R, A) & \to & \operatorname{Hom}_R(I, A) & \to & \operatorname{Ext}_R{}^1(R/I, A) & \to & 0 \\ & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \theta} & & & & \\ 0 & \to & A & \xrightarrow{\phi_A} & \operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, A) = A_{\mathcal{S}} & & & & \end{array}$$

with exact rows, θ is a monomorphism, and it suffices to show that $(\mathrm{Im}\ \theta)/\phi_A(A) = (A_{\mathcal{S}}/\phi_A(A))[I]$. Let $[f] \in \mathrm{Im}\ \theta$, $f: I \to A$. For $i \in I$, $r \in R$, $(f\bar{i})(r) = f(\bar{i}(r)) = f(ri) = rf(i) = f(i)(r)$, so $i[f] = [f\bar{i}] = [f(i)] = \phi_A(f(i)) \in \phi_A(A)$. Thus $I(\mathrm{Im}\ \theta) \subset \phi_A(A)$. Now let $x \in A_{\mathcal{S}}$ with $Ix \subset \phi_A(A)$ and define $g: I \to A : i \to \phi_A{}^{-1}(ix)$. Then for $i \in I$, $i[g] = \phi_A(g(i)) = ix$, so $I([g] - x) = 0$. Since $\mathcal{T}(A_{\mathcal{S}}) = 0$, this implies $x = [g] = \theta(g)$ and the proposition follows.

2.7. THEOREM. *Let R be an integral domain and $\mathcal{S}$ a strongly complete Serre class of $\mathcal{R}$. If R is integrally closed then $R_{\mathcal{S}}$ is integrally closed.*

PROOF. Suppose $x^n + s_{n-1}x^{n-1} + \cdots + s_1x + s_0 = 0$ with $s_i \in R_{\mathcal{S}}$ and $x \in Q$ (Q is the field of quotients of R). There is an $I \in \mathcal{T}(\mathcal{S})$ such that $Is_i \subset R$ for $i = 0, \cdots, n-1$, by 2.3. Let $y \in I$. Then $(yx)^n + (s_{n-1}y)(yx)^{n-1} + \cdots + s_1y^{n-1}(yx) + y^ns_0 = 0$ with $s_{n-k}y^k \in R$. Since R is integrally closed, $yx \in R$, and hence $Ix \subset R$. This means that $x \in R_{\mathcal{S}}$, and so $R_{\mathcal{S}}$ is integrally closed.

For an integral domain R, let R' be its integral closure. It is easy to see that for any strongly complete Serre class $\mathcal{S}$ of $\mathcal{R}$, $(R_{\mathcal{S}})' \subset (R')_{\mathcal{S}}$, though equality does not seem to hold in general. However, $((R_{\mathcal{S}})')_{\mathcal{S}} = (R')_{\mathcal{S}}$, which by 2.7 is integrally closed.

The *socle* of an R module is the submodule generated by the simple submodules. If M is a maximal ideal in R, the *M-socle* of an R module is the submodule generated by the simple submodules isomorphic to R/M. If R is a commutative Noetherian ring and S is the socle of an R module A, then S decomposes uniquely into the direct sum $\sum S_M$ of its *M-socles* S_M. Furthermore, S and S_M have unique maximal essential extensions E and E_M in A, and $E = \sum E_M$ [12]. Thus a natural definition of a "torsion" module over a commutative Noetherian ring is one whose socle is essential, and this class will now be examined.

2.8. THEOREM. *Let R be a commutative Noetherian ring and let $\mathcal{J}$ be the class of R modules whose socles are essential. Then $\mathcal{J}$ is a strongly complete Serre class. Moreover $\mathcal{T}(\mathcal{J}) = \bigcap_{P \in \mathcal{P}} \mathcal{T}_P$, where $\mathcal{P}$ is the set of prime ideals which are not maximal ideals, and is the smallest multiplicative (strongly complete) filter containing the maximal ideals of R.*

PROOF. Observe that an ideal I of R contains a product of maximal ideals if and only if every prime ideal containing I is maximal. The set $\mathcal{T}$ of such ideals is clearly $\bigcap \{\mathcal{T}_P \mid P$ is prime and not maximal$\}$, which is strongly complete (and multiplicative). Now $T \in \mathcal{S}(\mathcal{T})$ if and only if for $t \in T$, $0 : t \in \mathcal{T}$. Thus $T \in \mathcal{S}(\mathcal{T})$ if and only if $T \in \mathcal{J}$

[12, Theorem 1]. Hence $\mathcal{T}$ is a strongly complete Serre class, and $\mathcal{D}(\mathcal{T}) = \mathcal{D}(\mathcal{S}(\mathcal{D})) = \mathcal{D}$ is as described in the theorem.

2.9. Corollary. *Let R be a commutative Noetherian ring, A an R module, A_t the maximum essential extension in A of the socle of A, E an injective envelope of $A' = A/A_t$, and $\mathcal{T}$ the Serre class of left R modules with essential socles. Then*

$$A_{\mathcal{T}}/A' = (E/A')_t,$$

i.e., the cokernel of ϕ_A is the maximum essential extension in E/A' of the socle of E/A'.

2.10. Corollary. *Let R be a commutative Noetherian domain with field of quotients Q, and let $\mathcal{T}$ be the Serre class of modules with essential socles. Then $R_{\mathcal{T}}/R$ is the maximum essential extension of the socle of Q/R.*

Let R be a commutative Noetherian ring, M a maximal ideal of R, and $\mathcal{T}_M$ the class of R modules whose M-socle is essential. Then $\mathcal{T}_M$ is a strongly complete Serre class of $\mathcal{R}$ and $\mathcal{D}(\mathcal{T}_M) = \bigcap_{P \text{ prime}; P \neq M} \mathcal{D}_P$ is the smallest multiplicative filter containing M. Statements analogous to the corollaries above, with $\mathcal{T}$ replaced by $\mathcal{T}_M$, are also valid. Further, $A_{\mathcal{T}}/A' = \sum_M ((A')_{\mathcal{T}_M}/A')$, where M ranges over all maximal ideals of R and $A' = A/A_t$. In particular, if R is an integral domain with field of quotients Q, and $R_{\mathcal{T}}/R = \sum_M T_M/R$ is the unique primary decomposition of $(Q/R)_t$, then T_M is a subring of Q and $T_M = R_{\mathcal{T}_M}$.

3. **Quotient categories.** In significant cases, when $\mathcal{S}$ is a strongly complete Serre class in $\mathcal{R}$, the quotient category $\mathcal{R}/\mathcal{S}$ is equivalent to the category of all modules over the ring $R_{\mathcal{S}}$ [3, Corollaire 2, p. 414]. This phenomenon will be examined closer, and then attention will be turned to quotient categories $\mathcal{R}/\mathcal{B}$ with $\mathcal{B}$ bounded complete. First, however, some canonical functors are studied in a more general setting.

Let $\mathcal{S}$ be any Serre class of $\mathcal{R}$. The canonical exact functor from $\mathcal{R}$ to $\mathcal{R}/\mathcal{S}$ will be denoted by F. (See [3] or [16] for the missing definitions. Also the terminology and notation of §2 will be used freely.) Let $\mathcal{R}_{\mathcal{S}}$ denote the category of all left $R_{\mathcal{S}}$ modules. There is a canonical functor $L : \mathcal{R} \to \mathcal{R}_{\mathcal{S}}$ given by $L(M) = M_{\mathcal{S}} = \mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(R, M)$. This functor is called the *localizing functor*, and is clearly covariant, additive and left exact.

3.1. Lemma. *If $\mathcal{S}$ is a Serre class of $\mathcal{R}$ and $\mathcal{D}(\mathcal{S})$ had a cofinal set of finitely generated ideals, then $L : \mathcal{R} \to \mathcal{R}_{\mathcal{S}}$ commutes with direct sums.*

PROOF. Let $[f] \in (\sum M_\alpha)_{\mathcal{S}} = \operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, \sum M_\alpha)$. Then $f: I \to (\sum M_\alpha)/S$ where R/I, $S \in \mathcal{S}$, and it may be assumed that I is finitely generated. Hence there exist $\alpha_1, \cdots, \alpha_n$ such that $f(I) \subset (\sum_{i=1}^n M_{\alpha_i} + S)/S$. Now f induces maps $f_i: I \to M_{\alpha_i}/(M_{\alpha_i} \cap S)$, and hence yields an element $\sum_{i=1}^n [f_i]$ of $\sum_\alpha \operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, M_\alpha) = \sum_\alpha (M_\alpha)_{\mathcal{S}}$. The map $[f] \to \sum_i [f_i]$ is the inverse of the natural map $\sum_\alpha \operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, M_\alpha) \to \operatorname{Hom}_{\mathcal{R}/\mathcal{S}}(R, \sum_\alpha M_\alpha)$ which always exists.

There is another functor $T: \mathcal{R} \to \mathcal{R}_{\mathcal{S}}$ which is of interest. The ring $R_{\mathcal{S}}$ is a right R module with $xr = x\phi(r)$ for $x \in R_{\mathcal{S}}$ and $r \in R$, and for $M \in \mathcal{R}$, $R_{\mathcal{S}} \otimes_R M$ is a left R module with $r(x \otimes m) = (\phi(r)x) \otimes m$. Let $T(M) = R_{\mathcal{S}} \otimes_R M$, and for $f: M \to M'$, $T(f) = 1 \otimes f$. There is a natural transformation $\theta: T \to L$ given by $\theta_M: R_{\mathcal{S}} \otimes_R M \to M_{\mathcal{S}}$ where for $x \in R_{\mathcal{S}}$, $m \in M$, $\theta_M(x \otimes m) = x\phi(m) \in M_{\mathcal{S}}$.

3.2. THEOREM. *Let $\mathcal{S}$ be a Serre class of $\mathcal{R}$. The following are equivalent*:

(a) *The localization functor L is exact and $\mathcal{J}(\mathcal{S})$ has a cofinal set of finitely generated left ideals.*

(b) *The natural transformation $\theta: T \to L$ is an equivalence.*

(c) *$R_{\mathcal{S}}\phi(I) = R_{\mathcal{S}}$ for all $I \in \mathcal{J}(\mathcal{S})$.*

(d) *$\mathcal{R}_{\mathcal{S}} \cap \mathcal{S} = 0$.*

PROOF. Assume (a). By the lemma above, L commutes with direct sums. Since L is also exact, by [3, Proposition 1 bis, p. 404] L has an adjoint $L^*: \mathcal{R}_{\mathcal{S}} \to \mathcal{R}$. Then for $A \in \mathcal{R}_{\mathcal{S}}$, $L^*(A) \approx \operatorname{Hom}_R(R, L^*(A)) \approx \operatorname{Hom}_{R_{\mathcal{S}}}(L(R), A)$, and for $M \in \mathcal{R}$, $\operatorname{Hom}_{R_{\mathcal{S}}}(L(M), A) \approx \operatorname{Hom}_R(M, L^*(A)) \approx \operatorname{Hom}_R(M, \operatorname{Hom}_{R_{\mathcal{S}}}(R_{\mathcal{S}}, A)) \approx \operatorname{Hom}_R(R_{\mathcal{S}} \otimes_R M, A)$. The isomorphisms are all natural, whence L is equivalent to the functor $M \to R_{\mathcal{S}} \otimes_R M$. This equivalence is in fact the transformation θ. Thus (a) implies (b).

Assume (b). Then (c) follows from the diagram

$$\begin{tikzcd}
 & & 0 & \\
0 \arrow[r] & R_{\mathcal{S}}\phi(I) \arrow[r] & R_{\mathcal{S}} \arrow[u] & \\
0 \arrow[r] & R_{\mathcal{S}} \otimes_R I \arrow[r, "\theta_I"] & I_{\mathcal{S}} \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

which commutes and has exact rows and columns.

Let I be an ideal of R such that $R_{\mathcal{S}}\phi(I) = R_{\mathcal{S}}$. There exist $[f_1], \cdots, [f_n] \in R_{\mathcal{S}}$ and $i_1, \cdots, i_n \in I$ with $\sum_k [f_k]\phi(i_k) = 1$. For some $J \in \mathcal{F}(\mathcal{S})$, $f_k : J \to R/S$ with $S \in \mathcal{S}$ and $j + S = \sum_k f_k(j) i_k$ for $j \in J$. Thus $J \subset (\sum_k Ri_k) + S$, implying $\sum_k Ri_k + S \in \mathcal{F}(\mathcal{S})$ and hence that $\sum_k Ri_k \in \mathcal{F}(\mathcal{S})$. In particular, (c) implies that $\mathcal{F}(\mathcal{S})$ has a cofinal set of finitely generated left ideals. Still assuming (c), let $A \in \mathcal{R}$ and $[f] \in A_{\mathcal{S}} = \mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(R, A)$, $f : I \to A/A'$ with R/I, $A' \in \mathcal{S}$. There is a sum $\sum_k [f_k] \phi(i_k) = 1$ with $f_k : J \to R/S$ and $i_k \in I$. For $i \in I$ and $j \in J$, let $f(i) = \hat{f}(i) + A'$ and $f_k(j) = \hat{f}_k(j) + S$. Let S' be the submodule of I generated by $I \cap S$, Si_1, $Si_2, \cdots, Si_n$. Then $S' \in \mathcal{S}$ and the submodule A'' of A with $A''/A' = f(S')$ belongs to $\mathcal{S}$. Let $g_k : R/S \to A/A''$ be defined by $g_k(r + S) = r\hat{f}(i_k) + A''$ for $r \in R$. Then $[f_k][\hat{f}(i_k)] = [g_k f_k]$ and $\theta_A(\sum [f_k] \otimes \hat{f}(i_k)) = \sum_k [f_k][\hat{f}(i_k)] = [\sum_k g_k f_k]$. For $j \in J \cap I$,

$$\left(\sum_k g_k f_k\right)(j) = \sum_k g_k(\hat{f}_k(j) + S) = \sum_k \hat{f}_k(j)\hat{f}(i_k) + A''$$
$$= \hat{f}\left(\sum_k \hat{f}_k(j) i_k\right) + A'' = \hat{f}(j) + A'',$$

since $\sum_k \hat{f}_k(j) i_k = j + s$ for some $s \in S$, and $j \in I$, $j + s \in I$ imply $s \in I \cap S$ and hence $\hat{f}(s) \in A''$. It follows that $[f] = \theta_A(\sum_k [f_k] \otimes \hat{f}(i_k))$. Thus (c) implies θ_A is an epimorphism for $A \in \mathcal{R}$. Let $0 \to A \to B \to C \to 0$ be exact in $\mathcal{R}$. Now (a) follows from the commutative diagram

$$\begin{array}{ccccccc}
 & & R_{\mathcal{S}} \otimes_R A & \to & R_{\mathcal{S}} \otimes_R B & \to & R_{\mathcal{S}} \otimes_R C \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A_{\mathcal{S}} & \to & B_{\mathcal{S}} & \to & C_{\mathcal{S}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

which has exact rows and columns. Thus (c) implies (a).

If $\mathcal{R}_{\mathcal{S}} \cap \mathcal{S} = 0$, then the map $R/I \to R_{\mathcal{S}}/R_{\mathcal{S}}\phi(I)$ induced by ϕ must be zero for $I \in \mathcal{F}(\mathcal{S})$. Thus $\phi(1) = 1 \in R_{\mathcal{S}}\phi(I)$, and (c) holds. Now $\mathcal{R}_{\mathcal{S}} \cap \mathcal{S}$ is a Serre class of $\mathcal{R}_{\mathcal{S}}$. Thus it is not trivial if and only if it contains a nonzero cyclic $R_{\mathcal{S}}/J$. The R-homomorphism $R \to R_{\mathcal{S}}/J$ induced by ϕ has kernel $I \in \mathcal{F}(\mathcal{S})$. Thus $\phi(I) \subset J$ so that $R_{\mathcal{S}}\phi(I) \subset J$. Thus (c) implies (d).

3.3. Corollary. *If any of the equivalent conditions in* 3.2 *hold, then $R_{\mathcal{S}}$ is a flat R module.*

In view of the equivalent conditions in the theorem, in particular condition (c), and this corollary, the rings $R_{\mathcal{S}}$ do indeed behave as one expects a "ring of quotients" to behave. If S is a multiplicative set in the center of the ring R and $\mathcal{F}$ is the filter of those left ideals which contain an ideal of the form Rs with $s \in S$, then $\mathcal{S}(\mathcal{F}) = \mathcal{S}$ is a strongly complete Serre class, and $R_{\mathcal{S}}$ is the usual ring of quotients of R with respect to the set S. The ring $R_{\mathcal{S}}$ satisfies the equivalent conditions of the theorem above, and condition (c) just expresses the fact that each element $s \in S$ has an inverse in $R_{\mathcal{S}}$.

The homomorphism $\phi : R \to R_{\mathcal{S}}$ makes every $R_{\mathcal{S}}$ module into an R module. This operation composed with F yields a canonical exact functor $E : \mathcal{R}_{\mathcal{S}} \to \mathcal{R}/\mathcal{S}$. The functorial diagram

$$\begin{tikzcd} \mathcal{R} \arrow[rr, "L"] \arrow[dr, "F"'] & & \mathcal{R}_{\mathcal{S}} \arrow[dl, "E"] \\ & \mathcal{R}/\mathcal{S} & \end{tikzcd}$$

is not necessarily commutative. Suppose for example that $\mathcal{S}$ is not strongly complete and that $\mathcal{F}(\mathcal{S})$ has a cofinal set of finitely generated ideals. Then there exists a family of R modules $M_\alpha \in \mathcal{S}$ with $M = \sum_\alpha M_\alpha \notin \mathcal{S}$. Thus $F(M) \neq 0$, but $L(M) = M_{\mathcal{S}} = 0$ by 2.1 so $EL(M) = 0$, and EL and F cannot be equivalent. However if $\mathcal{S}$ is strongly complete the diagram does commute as is shown in the following proposition.

3.4. PROPOSITION. *The functors EL and F are equivalent if $\mathcal{S}$ is strongly complete.*

PROOF. Let $f : M \to N$ be an R-homomorphism. For $m \in M$,

$$\phi_N f(m) = \overline{f(m)} = f\bar{m} = L(f)(\bar{m}) = L(f)\phi_M(m).$$

Thus the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi_M"] & N \arrow[d, "\phi_N"] \\ M_{\mathcal{S}} \arrow[r, "L(f)"] & N_{\mathcal{S}} \end{tikzcd}$$

commutes. Now $EL(f)F(\phi_M) = [\phi_M][L(f)] = [L(f)\phi_M] = [\phi_N f] = F(\phi_N f)$. Define $\Phi : F \to EL$ by $\Phi_M = F(\phi_M) = [\phi_M]$. Then the diagram

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, "\Phi_M"] & F(N) \arrow[d, "\Phi_N"] \\ EL(M) \arrow[r, "EL(f)"] & EL(N) \end{tikzcd}$$

commutes, and Φ is a natural transformation of functors. If $\mathcal{S}$ is strongly complete, then by §2, Ker ϕ_M and Coker ϕ_M are in $\mathcal{S}$ for each M, whence $\Phi_M = [\phi_M]$ is an isomorphism and Φ is an equivalence.

When $\mathcal{S}$ is a strongly complete Serre class, the list of equivalent statements in 3.2 can be extended as follows.

3.5. THEOREM. *Let $\mathcal{S}$ be a strongly complete Serre class of $\mathcal{R}$. The following are equivalent.*

(a) *The functor E is a natural equivalence between the categories $\mathcal{R}_{\mathcal{S}}$ and $\mathcal{R}/\mathcal{S}$.*

(b) $R_{\mathcal{S}}\phi(I) = R_{\mathcal{S}}$ *for all* $I \in \mathcal{F}(\mathcal{S})$.

PROOF. Clearly (a) implies $\mathcal{R}_{\mathcal{S}} \cap \mathcal{S} = 0$, which by 3.2 implies (b). Again by 3.2, (b) implies the localization functor is exact and $\mathcal{F}(\mathcal{S})$ has a cofinal set of finitely generated ideals. By [3, Corollaire 2, p. 414], this implies (a).

From its definition, the localization functor L is exact if and only if R is projective in $\mathcal{R}/\mathcal{S}$. Also L may be exact without E being an equivalence, as shown by letting $R = \prod_{\aleph_0} Z$ and $\mathcal{F}$ those ideals of R containing $\sum_{\aleph_0} Z$. In this example $\sum_{\aleph_0} Z$ is a projective ideal of R and $\{\sum_{\aleph_0} Z\}$ a cofinal subset of $\mathcal{F}$. Moreover $\sum_{\aleph_0} Z$ is not finitely generated so E cannot be an equivalence by 3.5 and 3.2. It is easy to observe that whenever $\mathcal{F}$ is a filter containing a cofinal set of projective left ideals, R is projective in $\mathcal{R}/\mathcal{S}$ for $\mathcal{F} = \mathcal{F}(\mathcal{S})$.

For $\mathcal{S}$ strongly complete, let $\mathcal{S}' = \mathcal{R}_{\mathcal{S}} \cap \mathcal{S}$. That is, $\mathcal{S}'$ is the class of $R_{\mathcal{S}}$ modules that are in $\mathcal{S}$ when considered as R modules. Proposition 3, p. 413 in [3] asserts that the functor E induces an equivalence between the categories $\mathcal{R}_{\mathcal{S}}/\mathcal{S}'$ and $\mathcal{R}/\mathcal{S}$. When $R_{\mathcal{S}} = R$, this proposition yields no information. In fact, in the example in the previous paragraph, $R_{\mathcal{S}(\mathcal{F})} = R$, as can be easily checked. However, for $\mathcal{S}$ strongly complete, $\mathcal{R}/\mathcal{S}$ is always equivalent to a subcategory of $\mathcal{R}_{\mathcal{S}}$ (as well as a subcategory of $\mathcal{R}$ itself), as will now be shown. Of course this subcategory will be $\mathcal{R}_{\mathcal{S}}$ itself if and only if $\mathcal{S}' = 0$. Further, the significance of the localization functor being exact is readily seen in terms of this subcategory.

3.6. LEMMA. *Let $\mathcal{S}$ be a strongly complete Serre class of $\mathcal{R}$. Then* $\mathrm{Hom}_{\mathcal{R}/\mathcal{S}}(A_{\mathcal{S}}, B_{\mathcal{S}}) \approx \mathrm{Hom}_R(A_{\mathcal{S}}, B_{\mathcal{S}})$, *for all* $A, B \in \mathcal{R}$.

PROOF. Let $A' \subset A_{\mathcal{S}}$ with $A_{\mathcal{S}}/A' \in \mathcal{S}$. Since $\mathcal{S}$ is strongly complete, $\mathrm{Hom}_R(A_{\mathcal{S}}/A', B_{\mathcal{S}}) = 0 = \mathrm{Ext}_R{}^1(A_{\mathcal{S}}/A', B_{\mathcal{S}})$. Thus $\mathrm{Hom}_R(A_{\mathcal{S}}, B_{\mathcal{S}}) \to \mathrm{Hom}_R(A', B_{\mathcal{S}})$, induced by the inclusion $A' \subset A_{\mathcal{S}}$, is an isomorphism. The lemma follows.

3.7. Lemma. *Let $\mathcal{S}$ be a strongly complete Serre class, and let $A, B \in \mathcal{R}_{\mathcal{S}}$ such that $\phi_A : A \to A_{\mathcal{S}}$ and $\phi_B : B \to B_{\mathcal{S}}$ are isomorphisms. Then*

$$\mathrm{Hom}_{R_{\mathcal{S}}}(A, B) = \mathrm{Hom}_R(A, B).$$

Proof. Let $x \in R_{\mathcal{S}}$ and $a \in A$, with $x = [g], g : I \to R$. Then $x\phi_A(a) = [g]\ [\bar{a}] = [\bar{a}g]$, and for $i \in I$, $\bar{a}g(i) = g(i)a = \phi(g(i))(a)$. Also $\phi_A(xa) = [\overline{xa}]$, and $\overline{xa}(i) = i(xa) = (\phi(i)x)a = [g\bar{i}]\,a = \phi(g(i))a$. Thus ϕ_A is an $R_{\mathcal{S}}$-homomorphism. Now let $f \in \mathrm{Hom}_R(A, B)$, $x = [g] \in R_{\mathcal{S}}$ with $g : I \to R$, $I \in \mathcal{F}(\mathcal{S})$. Define $\hat{f} : A_{\mathcal{S}} \to B_{\mathcal{S}} : \phi_A(a) \to \phi_B(f(a))$. This clearly is an R-homomorphism. Now $\hat{f}(x\phi_A(a)) = \hat{f}(\phi_A(xa)) = \phi_B f(xa) = [\overline{f(xa)}]$, and $x\hat{f}(\phi_A(a)) = x\phi_B(f(a)) = [g]\ [\overline{f(a)}] = [\overline{f(a)}g]$. For $i \in I$, $\overline{f(xa)}(i) = if(xa) = f(i(xa)) = f(g(i)a) = g(i)f(a) = \overline{f(a)}g(i)$. Thus $x\hat{f} = \hat{f}x$ and $\hat{f}$ is an $R_{\mathcal{S}}$-homomorphism. It follows that $f = \phi_B{}^{-1}\hat{f}\phi_A$ is an $R_{\mathcal{S}}$-homomorphism and hence that $\mathrm{Hom}_{R_{\mathcal{S}}}(A, B) = \mathrm{Hom}_R(A, B)$.

Let $\mathcal{C}_{\mathcal{S}}$ be the full subcategory of $\mathcal{R}$ whose objects are those modules A such that $\phi_A : A \to A_{\mathcal{S}}$ is an isomorphism. Assuming $\mathcal{S}$ is a strongly complete Serre class, the category $\mathcal{C}_{\mathcal{S}}$ is a full subcategory of $\mathcal{R}_{\mathcal{S}}$ in view of 3.7. Also the isomorphisms in 3.6 and 3.7 are functorial, and $\phi_A : A \to A_{\mathcal{S}}$ induces an isomorphism in $\mathcal{R}/\mathcal{S}$. These remarks establish

3.8. Theorem. *If $\mathcal{S}$ is a strongly complete Serre class of $\mathcal{R}$, then $\mathcal{R}/\mathcal{S}$ is equivalent in a natural way to a full subcategory of $\mathcal{R}$, the canonical functor $F : \mathcal{R} \to \mathcal{R}/\mathcal{S}$ induces an equivalence between $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{R}/\mathcal{S}$, and considering $\mathcal{C}_{\mathcal{S}}$ as a subcategory of $\mathcal{R}_{\mathcal{S}}$, the functor $E : \mathcal{R}_{\mathcal{S}} \to \mathcal{R}/\mathcal{S}$ induces an equivalence between $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{R}/\mathcal{S}$.*

It is assumed that $\mathcal{S}$ is strongly complete until mentioned otherwise (after 3.14).

It is perhaps more appropriate to consider $\mathcal{C}_{\mathcal{S}}$ as a subcategory of $\mathcal{R}_{\mathcal{S}}$. Note that $\mathcal{C}_{\mathcal{S}} = \mathcal{R}_{\mathcal{S}}$ if and only if $\mathcal{R}_{\mathcal{S}} \cap \mathcal{S} = 0$, and that the categories $\mathcal{R}_{\mathcal{S}}/(\mathcal{R}_{\mathcal{S}} \cap \mathcal{S})$ and $\mathcal{C}_{\mathcal{S}}$ are equivalent. It is interesting to relate monomorphisms, epimorphisms, kernels, cokernels, etc., in $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{R}_{\mathcal{S}}$. It is not true, for example, that an epimorphism in $\mathcal{C}_{\mathcal{S}}$ is always an epimorphism in $\mathcal{R}_{\mathcal{S}}$. The significance of the exactness of the localization functor will become clear in these considerations. Using the exact sequence

$$0 \to \mathrm{Hom}_R(A_{\mathcal{S}}/A, A) \to \mathrm{Hom}_R(A_{\mathcal{S}}, A)$$
$$\to \mathrm{Hom}_R(A, A) \to \mathrm{Ext}_R(A_{\mathcal{S}}/A, A),$$

it is easy to see that $\mathcal{C}_{\mathcal{S}}$ consists of those R modules A such that $\mathrm{Hom}_R(S, A) = 0 = \mathrm{Ext}_R{}^1(S, A)$ for all $S \in \mathcal{S}$.

3.9. LEMMA. *Let $A, B \in \mathcal{C}_{\mathcal{S}}$ and $f \in \mathrm{Hom}_R(A, B)$. Then* $\mathrm{Ker}_{\mathcal{R}}\, f \in \mathcal{C}_{\mathcal{S}}$, *and hence is a kernel of f in $\mathcal{C}_{\mathcal{S}}$.*

PROOF. Let $K = \mathrm{Ker}_{\mathcal{R}}\, f$. The diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & K \arrow[r] \arrow[d, "\phi_K"] & A \arrow[r, "f"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\
0 \arrow[r] & K_{\mathcal{S}} \arrow[r] & A_{\mathcal{S}} \arrow[r, "L(f)"] \arrow[d] & B_{\mathcal{S}} \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

is commutative with exact rows and columns. By the 5-lemma, ϕ_K is an isomorphism. The lemma follows readily.

3.10. LEMMA. *Let $A, B \in \mathcal{C}_{\mathcal{S}}$, $f \in \mathrm{Hom}_{\mathcal{R}}(A, B)$. Then $B \rightarrow (B/\mathrm{Im}_{\mathcal{R}}\, f)_{\mathcal{S}}$ is a cokernel in $\mathcal{C}_{\mathcal{S}}$ of f.*

PROOF. Since the sequence $A \xrightarrow{f} B \rightarrow B/\mathrm{Im} f \rightarrow 0$ is exact in $\mathcal{R}$, the sequence $A_{\mathcal{S}} \rightarrow B_{\mathcal{S}} \rightarrow (B/\mathrm{Im}\, f)_{\mathcal{S}} \rightarrow 0$ is exact in $\mathcal{R}/\mathcal{S}$, and hence exact in $\mathcal{C}_{\mathcal{S}}$.

3.11. PROPOSITION. *A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}_{\mathcal{S}}$ is exact in $\mathcal{C}_{\mathcal{S}}$ if and only if* $\mathrm{Ker}_{\mathcal{R}}\, g = (\mathrm{Im}_{\mathcal{R}}\, f)_{\mathcal{S}}$.

PROOF. By 3.9 and 3.10, $\mathrm{Ker}_{\mathcal{R}}\, g = \mathrm{Ker}_{\mathcal{C}_{\mathcal{S}}}\, g$ and $\mathrm{Im}_{\mathcal{C}_{\mathcal{S}}}\, f = \mathrm{Ker}_{\mathcal{R}}(B \rightarrow (B/\mathrm{Im}_{\mathcal{R}} f)_{\mathcal{S}})$. Now $0 \rightarrow \mathrm{Im}_{\mathcal{R}} f \rightarrow B \rightarrow B/\mathrm{Im}_{\mathcal{R}} f \rightarrow 0$ exact in $\mathcal{R}$ implies $0 \rightarrow (\mathrm{Im}_{\mathcal{R}} f)_{\mathcal{S}} \rightarrow B_{\mathcal{S}} \rightarrow (B/\mathrm{Im}_{\mathcal{R}} f)_{\mathcal{S}}$ is exact in $\mathcal{R}$, whence $\mathrm{Im}_{\mathcal{C}_{\mathcal{S}}}\, f = (\mathrm{Im}_{\mathcal{R}}\, f)_{\mathcal{S}}$. The proposition follows.

From 3.11 follows readily

3.12. COROLLARY. *Let $A, B, C \in \mathcal{C}_{\mathcal{S}}$. Then*

(a) *$f : A \rightarrow B$ is a monomorphism in $\mathcal{C}_{\mathcal{S}}$ if and only if f is a monomorphism in $\mathcal{R}$ (or $\mathcal{R}_{\mathcal{S}}$).*

(b) *$f : A \rightarrow B$ is an epimorphism in $\mathcal{C}_{\mathcal{S}}$ if and only if $B/\mathrm{Im}_{\mathcal{R}}\, f \in \mathcal{S}$.*

(c) *$0 \rightarrow A \rightarrow B \rightarrow C$ is exact in $\mathcal{C}_{\mathcal{S}}$ if and only if it is exact in $\mathcal{R}$ (or $\mathcal{R}_{\mathcal{S}}$).*

The fact that kernels and monomorphisms in $\mathcal{C}_{\mathcal{S}}$ are kernels and monomorphisms respectively in $\mathcal{R}_{\mathcal{S}}$ (or $\mathcal{R}$) is a reflection of the left exactness of the localization functor L. Cokernels and epimorphisms in $\mathcal{C}_{\mathcal{S}}$ may indeed not be cokernels and epimorphisms respectively in $\mathcal{R}_{\mathcal{S}}$. A subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is called an *exact subcategory* if the natural inclusion functor $\mathcal{C} \to \mathcal{D}$ is exact. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are abelian, a map in $\mathcal{C}$ will be epimorphism if and only if it is an epimorphism in $\mathcal{D}$.

3.13. THEOREM. *$\mathcal{C}_{\mathcal{S}}$ is an exact subcategory of $\mathcal{R}_{\mathcal{S}}$ (and hence of $\mathcal{R}$) if and only if the localization functor $L : \mathcal{R} \to \mathcal{R}_{\mathcal{S}}$ is exact.*

PROOF. Let $I : \mathcal{C}_{\mathcal{S}} \to \mathcal{R}_{\mathcal{S}}$ be the inclusion functor. There is a functor $L' : \mathcal{R} \to \mathcal{C}_{\mathcal{S}}$ with $IL' = L$. Moreover, L' is an exact functor. Thus I exact implies L exact. Suppose I is not exact. Then by 3.12(c), there is a $\mathcal{C}_{\mathcal{S}}$ epimorphism $f : A \to B$ such that $\mathrm{Im}_{\mathcal{R}} f \neq (\mathrm{Im}_{\mathcal{R}} f)_{\mathcal{S}} = B$. The sequence

$$0 \to \mathrm{Ker}_{\mathcal{R}} f \to A \to \mathrm{Im}_{\mathcal{R}} f \to 0$$

is exact in $\mathcal{R}$ but

$$0 \to (\mathrm{Ker}_{\mathcal{R}} f)_{\mathcal{S}} \to A_{\mathcal{S}} \to (\mathrm{Im}_{\mathcal{R}} f)_{\mathcal{S}} \to 0$$

is not exact in $\mathcal{R}_{\mathcal{S}}$. Hence L is not exact.

3.14. COROLLARY. *If the localization functor $L : \mathcal{R} \to \mathcal{R}_{\mathcal{S}}$ is exact and $\mathcal{G}(\mathcal{S})$ had a cofinal set of finitely generated left ideals, then $\mathcal{R}_{\mathcal{S}}$ is equivalent to a full exact subcategory of $\mathcal{R}$.*

PROOF. By 3.2, $\mathcal{R}_{\mathcal{S}} \cap \mathcal{S} = 0$, whence $\mathcal{C}_{\mathcal{S}} = \mathcal{R}_{\mathcal{S}}$. Apply 3.13.

Attention will now be turned to quotient categories $\mathcal{R}/\mathcal{B}$, where $\mathcal{B}$ is a bounded complete Serre class. Quotient categories of this type have important applications in the study of quasi-isomorphisms of torsion free abelian groups [16]. It seems that the concept of bounded complete Serre classes is natural only when R is commutative. In any case, *R is assumed to be commutative* in the considerations of $\mathcal{R}/\mathcal{B}$ below.

In the remainder of this section $\mathcal{B}$ denotes a bounded complete Serre class of $\mathcal{R}$. There are two other categories naturally associated with $\mathcal{R}/\mathcal{B}$. Let $\mathcal{T}_{\mathcal{B}}$ be the category with objects the R modules and with $\mathrm{Hom}_{\mathcal{H}_{\mathcal{B}}}(A, B) = R_{\mathcal{B}} \otimes_R \mathrm{Hom}_R(A, B)$. Let $\mathcal{H}_{\mathcal{B}}$ be the category with objects the R modules and with $\mathrm{Hom}_{\mathcal{H}_{\mathcal{B}}}(A, B) = \mathrm{Hom}_{\mathcal{R}/\mathcal{B}}(R, \mathrm{Hom}_R(A, B)) = (\mathrm{Hom}_R(A, B))_{\mathcal{B}}$. With the obvious definitions of composition of maps, both $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{H}_{\mathcal{B}}$ are additive categories. There are natural functors relating the categories $\mathcal{R}/\mathcal{B}$, $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{H}_{\mathcal{B}}$, and the main objective is to determine when these functors are equivalences.

Note that for $[f] \in \mathrm{Hom}_{\mathscr{R}/\mathscr{B}}(A, B)$, it may be assumed that $f: IA \to B/B[I]$ for some $I \in \mathscr{F}(\mathscr{B})$, where $B[I] = \{b \in B \mid Ib = 0\}$. Also $\mathrm{Hom}_R(A, B)[I] = \mathrm{Hom}_R(A, B[I])$ for all ideals I of R.

Let $[f] \in \mathrm{Hom}_{\mathscr{R}/\mathscr{B}}(R, \mathrm{Hom}_R(A, B))$ with

$$f : I \to \mathrm{Hom}_R(A, B)/\mathrm{Hom}_R(A, B[I]), \qquad I \in \mathscr{F}(\mathscr{B}).$$

Define $h_f: IA \to B/B[I]$ by $h_f(ia) = \hat{f}(i)(a) + B[I]$, where $f(i) = \hat{f}(i) + \mathrm{Hom}_R(A, B[I])$. Now defining $H(A) = A$, and $H([f]) = [h_f]$ yields an additive covariant functor $H : \mathscr{H}_{\mathscr{B}} \to \mathscr{R}/\mathscr{B}$.

3.15. Theorem. *Let $\mathscr{B}$ be a bounded complete Serre class in $\mathscr{R}$, with $\mathscr{R}$ commutative. Then the functor $H : \mathscr{H}_{\mathscr{B}} \to \mathscr{R}/\mathscr{B}$ is an equivalence. In particular, $\mathscr{H}_{\mathscr{B}}$ is an abelian category.*

Proof. It suffices to show that H induces an isomorphism

$$\mathrm{Hom}_{\mathscr{R}/\mathscr{B}}(R, \mathrm{Hom}_R(A, B)) \to \mathrm{Hom}_{\mathscr{R}/\mathscr{B}}(A, B).$$

Let $[g] \in \mathrm{Hom}_{\mathscr{R}/\mathscr{B}}(A, B)$, $g : IA \to B/B[I]$, $I \in \mathscr{F}(\mathscr{B})$. Define $\hat{g} : I \to \mathrm{Hom}_R(A, B/B[I])$ by $(\hat{g}(i))(a) = g(ia)$ for $i \in I$, $a \in A$. The exact sequence

$$0 \to \mathrm{Hom}_R(A, B[I]) \to \mathrm{Hom}_R(A, B)$$
$$\to \mathrm{Hom}_R(A, B/B[I]) \to \mathrm{Ext}_R{}^1(A, B[I])$$

induces a natural monomorphism

$$\mathrm{Hom}_R(A, B)/\mathrm{Hom}_R(A, B[I]) \xrightarrow{\alpha} \mathrm{Hom}_R(A, B/B[I])$$

with its cokernel annihilated by I, since $I(\mathrm{Ext}_R{}^1(A, B[I])) = 0$. Since $\mathscr{B}$ is a bounded complete Serre class, this cokernel is in $\mathscr{B}$. Thus $I' = \hat{g}^{-1}(\alpha)$ is an ideal belonging to $\mathscr{F}(\mathscr{B})$. The diagram

$$\begin{tikzcd}
I' \arrow[r] \arrow[d, dashed, "\tilde{g}"] & I \arrow[d, "\hat{g}"] \\
\mathrm{Hom}_R(A, B)/\mathrm{Hom}_R(A, B[I]) \arrow[r, "\alpha"] \arrow[d, "k"] & \mathrm{Hom}_R(A, B/B[I]) \\
\mathrm{Hom}_R(A, B)/\mathrm{Hom}_R(A, B[I'])
\end{tikzcd}$$

can be filled in uniquely (where k is induced by the inclusion $(I' \subseteq I)$, and the correspondence $[g] \to [k\tilde{g}] \in \mathrm{Hom}_{\mathscr{R}/\mathscr{B}}(R, \mathrm{Hom}_R(A, B))$ is an inverse of the homomorphism $\mathrm{Hom}_{\mathscr{R}/\mathscr{B}}(R, \mathrm{Hom}_R(A, B)) \to$

$\mathrm{Hom}_{\mathcal{R}/\mathcal{B}}(A, B)$ induced by H). This concludes the proof.

One may prove directly that $\mathcal{H}_{\mathcal{B}}$ is an abelian category, though it is quite laborious. For example, the proof of the existence of kernels proceeds as follows. Let $[f] \in \mathrm{Hom}_{\mathcal{H}_{\mathcal{B}}}(A, B) = \mathrm{Hom}_{\mathcal{R}/\mathcal{B}}(R, \mathrm{Hom}_R(A, B))$ with $f : I \to \mathrm{Hom}_R(A, B)/\mathrm{Hom}_R(A, B[I])$, $I \in \mathcal{I}(\mathcal{B})$, and $f(i) = \hat{f}(i) + \mathrm{Hom}_R(A, B[I])$ for $i \in I$. Let $K = \bigcap_{i \in I} \mathrm{Ker}_{\mathcal{R}} \hat{f}(i)$ with $j : K \to A$ the inclusion map, and define $k_f : R \to \mathrm{Hom}_R(K, A)$ by $k_f(1) = j$. Then $[k_f] \in \mathrm{Hom}_{\mathcal{H}_{\mathcal{B}}}(K, A)$ and it can be checked that K with $[k_f]$ is a kernel in $\mathcal{H}_{\mathcal{B}}$ for $[f]$.

A functor $D : \mathcal{T}_{\mathcal{B}} \to \mathcal{H}_{\mathcal{B}}$ is defined as follows. For $[f] \in \mathcal{R}_{\mathcal{B}}$ and $g \in \mathrm{Hom}_R(A, B)$ with $f : I \to R/R[I]$, $I \in \mathcal{I}(\mathcal{B})$, let $f(i) = \hat{f}(i) + R[I]$, and define $g_f : I \to \mathrm{Hom}_R(A, B)/\mathrm{Hom}_R(A, B)[I]$ by $g_f(i) = \hat{f}(i)g + \mathrm{Hom}_R(A, B)[I]$, where $(\hat{f}(i)g)(a) = \hat{f}(i)g(a)$. Let $D(A) = A$ for $A \in \mathcal{T}_{\mathcal{B}}$ and $D([f] \otimes g) = [g_f]$. Then D is an additive covariant functor. Also D composed with H yields a canonical functor $\mathcal{T}_{\mathcal{B}} \to \mathcal{R}/\mathcal{B}$.

3.16. THEOREM. *Let $\mathcal{B}$ be a bounded complete Serre class of $\mathcal{R}$, with R a commutative ring. The following are equivalent.*

(a) *$R_{\mathcal{B}}\phi(I) = R_{\mathcal{B}}$ for all $I \in \mathcal{I}(\mathcal{B})$.*

(b) *The canonical functor $HD : \mathcal{T}_{\mathcal{B}} \to \mathcal{R}/\mathcal{B}$ is an equivalence.*

PROOF. By 3.2, (a) is equivalent to the natural transformation $\theta : T \to L$ being an equivalence. One need only observe that the map $R_{\mathcal{B}} \otimes_R \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_{\mathcal{R}/\mathcal{B}}(R, \mathrm{Hom}_R(A, B))$ induced by the functor D is the same as the map $\theta_{\mathrm{Hom}_{\mathcal{R}}(A,B)}$. This together with 3.15 shows that (a) is equivalent to (b).

4. **Injectives relative to Serre classes.** The main purpose of this section is to generalize some of the results of [9]. Throughout, $\mathcal{S}$ will be a strongly complete Serre class of $\mathcal{R}$, and in the case $\mathcal{S} = \mathcal{R}$ various results in [9] are obtained. Let P be a set of primes. A P-group is an abelian group each element of which has order a product of powers of primes in P. Specializing $\mathcal{S}$ to the class of P-groups yields the principal results of [5].

4.1. DEFINITION. A module $M \in \mathcal{R}$ is $\mathcal{S}$-injective if for each exact sequence

$$0 \to A \to B \to S \to 0$$

of R modules with $S \in \mathcal{S}$, the associated sequence

$$0 \to \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to 0$$

is exact.

According to Proposition 2.4, the following are equivalent.

(a) M is $\mathcal{S}$-injective.

(b) $\mathrm{Ext}_R{}^1(S, M) = 0$ for all $S \in \mathcal{S}$.

(c) For each $I \in \mathcal{I}(\mathcal{S})$, the sequence $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I, M) \to 0$ is exact.

(d) $\mathcal{I}(\bar{M}/M) = 0$, where $\bar{M}$ is an injective envelope of M.

As before, $\mathcal{I} = \mathcal{I}(\mathcal{S})$ denotes the set of ideals I of R such that $R/I \in \mathcal{S}$, and for $M \in \mathcal{R}$, $\mathcal{I}M = \{m \in M \mid (0 : m) \in \mathcal{I}(\mathcal{S})\} = \{m \in M \mid Rm \in \mathcal{S}\}$ is the maximum submodule of M which belongs to $\mathcal{S}$. It is interesting to note that $\mathcal{S}$ is closed under injective envelopes if and only if $S \in \mathcal{S}$ and S $\mathcal{S}$-injective implies S is injective.

The module $\hat{M}$ obtained in 4.2 will play the role that the injective envelope of M plays in [9], and is an *$\mathcal{S}$-injective envelope* of M.

4.2. PROPOSITION. *Let $M \in \mathcal{R}$ and let $\bar{M}$ be an injective envelope of M. Let $\hat{M}$ be the (unique) largest submodule of $\bar{M}$ such that $\hat{M}/M \in \mathcal{S}$. Then $\hat{M}$ is $\mathcal{S}$-injective and is (up to isomorphism) the unique smallest $\mathcal{S}$-injective module containing M.*

Proposition 4.2 is proved by standard arguments.

4.3. DEFINITION. Let $M \in \mathcal{R}$. Then M is *quasi-$\mathcal{S}$-injective* if $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(N, M) \to 0$ is exact for all submodules N of M such that $M/N \in \mathcal{S}$.

4.4. THEOREM. *Let $M \in \mathcal{R}$ and $E = \mathrm{Hom}_R(\hat{M}, \hat{M})$. Then M is quasi-$\mathcal{S}$-injective if and only if M is an E submodule of $\hat{M}$; that is, if and only if M is a fully invariant submodule of $\hat{M}$.*

PROOF. Suppose M is an E submodule of $\hat{M}$ and $N \subset M$ with $M/N \in \mathcal{S}$. Let $f \in \mathrm{Hom}_R(N, M)$. Since $\hat{M}/M \in \mathcal{S}$, $\hat{M}/N \in \mathcal{S}$ and $\hat{M}$ $\mathcal{S}$-injective imply there exists an extension $\hat{f} : \hat{M} \to \hat{M}$ of f. Then $\hat{f}(M) \subset M$ and $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(N, M) \to 0$ is exact.

Now suppose M is quasi-$\mathcal{S}$-injective and let $f \in \mathrm{Hom}_R(\hat{M}, \hat{M})$. Let $N = M \cap f^{-1}(M)$. Then the monomorphism $M/N \to \hat{M}/M$ induced by f implies $M/N \in \mathcal{S}$. Thus $f : N \to M$ can be extended to $g : M \to M$, and since $\hat{M}$ is $\mathcal{S}$-injective and $\hat{M}/M \in \mathcal{S}$, g can be extended to $\hat{g} : \hat{M} \to \hat{M}$. If $N \neq M$ then $(\hat{g} - f)(M) \neq 0$, and M being essential in $\hat{M}$, $(\hat{g} - f)(M) \cap M \neq 0$. There are m, $m' \in M$ with $(\hat{g} - f)(m) = m' \neq 0$. Then $f(m) = \hat{g}(m) - m' = g(m) - m' \in M$ implies $m \in N$ contradicting $(\hat{g} - f)(m) \neq 0$. Thus $N = M = M \cap f^{-1}(M)$ and $f(M) \subset M$ so M is an E submodule of $\hat{E}$.

4.5. COROLLARY. *Each $M \in \mathcal{R}$ has a unique (up to isomorphism) minimal quasi-$\mathcal{S}$-injective essential extension (called the quasi-$\mathcal{S}$-injective envelope of M).*

PROOF. The desired extension is the (E, R) submodule of $\hat{M}$ generated by M.

4.6. DEFINITION. Let $M \in \mathcal{R}$. Then $M^* = \{m \in M \mid 0 : m \in \mathcal{F}(\mathcal{S})$ and $0 : m$ is essential in $R\}$ is the $\mathcal{S}$-singular submodule of M.

It is easy to show that the set of ideals $\mathcal{F}_e = \{I \mid I \in \mathcal{F}(\mathcal{S})$ and I is essential in $R\}$ is a filter. Moreover if $R^* = 0$, $\mathcal{F}_e$ is a strongly complete filter.

The following lemma is fairly well known.

4.7. LEMMA. *If N is an essential submodule of M, then $N : m = 0 : (m + N) = \{r \in R \mid rm \in N\}$ is an essential left ideal of R for all $m \in M$. Furthermore, if $N^* = 0$ then $M^* = 0$.*

4.8. PROPOSITION. *Let $M \in \mathcal{R}$. Then $\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(M, \hat{M})$ is an isomorphism if and only if M is quasi-$\mathcal{S}$-injective. If $M^* = 0$ then $\operatorname{Hom}_R(\hat{M}, \hat{M}) \to \operatorname{Hom}_R(M, \hat{M})$ is an isomorphism.*

PROOF. The first assertion follows from 4.4. Let $M^* = 0$. Then $\operatorname{Hom}_R(\hat{M}/M, \hat{M}) = 0$ by 4.7, and $\operatorname{Ext}_R{}^1 (\hat{M}/M, \hat{M}) = 0$ since $\hat{M}/M \in \mathcal{S}$. Thus $\operatorname{Hom}_R(\hat{M}, \hat{M}) \to \operatorname{Hom}_R(M, \hat{M})$ is an isomorphism.

Notice that by 4.8, if $M^* = 0$ then M is quasi-$\mathcal{S}$-injective if and only if $\operatorname{Hom}_R(M, M)$ is naturally isomorphic to $\operatorname{Hom}_R(\hat{M}, \hat{M})$.

4.9. PROPOSITION. *If $M^* = 0$ and $M/N \in \mathcal{S}$ then N has a unique maximal essential extension in M.*

PROOF. Let $L = \{x \in M \mid N : x$ is essential in $R\}$. Note that $N : x \in \mathcal{F}(\mathcal{S})$ for all $x \in M$ since $M/N \in \mathcal{S}$. Now if $x, y \in L$, $r \in R$, then $N : (x + y) \supset (N : x) \cap (N : y)$ which is essential and $N : rx = (N : x) : r$ which is essential. Thus L is a submodule of M, and clearly $L \supset N$. Now if K is an essential extension of N in M, by 4.7 if $k \in K$ then $(N : k)$ is essential. Thus $K \subset L$. If $0 \neq x \in L$, $M^* = 0$ implies $(N : x)x \neq 0$ whence $Rx \cap N \neq 0$. Thus N is essential in L.

In [9], the notion of a *closed* submodule plays a fundamental role. This notion in the present context is given in

4.10. DEFINITION. Let $\mathcal{L}_{\mathcal{S}}{}^c(M)$ denote the set of submodules N of M such that $(M/N)^* = 0$. Such submodules are called *$\mathcal{S}$-closed* in M. For a submodule N of M, the *$\mathcal{S}$-closure* of N in M is the submodule N^c of M given by $N^c/N = (M/N)^*$.

It is easy to see that N^c is $\mathcal{S}$-closed in M. The lattice $\mathcal{L}_{\mathcal{S}}{}^c(M)$ is of greatest interest when $M^* = 0$. It is in this case, for example, when every submodule of M is essential in its $\mathcal{S}$-closure. Some basic properties of $\mathcal{S}$-closure are given in

4.11. PROPOSITION. *Let $M \in \mathcal{R}$ with $N, N_i, N_\alpha, \cdots$ submodules of M. Then*

(a) *If N_α is $\mathcal{S}$-closed in M then $\bigcap_\alpha N_\alpha$ is $\mathcal{S}$-closed in M.*

(b) *$N^c = \bigcap \{C \mid M \supset C \supset N$ and C is $\mathcal{S}$-closed in $M\}$.*

(c) *N^c is the unique maximal essential extension of $N + M^*$ in M', where $M'/N = \mathcal{Q}(M/N)$.*

(d) *$(N^c)^c = N^c$.*

(e) *$(N : m)^c = N^c : m$ for $m \in M$.*

(f) *$(\bigcap_{i=1}^r N_i)^c = \bigcap_{i=1}^r N_i^c$.*

(g) *Let $M^* = 0$, $m \in M$. Then $Im = 0$ implies $I^c m = 0$.*

(h) *Let I be a left ideal of R. If $M^* = 0$ then $M(I) = M(I^c)$, where $M(I)$ denotes the elements of M weakly annihilated by I (see* 1.11 *for definition of weakly annihilate).*

(i) *If C is $\mathcal{S}$-closed in M then $C : S$ is $\mathcal{S}$-closed in R for any subset S of M.*

PROOF. There is a monomorphism $M/(\bigcap_\alpha N_\alpha) \to \prod_\alpha (M/N_\alpha)$. Since $(\prod_\alpha (M/N_\alpha))^* \subset \prod_\alpha (M/N_\alpha)^* = 0$, it follows that $(M/(\bigcap_\alpha N_\alpha))^* = 0$ and $\bigcap_\alpha N_\alpha$ is $\mathcal{S}$-closed in M. The proofs of (b) and (d) are almost immediate.

Let $M'/N = \mathcal{Q}(M/N)$. It is easy to show that $M'/(N + M^*) = \mathcal{Q}(M/(N + M^*))$. Since $(M/M^*)^* = 0$, $(M'/M^*)^* = 0$ and by 4.9, $(N + M^*)/M^*$ has a unique maximal essential extension L/M^* in M'/M^*. It follows that $N + M^*$ is essential in L. Moreover $L/M^* = \{x + M^* \in M'/M^* \mid ((N + M^*)/M^*) : (x + M^*)$ is essential in $R\}$, so $(M'/(N + M^*))^* = L/(N + M^*)$ and $L = (N + M^*)^c$. Now $N^c \subset L$, but $N^c \supset N + M^*$ implies $N^c = (N^c)^c \supset (N + M^*)^c = L$ so $N^c = L$, and since L contains any essential extension of $N + M^*$ in M' by 4.7, $N^c = L$ is the maximum essential extension of $N + M^*$ in M'.

To prove (e), $x \in (N : m)^c$ if and only if there exists an essential ideal J, $J \in \mathcal{Q}(\mathcal{S})$, with $Jx \subset N : m$ if and only if $xm \in N^c$ if and only if $x \in N^c : m$. For (f), the inclusion $(\bigcap_{i=1}^r N_i)^c \subset \bigcap_{i=1}^r N_i^c$ is clear (this does not even depend on the intersection being finite). Let $x \in \bigcap_{i=1}^r N_i^c$. Then there exist essential ideals $J_1, \cdots, J_r$ of $\mathcal{Q}(\mathcal{S})$ with $J_i x \subset N_i$. Now $J = \bigcap_{i=1}^r J_i$ is essential and in $\mathcal{Q}(\mathcal{S})$, and $Jx \subset \bigcap_{i=1}^r N_i$ implies $x \in (\bigcap_{i=1}^r N_i)^c$.

Suppose $M^* = 0$ and $Im = 0$ for some ideal I of R, $m \in M$. Let $r \in I^c$. Then $I : r$ is essential and in $\mathcal{Q}(\mathcal{S})$, and $(I : r)rm \subset Im = 0$ implies $rm = 0$. This proves (g).

Suppose $M^* = 0$ and let I be a left ideal of R. The inclusion $I \subset I^c$ implies $M(I) \supset M(I^c)$. Let $x \in M(I)$. This means there exist elements $r_1, \cdots, r_n \in R$ with $(\bigcap_{i=1}^n (I : r_i))x = 0$. Then

$(\bigcap_{i=1}^{n}(I:r_i))^c x = 0$. But $(\bigcap_{i=1}^{n}(I:r_i))^c = \bigcap_{i=1}^{n}(I:r_i)^c = \bigcap_{i=1}^{n}(I^c:r_i)$, so $x \in M(I^c)$ and $M(I^c) = M(I)$.

Let C be $\mathcal{S}$-closed in a module M, and S any subset of M. Since $(M/C)^* = 0$, $(C:S)(s+C) = 0$ for $s \in S$ implies $(C:S)^c(s+C) = 0$, and (i) follows.

4.12. PROPOSITION. *Let M be quasi-$\mathcal{S}$-injective. If N is $\mathcal{S}$-closed in M and $M/N \in \mathcal{S}$ then N is an absolute direct summand of M. In particular, N is quasi-$\mathcal{S}$-injective.*

PROOF. Let A be a maximal submodule of M such that $A \cap N = 0$. There is a homomorphism $f: A \oplus N \to M : (a+n) \to n$ for $a \in A$, $n \in N$. Since $M/(A \oplus N) \in \mathcal{S}$ and M is quasi-$\mathcal{S}$-injective f can be extended to a map $M \to M$. Now for $m \in M$, $(A \oplus N : m) \in \mathcal{F}_e$ by 4.7 so $(N : f(m)) \in \mathcal{F}_e$. Then $(M/N)^* = 0$ implies $f(m) \in N$. Thus $f: M \to N$ with $f(n) = n$ for all $n \in N$, implying $M = N \oplus \operatorname{Ker} f$. Moreover $A \subset \operatorname{Ker} f$ with A maximally disjoint from N implies $A = \operatorname{Ker} f$, so $M = N \oplus A$.

4.13. THEOREM. *Let M be quasi-$\mathcal{S}$-injective. If N and N' are $\mathcal{S}$-closed in M and $M/N \in \mathcal{S}$ then $N + N'$ is $\mathcal{S}$-closed in M.*

PROOF. By 4.12, $(N+N')^c$ is quasi-$\mathcal{S}$-injective. Also $N+N'$ is $\mathcal{S}$-closed in M if and only if it is $\mathcal{S}$-closed in $(N+N')^c$, so it may be assumed that $(N+N')^c = M$. Let $A \subset N'$ be maximal such that $A \cap N = 0$. By 4.12, $M = N + B$ with $B \supset A$. Then $B \cap N' = A$ so A is $\mathcal{S}$-closed in M by 4.11(a). Moreover, B is quasi-$\mathcal{S}$-injective, being a summand of M, so by 4.12, $B = A \oplus C$ for some C, and $M = (N \oplus A) \oplus C$ implying $N \oplus A$ is $\mathcal{S}$-closed in M. Let $x \in N + N'$ and suppose $Rx \cap (N \oplus A) = 0$ with $x \neq 0$. Then $N:x$ is in $\mathcal{F}(\mathcal{S})$ so it must not be essential, and there is a nonzero ideal I of R with $I \cap (N:x) = 0$. Now $x = n + n'$ with $n \in N$ and $n' \in N'$ and $n' \notin A$. Thus $(In' + A) \cap N \neq 0$ and $jn' + a = n_1 \neq 0$ for some $j \in I$, $a \in A$, $n_1 \in N$. Since $A \cap N = 0$, in particular $j \neq 0$. But $jx = jn + jn' = jn + n_1 - a \in Rx \cap (A \oplus N) = 0$, contradicting $I \cap (N:x) = 0$. Thus $A \oplus N$ is essential in $N + N'$. Then since $(N+N')/(N \oplus A)$ is in $\mathcal{S}$, $(N+N')/(N \oplus A) = ((N+N')/(N \oplus A))^* \subset (M/(N \oplus A))^* = 0$, so $N + N' = N \oplus A$ is $\mathcal{S}$-closed in M.

4.14. COROLLARY. *Let M be quasi-$\mathcal{R}$-injective. If N is $\mathcal{S}$-closed in M then N is quasi-$\mathcal{S}$-injective.*

PROOF. Let $S/N = \mathcal{F}(M/N)$. Since $\mathcal{F}(M/S) = 0$ it is easy to check that S is quasi-$\mathcal{S}$-injective. Now N is $\mathcal{S}$-closed in S with $S/N \in \mathcal{S}$ so, by 4.12, N is quasi-$\mathcal{S}$-injective.

4.15. COROLLARY. *Let M be quasi-$\mathcal{R}$-injective. If N and N' are $\mathcal{S}$-closed in M and $(N + N')/N \in \mathcal{S}$ then $N + N'$ is $\mathcal{S}$-closed in M.*

PROOF. Let $S/(N + N') = \mathcal{T}(M/(N + N'))$. Then S is quasi-$\mathcal{S}$-injective, and the exact sequence

$$0 \to (N + N')/N \to S/N \to S/(N + N') \to 0$$

with $(N + N')/N$ and $S/(N + N')$ in $\mathcal{S}$ implies $S/N \in \mathcal{S}$. Then by 4.13, $N + N'$ is $\mathcal{S}$-closed in S and hence in M.

4.16. PROPOSITION. *Let N be an $\mathcal{S}$-closed submodule of an R module M and let $k \in \mathrm{Hom}_R(M', M)$. Then $k^{-1}(N)$ is $\mathcal{S}$-closed in M'.*

PROOF. Let $x \in (k^{-1}(N))^c$. Then $I = (k^{-1}(N) : x)$ is essential in R and is in $\mathcal{T}(\mathcal{S})$. Hence $Ik(x) = k(Ix) \subset k(k^{-1}(N)) \subset N$ with N $\mathcal{S}$-closed in M implies $k(x) \in N$, so $x \in k^{-1}(N)$. Thus $k^{-1}(N) = (k^{-1}(N))^c$ is $\mathcal{S}$-closed in M'.

For an R module M, let $E(M) = \mathrm{Hom}_R(M, M)$ and $\mathcal{E}_{\mathcal{S}}(M) = \{N \mid N \text{ is an } E(M) \text{ submodule of } M, 0 : N \in \mathcal{T}(\mathcal{S}) \text{ and } N = M[0 : N]\}$ where $M[I] = \{m \in M \mid Im = 0\}$. The module M is called *$\mathcal{S}$-faithful* if it is not annihilated by any nonzero ideal which belongs to $\mathcal{S}$.

4.17. THEOREM. *Let M be an $\mathcal{S}$-faithful, $\mathcal{S}$-injective R module with $M^* = 0$. Then $\mathcal{E}_{\mathcal{S}}(M)$ is lattice dual isomorphic to $\mathcal{L}_{\mathcal{S}}{}^c(R) \cap \mathcal{T}(\mathcal{S})$ under the correspondence $N \to 0 : N$.*

PROOF. By 4.11(i), $0 : N$ is $\mathcal{S}$-closed in R for any $N \subset M$, since $M^* = 0$. Thus for $N \in \mathcal{E}_{\mathcal{S}}(M)$, $0 : N$ is in $\mathcal{L}_{\mathcal{S}}{}^c(R) \cap \mathcal{T}(\mathcal{S}) = \mathcal{T}^c(\mathcal{S})$. Let $I \in \mathcal{T}^c(\mathcal{S})$. Then $M[I]$ is an $E(M)$ submodule of M, and $I \subset (0 : M[I])$ implies $(0 : M[I]) \in \mathcal{T}(\mathcal{S})$. It is easily checked that $M[I] = M[0 : M[I]]$, hence $M[I] \in \mathcal{E}_{\mathcal{S}}(M)$. Let $J = 0 : M[I]$ and let $B \subset J$ be maximal such that $I \cap B = 0$. Assume $B \neq 0$. The monomorphism $B \to R/I$ implies $B \in \mathcal{S}$ so there exists an $x \in M$ such that $Bx \neq 0$. The map $f : I \oplus B \to M$ with $f(I) = 0$ and $f(b) = bx$ for $b \in B$ can be extended to an R-homomorphism $f : R \to M$ since $R/(I \oplus B) \in \mathcal{S}$. Now $y = f(1)$ satisfies $by = bx$ for $b \in B$ and $Iy = 0$. Thus $By \neq 0$, but $y \in M[I]$ implies $Jy = 0$ which is a contradiction since $B \subset J$. Thus $B = 0$ and I is essential in J. Then I $\mathcal{S}$-closed and $J/I \in \mathcal{S}$ implies $I = J = 0 : M[I]$. The theorem follows.

For an ideal I of R, $M(I)$ denotes the submodule of M which is weakly annihilated by I (see 1.11). The module M is *strongly $\mathcal{S}$-faithful* if $M \neq M(I)$ for all nonzero ideals I of R such that $I \in \mathcal{S}$.

4.18. THEOREM. *Let M be a strongly $\mathcal{S}$-faithful, $\mathcal{S}$-injective R module with $M^* = 0$, and let $\mathcal{B}_{\mathcal{S}}(M) = \{N \mid N = M(I) \text{ for some } I \in \mathcal{T}(\mathcal{S})\}$. Then $\mathcal{B}_{\mathcal{S}}(M)$ is lattice dual isomorphic to $\mathcal{L}_{\mathcal{S}}{}^c(R) \cap \mathcal{T}(\mathcal{S})$.*

PROOF. Suppose $I, J \in \mathcal{I}(\mathcal{S})$ are both $\mathcal{S}$-closed in R and $M(I) = M(J)$. Let $B \subset J$ such that $I \cap B = 0$ and assume $B \neq 0$. The monomorphism $B \to R/I$ implies $B \in \mathcal{S}$ so there exists an $x \in M$ with $x \notin M(B)$. The map $F: I \oplus B \to M$ with $f(I) = 0$ and $f(b) = bx$ for $b \in B$ can be extended to an R-homomorphism $f: R \to M$, and letting $y = f(1)$, $Iy = 0$ and $by = bx$ for $b \in B$. Thus $y \in M(I) = M(J) \subset M(B)$, so there exist $r_1, \cdots, r_n \in R$ with $(\bigcap_{i=1}^n (B : r_i))y = 0$. Let $r_0 = 1$, then $(\bigcap_{i=0}^n (B : r_i))y = 0$ with $\bigcap_{i=0}^n (B : r_i) \subset B$ implying $(\bigcap_{i=0}^n (B : r_i))x = 0$. This contradiction establishes that $B = 0$. Now let $a \in I + J$, $a \neq 0$ and suppose $Ra \cap I = 0$. Then $a = i + j$ with $i \in I$, $j \in J$, and $I : j \in \mathcal{I}(\mathcal{S})$ with I $\mathcal{S}$-closed and $j \notin I$ implies $I : j$ is not essential in R. Let $K \neq 0$ with $K \cap (I : j) = 0$. Then $I \cap Kj = 0$ but $Kj \neq 0$ with $Kj \subset J$ contradicting what was proved earlier. Thus I is essential in $I + J$, and then $(I + J)/I \in \mathcal{S}$ and I $\mathcal{S}$-closed implies $I = I + J$ so $J \subset I$. It follows that $J = I$.

Let $I \in \mathcal{I}(\mathcal{S})$. Clearly $M(I^c) \subset M(I)$. Conversely if $m \in M(I)$ with $(\bigcap_{i=1}^n (I : r_i))m = 0$ then $(\bigcap_{i=1}^n (I : r_i))^c m = 0$, and $(\bigcap_{i=1}^n (I : r_i))^c = \bigcap_{i=1}^n (I : r_i)^c = \bigcap_{i=1}^n (I^c : r_i)$ so $m \in M(I^c)$. The theorem follows.

Let $R_e = \{r \in R \mid (0 : r) \text{ is essential in } R\}$. If $R_e = 0$ then the set $\mathcal{I}$ of ideals of R which are essential in R is a strongly complete filter of ideals. Letting $\mathcal{S} = \mathcal{S}(\mathcal{I})$, $R_{\mathcal{S}}$ is an injective R module, in fact an injective envelope of R. (See [3, pp. 416–421].)

4.19. THEOREM. *Suppose $R_e = 0$, and let $\mathcal{I}$ be the filter of all essential ideals of R. The following are equivalent.*

(a) *$R_{\mathcal{S}(\mathcal{I})}$ is a semisimple ring.*

(b) *There does not exist an infinite family of left ideals of R whose sum is direct.*

(c) *$\mathcal{I}$ has a cofinal set of finitely generated ideals.*

(d) *$R_{\mathcal{S}}I = R_{\mathcal{S}}$ for all $I \in \mathcal{I}$.*

PROOF. The equivalence of (a) and (b) is in [3, Lemme 6, p. 418]. The equivalence of (b) and (c) is easily verified.

Let $a \in R_{\mathcal{S}}$. The homomorphism $\bar{a}: R_{\mathcal{S}} \to R_{\mathcal{S}} : x \to xa$ induces an exact sequence

$$0 \to K \to R_{\mathcal{S}} \xrightarrow{\bar{a}} R_{\mathcal{S}}a \to 0.$$

Now $(R_{\mathcal{S}}/K)^* \simeq (R_{\mathcal{S}}a)^* = 0$ so K is $\mathcal{S}$-closed in $R_{\mathcal{S}}$. This means K is a maximal essential extension of itself in $R_{\mathcal{S}}$, so since $R_{\mathcal{S}}$ is injective, K is a direct summand of $R_{\mathcal{S}}$. It follows that $R_{\mathcal{S}}a$ is an injective R module. Now let I be a finitely generated and essential ideal of R, with generators $a_1, \cdots, a_n$. Then $R_{\mathcal{S}}I = R_{\mathcal{S}}a_1 + \cdots + R_{\mathcal{S}}a_n$ is essential in $R_{\mathcal{S}}$. The module $R_{\mathcal{S}}a_1 \oplus \cdots \oplus R_{\mathcal{S}}a_n$ is injective, and the exact sequence

$$0 \to L \to R_{\mathcal{S}}a_1 \oplus \cdots \oplus R_{\mathcal{S}}a_n \to R_{\mathcal{S}}I \to 0$$

splits since L is $\mathcal{S}$-closed, implying $R_{\mathcal{S}}I$ is injective. It follows that $R_{\mathcal{S}}I = R_{\mathcal{S}}$ and the equivalence of (c) and (d) follows.

Additional properties equivalent to (a), (b), (c) and (d) are stated in Theorems 3.2 and 3.5.

Two types of regular elements will be considered. Let $B_{\mathcal{S}}(R) = \{a \in R \mid Ra \in \mathcal{F}(\mathcal{S})\}$ and $D_{\mathcal{S}}(R) = \{a \in R \mid 0 : a \in \mathcal{S}\}$. The standard notions are realized when $R_e = 0$ and $\mathcal{F}$ is the set of all essential ideals of R.

4.20. Proposition. *$B_{\mathcal{S}}(R)$ and $D_{\mathcal{S}}(R)$ are multiplicative sets. Moreover, $a \in D_{\mathcal{S}}(R)$ if and only if $\mathcal{F}R : a = \mathcal{F}R$.*

Proof. Let a, $b \in B_{\mathcal{S}}(R)$. The epimorphism $R/Rb \to Ra/Rba$: $R + Rb \to ra + Rba$ implies $Ra/Rba \in \mathcal{S}$. Then the exact sequence

$$0 \to Ra/Rba \to R/Rba \to R/Ra \to 0$$

implies $R/Rba \in \mathcal{S}$, so $ba \in B_{\mathcal{S}}(R)$.

Let a, $b \in D_{\mathcal{S}}(R)$. The map $(0 : ab) \to (0 : b)$ with $x \to xa$ yields an exact sequence $0 \to (0 : a) \to (0 : ab) \to (0 : b) \to 0$, implying $(0 : ab) \in \mathcal{S}$, so $ab \in D_{\mathcal{S}}(R)$.

Since $\mathcal{F}R$ is a two-sided ideal, $\mathcal{F}R \subset \mathcal{F}R : a$ for any $a \in R$. Suppose $0 : a \in \mathcal{S}$ and let $x \in \mathcal{F}R : a$. Then $0 : xa \in \mathcal{F}(\mathcal{S})$ and $(0 : xa)/(0 : x) \simeq (0 : a) \in \mathcal{S}$ implies $0 : x \in \mathcal{F}(\mathcal{S})$. Thus $x \in \mathcal{F}R$. The converse follows from the inclusion $0 : a \subset \mathcal{F}R : a$.

Let $U_{\mathcal{S}}(R) = \{r \in R \mid \phi(r) \text{ is a unit in } R_{\mathcal{S}}\}$.

4.21. Theorem. $U_{\mathcal{S}}(R) = B_{\mathcal{S}}(R) \cap D_{\mathcal{S}}(R)$.

Proof. Let $a \in B_{\mathcal{S}}(R) \cap D_{\mathcal{S}}(R)$ and define $f : Ra \to R/(0 : a)$ by $f(ra) = r + (0 : a)$. Then $[f] \in R_{\mathcal{S}}$ and for $r \in R$, $\bar{a}f(ra) = \bar{a}(r + (0 : a)) = ra + (0 : a)$, $f\bar{a}(r) = f(ra) = r + (0 : a)$. Thus $[f]\phi(a) = [\bar{a}f] = 1 = [f\bar{a}] = \phi(a)[f]$ implying $a \in U_{\mathcal{S}}(R)$.

Suppose $b \in U_{\mathcal{S}}(R)$, and $\phi(b)x = 1$. Then $\phi(0 : b) = \phi(0 : b)\phi(b)x = \phi((0 : b)b)x = 0$ implying $(0 : b) \subset \mathcal{F}R$ so $(0 : b) \in \mathcal{S}$. Now $x = [g]$ with $g : I \to R/\mathcal{F}R$ for some $I \in \mathcal{F}(\mathcal{S})$, and for $i \in I$ with $g(i) = r_i + \mathcal{F}R$, $\bar{b}g(i) = r_ib + \mathcal{F}R = i + \mathcal{F}R$. Thus $I \subset Rb + \mathcal{F}R$ implying $Rb + \mathcal{F}R \in \mathcal{F}(\mathcal{S})$ and hence $Rb \in \mathcal{F}(\mathcal{S})$. This completes the proof.

4.22. Corollary. *If $I \cap B_{\mathcal{S}}(R) \cap D_{\mathcal{S}}(R) \neq \emptyset$ for each $I \in \mathcal{F}(\mathcal{S})$ then $R_{\mathcal{S}} = \{\phi(b)^{-1}\phi(a) \mid a \in R, b \in B_{\mathcal{S}}(R) \cap D_{\mathcal{S}}(R)\}$.*

PROOF. Let $[f] \in R_{\mathcal{S}}$, $f : I \to R/\mathcal{T}R$ with $I \in \mathcal{T}(\mathcal{S})$. Let $b \in I \cap B_{\mathcal{S}}(R) \cap D_{\mathcal{S}}(R)$, and $f(b) = a + \mathcal{T}R$. Now $\phi(b)^{-1} = [g]$ where $g : Rb \to R/\mathcal{T}R : rb \to r + \mathcal{T}R$, and $\phi(b)^{-1}\phi(a) = [g]\,[\bar{a}] = [\bar{a}g]$. Then for $r \in R$, $\bar{a}g(rb) = ra + \mathcal{T}R = rf(b) = f(rb)$ so $[f] = [\bar{a}g] = \phi(b)^{-1}\phi(a)$. The other inclusion follows from 4.21.

This condition is equivalent to the conditions in [3, p. 415], that $B_{\mathcal{S}}(R) \cap D_{\mathcal{S}}(R)$ be a multiplicative set such that the ring of quotients with respect to that set exists. In particular, if the hypothesis of 4.22 is satisfied, $R_{\mathcal{S}}$ is flat and the localization functor is exact and is isomorphic to the functor $M \to R_{\mathcal{S}} \otimes_R M$.

In order to compare $B_{\mathcal{S}}(R)$ and $D_{\mathcal{S}}(R)$ some assumptions must be made.

4.23. PROPOSITION. *If the ascending chain condition holds for the set of ideals* $\{(\mathcal{T}R : a) \mid a \in B_{\mathcal{S}}(R)\}$ *then* $B_{\mathcal{S}}(R) \subset D_{\mathcal{S}}(R)$.

PROOF. Let $a \in B_{\mathcal{S}}(R)$, and let $I_n = \mathcal{T}R : a^n$. Then $I_1 \subset I_2 \subset \cdots$ whence there exists an n such that $I_n = I_{n+1} = \cdots$. Now $a^n \in B_{\mathcal{S}}(R)$ and $\mathcal{T}R : a^n = \mathcal{T}R : a^{2n}$. If $x \in (\mathcal{T}R : a^n) \cap Ra^n$ then $x = ra^n$ with $ra^{2n} \in \mathcal{T}R$, whence $r \in \mathcal{T}R : a^{2n} = \mathcal{T}R : a^n$ and $x = ra^n$ is in $\mathcal{T}R$. Thus $(\mathcal{T}R : a^n) \cap Ra^n \in \mathcal{S}$. But $Ra^n \in \mathcal{T}(\mathcal{S})$ so $R/Ra^n \in \mathcal{S}$, and the exact sequence

$$0 \to (\mathcal{T}R : a^n) \cap Ra^n \to \mathcal{T}R : a^n \to R/Ra^n$$

implies $\mathcal{T}R : a^n \in \mathcal{S}$. Since $0 : a \subset \mathcal{T}R : a^n$, $0 : a \in \mathcal{S}$ and $a \in D_{\mathcal{S}}(R)$.

Reasonable conditions for the inclusion in 4.23 to be an equality seem to be elusive. In the special case $R_e = 0$ and $\mathcal{T}$ is the filter of all essential ideals, the a.c.c. on $\mathcal{L}_{\mathcal{S}}{}^c(R)$ will suffice. (See [9, Theorem 3.4].)

5. Concordant and harmonic functors.

Let $\mathcal{A}$ be an abelian category and I the identity functor $\mathcal{A} \to \mathcal{A}$. A *subfunctor* of the identity functor I is a (covariant, additive) functor $S : \mathcal{A} \to \mathcal{A}$ together with a natural transformation $\alpha : S \to I$ such that $\alpha_A : S(A) \to I(A) = A$ is a monomorphism for all $A \in \mathcal{A}$. Dually, a *quotient functor* of I is a functor $Q : \mathcal{A} \to \mathcal{A}$ together with a natural transformation $\beta : I \to Q$ such that $\beta_A : I(A) \to Q(A)$ is an epimorphism for all $A \in \mathcal{A}$. With a subfunctor (S, α) of I there is associated a quotient functor (S^*, α^*) of I via $S^*(A) = \operatorname{Cok} \alpha_A$, $\alpha_A{}^* = \operatorname{cok} \alpha_A$, and with a quotient functor (Q, β) there is associated a subfunctor (Q^*, β^*) of I via $Q^*(A) = \operatorname{Ker} \beta_A$ and $\beta_A{}^* = \ker \beta_A$. It is readily seen that $S^{**} = S$ and $Q^{**} = Q$.

5.1. Definition. A functor $S : \mathcal{A} \to \mathcal{A}$ is *concordant* if S is a left exact subfunctor of the identity functor. A functor $Q : \mathcal{A} \to \mathcal{A}$ is *harmonic* if Q is a right exact quotient functor of the identity functor.

The purpose of this section is to show how concordant and harmonic functors give rise in natural ways to relative homological algebras, and to characterize such functors in terms of certain subclasses of $\mathcal{A}$. In particular, it will be shown that if $\mathcal{A} = \mathcal{R}$ for a ring R, then concordant functors are in natural one-one correspondence with the filters of left ideals of R, and harmonic functors are in natural one-one correspondence with the two-sided ideals of R. Thus concordant functors are intimately related to the strongly complete additive classes studied in §1.

Two significant examples are the following. Let J be a left ideal of the ring R. Let $S(A) = A(J)$, the submodule of A consisting of elements which are weakly annihilated by J (see 1.11). Then $S : \mathcal{R} \to \mathcal{R}$ is concordant. Let $Q(A) = A/JA$. Then Q is harmonic.

5.2. Proposition. *Let S be a subfunctor of the identity. The following are equivalent.*

(a) *S is concordant.*

(b) *S^* preserves monomorphisms.*

(c) *If A is a subobject of B then $A \cap S(B) = S(A)$.*

Proof. Let $0 \to A \to B \to C \to 0$ be exact. Then the kernel of $S(B) \to S(C)$ is $A \cap S(B)$, which is $S(A)$ if and only if $A/S(A) \to B/S(B)$ is a monomorphism. The proposition follows.

The terminology concordant is inspired by 5.2(c). Dually, one can show

5.3. Proposition. *Let Q be a quotient functor of the identity. The following are equivalent.*

(a) *Q is harmonic.*

(b) *Q^* preserves epimorphisms.*

(c) *If A is a subobject of B, then $Q^*(B/A) = (Q^*(B) + A)/A$.*

5.4. Definition. Let S be a subfunctor of the identity, and $0 \to A \to B \to C \to 0$ be exact. This sequence is *S-pure*, or A is *S-pure in B*, if $0 \to S(A) \to S(B) \to S(C) \to 0$ is exact. Similarly, if Q is a quotient functor of the identity, then this sequence is Q-pure, or A is Q-pure in B, if $0 \to Q(A) \to Q(B) \to Q(C) \to 0$ is exact.

5.5. Corollary. *Let S be concordant, Q harmonic, and let $0 \to A \to B \to B/A \to 0$ be exact. Then A is S-pure in B if and only if A is S^*-pure in B if and only if $S(B/A) = (S(B) + A)/A$, and A is*

Q-pure in B if and only if A is Q^*-pure in B if and only if $A \cap Q^*(B) = Q^*(A)$.

PROOF. If $T =$ S or Q^*, the sequence

$$0 \to A \cap T(B) \to T(B) \to (T(B) + A)/A \to 0$$

is always exact, using the natural isomorphism $T(B)/(A \cap T(B)) \approx (T(B) + A)/A$.

The corollary follows, using 5.2 and 5.3.

It is useful to note that if T is any subfunctor of the identity, and $A \subset B$, then $T(A) \subset A \cap T(B)$ and $(T(B) + A)/A \subset T(B/A)$.

5.6. THEOREM. *Let S be concordant and Q harmonic. If A, B, $C \in \mathcal{A}$ with $A \subset B \subset C$, and $T =$ S, S*, Q, or Q^*, then*

(a) *If a short exact sequence is T-pure, then any equivalent short exact sequence is T-pure.*

(b) *If A is a direct summand of B, then A is T-pure in B.*

(c) *If A is T-pure in C, then A is T-pure in B.*

(d) *If A is T-pure in C and B/A is T-pure in C/A, then B is T-pure in C.*

(e) *If B is T-pure in C, then B/A is T-pure in C/A.*

(f) *If A is T-pure in B and B is T-pure in C, then A is T-pure in C.*

PROOF. Since T is additive, (a) and (b) are immediate. Suppose $T =$ S; that is, that T is concordant.

(c) Suppose A is T-pure in C. Then

$$\begin{aligned} T(B/A) &= (B/A) \cap T(C/A) = (B/A) \cap ((T(C) + A)/A) \\ &= (B \cap (T(C) + A))/A = ((B \cap T(C)) + A)/A \\ &= (T(B) + A)/A. \end{aligned}$$

(d) Suppose A is T-pure in C and B/A is T-pure in C/A. Then

$$\begin{aligned} T(C/B) &\cong T((C/A)/(B/A)) = (T(C/A) + (B/A))/(B/A) \\ &= (((T(C) + A)/A) + (B/A))/(B/A) \\ &= ((T(C) + B)/A)/(B/A) \cong (T(C) + B)/B. \end{aligned}$$

One may easily check that these natural isomorphisms imply $T(C/B) = (T(C) + B)/B$.

(e) Suppose B is T-pure in C. Then

$$\begin{aligned} T((C/A)/(B/A)) &\cong T(C/B) = (T(C) + B)/B \\ &\cong ((T(C) + B)/A)/(B/A) \\ &= (((T(C) + A)/A) + (B/A))/(B/A) \\ &\subseteq (T(C/A) + (B/A))/(B/A) \subseteq T((C/A)/(B/A)). \end{aligned}$$

It follows that $T((C/A)/(B/A)) = (T(C/A) + (B/A))/(B/A)$.

(f) Suppose A is T-pure in B and B is T-pure in C. Write $T(C/A) = D/A$. By (e), $B/A \cap T(C/A) = T(B/A) = (T(B) + A)/A$, so that $B \cap D = T(B) + A$. The natural projection $C/A \to C/B$ induces the diagram

$$\begin{array}{ccc} T(C/A) & \to & T(C/B) = (T(C) + B)/B \\ \| & & \| \\ D/A & \to & (D + B)/B \end{array}$$

so that $D + B = T(C) + B$. Now, since $T(C) \subset D$ we have $D = D \cap (T(C) + B) = (D \cap B) + T(C) = T(C) + A$, that is, $T(C/A) = D/A = (T(C) + A)/A$.

If T is harmonic, (c)–(f) follow in a dual fashion.

The properties expressed in 5.6 are simply a restatement of the axioms for a proper class [11, p. 368]. Consequently, for each pair A, B of objects of $\mathcal{A}$ and for each positive integer n is associated an abelian group $\mathrm{Ext}_T{}^n(A, B)$. The elements of $\mathrm{Ext}_T{}^1(A, B)$ are the usual equivalence classes of T-pure sequences of the form $0 \to B \to X \to A \to 0$. Assuming that V is concordant or V^* is harmonic, it is not difficult to show that

$$\mathrm{Ext}_V{}^1(A, B) = \mathrm{Ker}\,(\mathrm{Ext}^1(A, B) \to \mathrm{Ext}^1(V(A), V^*(B))).$$

From [11, Theorem 5.1, p. 372] follows

5.7. Corollary. *Let S be concordant and Q harmonic. If $T = S$, S^*, Q, or Q^* and $0 \to A \to B \to C \to 0$ is a T-pure exact sequence, then for any $X \in \mathcal{A}$, there exist exact sequences*

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{A}}(X, A) \to \mathrm{Hom}_{\mathcal{A}}(X, B) \to \mathrm{Hom}_{\mathcal{A}}(X, C) \\ &\to \mathrm{Ext}_T{}^1(X, A) \to \mathrm{Ext}_T{}^1(X, B) \to \mathrm{Ext}_T{}^1(X, C) \to \mathrm{Ext}_T{}^2(X, A) \to \cdots \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{A}}(C, X) \to \mathrm{Hom}_{\mathcal{A}}(B, X) \to \mathrm{Hom}_{\mathcal{A}}(A, X) \\ &\to \mathrm{Ext}_T{}^1(C, X) \to \mathrm{Ext}_T{}^1(B, X) \to \mathrm{Ext}_T{}^1(A, X) \to \mathrm{Ext}_T{}^2(C, X) \to \cdots. \end{aligned}$$

For T concordant or T harmonic, those exact sequences $0 \to A \to B \to C \to 0$ for which $0 \to T(A) \to T(B) \to T(C) \to 0$ is *splitting* exact form a proper class, that is, satisfy (a)–(f) of 5.6. However, the resulting relative homological algebras are special cases of those studied in [15]. Indeed, if T is a subfunctor of the identity such that $T^2 = T$ and $\mathcal{C}_T = \{A \in \mathcal{A} \mid T(A) = A\}$, then $0 \to T(A) \to T(B) \to$

$T(C) \to 0$ is split exact for $0 \to A \to B \to C \to 0$ exact if and only if this latter sequence is $\mathcal{C}_T$-pure in the sense of [15]. Dual statements hold for quotient functors of the identity.

Attention is turned now to characterizing concordant and harmonic functors. Let S be concordant. As before, for $A \in \mathcal{A}$, $S(A)$ is identified with the subobject $\alpha_A : S(A) \to A$ of A. Let $\mathcal{C}_S$ denote the class of objects $A \in \mathcal{A}$ such that $S(A) = A$.

5.8. Lemma. *Let* S *and* S′ *be concordant. Then*

(a) $S^2 = S$.

(b) $\mathcal{C}_S$ *is closed under subobjects, quotient objects and (finite) direct sums.*

(c) *If* $\mathcal{C}_S = \mathcal{C}_{S'}$ *then* $S = S'$.

Proof. For $A \in \mathcal{A}$, $S(A) \subset A$ and, by 5.2, $S(A) = S(A) \cap A = S(S(A)) = S^2(A)$. Hence $S^2 = S$. If $B \subset A \in \mathcal{C}_S$ then $S(B) = B \cap S(A) = B \cap A = B$, whence $B \in \mathcal{C}_S$. Also $S(A/B) \supset (S(A) + B)/B = (A + B)/B = A/B$, whence $A/B \in \mathcal{C}_S$. Since S is additive, $S(A_1 \oplus A_2) = A_1 \oplus A_2$ for $A_1, A_2 \in \mathcal{C}_S$, and (b) follows.

Suppose $\mathcal{C}_S = \mathcal{C}_{S'}$. For $A \in \mathcal{S}$, $S(S(A)) = S(A) \in \mathcal{C}_S$, so that $S(A) = S'(S(A)) \subset S'(A)$. By symmetry, $S(A) = S'(A)$ and (c) follows.

Suppose $\mathcal{A}$ has arbitrary infinite direct sums. As in §1 with categories of modules, a nonempty subclass $\mathcal{C}$ of $\mathcal{A}$ is called a *strongly complete additive class* if it is closed under subobjects, quotient objects, and arbitrary direct sums. For such a class $\mathcal{C}$, and $A \in \mathcal{A}$, A has a maximum subobject which is in $\mathcal{C}$. Define $S_{\mathcal{C}}(A)$ to be this subobject.

5.9. Lemma. *If* $\mathcal{C}$ *is a strongly complete additive class in the category* $\mathcal{A}$ *with infinite direct sums, then* $S_{\mathcal{C}}$ *is concordant.*

Proof. This follows readily from 5.2.

Now 5.8 and 5.9 yield

5.10. Theorem. *Let* $\mathcal{A}$ *be an abelian category with arbitrary infinite direct sums. Then* $S \to \mathcal{C}_S$ *is a natural one-one correspondence between the concordant functors on* $\mathcal{A}$ *and the strongly complete additive classes of* $\mathcal{A}$. *The inverse of this correspondence is* $\mathcal{C} \to S_{\mathcal{C}}$.

From 1.10, follows

5.11. Corollary. *Let R be a ring. Then the concordant functors on* $\mathcal{R}$ *are in natural one-one correspondence with the filters of left ideals of R.*

In a dual fashion, harmonic functors can be characterized.

5.12. LEMMA. *Let Q and Q' be harmonic, and $\mathcal{C}_Q$ be the class of objects $A \in \mathcal{A}$ such that $Q(A) = A$. Then*

(a) $Q^2 = Q$.

(b) *$\mathcal{C}_Q$ is closed under subobjects, quotient objects and finite direct products.*

(c) *If $\mathcal{C}_Q = \mathcal{C}_{Q'}$ then $Q = Q'$.*

PROOF. For $A \in \mathcal{A}$,

$$Q^2(A) = Q(A/Q^*(A)) = (A/Q^*(A))/Q^*(A/Q^*(A))$$
$$= (A/Q^*(A))/((Q^*(A) + Q^*(A))/Q^*(A)) = A/Q^*(A) = Q(A),$$

using 5.3(c). Thus (a) holds.

If $A \subset B = Q(B)$, $Q^*(A) \to Q^*(B) = 0$ is a monomorphism, Q^* being a subfunctor of the identity, so that $Q^*(A) = 0$ and $Q(A) = A$. Also $0 = Q^*(B) \to Q^*(B/A) \to 0$ is exact by 5.3(b), so $Q^*(B/A) = 0$ and $Q(B/A) = B/A$. If $Q(A_i) = A_i$, then $Q(A_1 \oplus A_2) = Q(A_1) \oplus Q(A_2) = A_1 \oplus A_2$, and (*b*) follows.

Suppose $\mathcal{C}_Q = \mathcal{C}_{Q'}$, and $A \in \mathcal{A}$. Then $Q^*(A/Q^*(A)) = 0$, so $Q'^*(A/Q^*(A)) = (Q'^*(A) + Q^*(A))/Q^*(A) = 0$ whence $Q'^*(A) \subset Q^*(A)$. By symmetry, $Q'^*(A) = Q^*(A)$, and it follows that $Q = Q'$.

Let $\mathcal{A}$ have infinite products. Call a nonempty subclass $\mathcal{C}$ of $\mathcal{A}$ a *strongly complete multiplicative class* if $\mathcal{C}$ is closed under subobjects, quotient objects, and arbitrary products. For $A \in \mathcal{A}$, any set $\{A_\alpha\}$ of subobjects of A has an intersection in A, namely the kernel of the natural map $A \to \prod_\alpha (A/A_\alpha)$. Define

$$Q_{\mathcal{C}}(A) = \bigcap \{B \mid B \subset A \text{ and } A/B \in \mathcal{C}\}.$$

The following lemma is straightforward.

5.13. LEMMA. *If $\mathcal{C}$ is a strongly complete multiplicative class in an abelian category $\mathcal{A}$ having infinite products, then $Q_{\mathcal{C}}$ is harmonic.*

The dual to 5.10 is

5.14. THEOREM. *Let $\mathcal{A}$ be an abelian category with arbitrary products. Then $Q \to \mathcal{C}_Q$ is a natural one-one correspondence between the harmonic functors on $\mathcal{A}$ and the strongly complete multiplicative classes of $\mathcal{A}$. The inverse of this correspondence is $\mathcal{C} \to Q_{\mathcal{C}}$.*

Strongly complete multiplicative classes in the category $\mathcal{R}$ of R modules are readily determined. Let $\mathcal{C}$ be such a subclass of $\mathcal{R}$. Then $\mathcal{F}(\mathcal{C})$ is a filter of R and $\prod_{I \in \mathcal{F}(\mathcal{C})} R/I \in \mathcal{C}$. The class $\mathcal{C}$ is a strongly complete additive class so by 1.10, $\mathcal{C} = \mathcal{S}(\mathcal{F}(\mathcal{C}))$. Hence there is an $I_{\mathcal{C}} \in \mathcal{F}(\mathcal{C})$ such that $I_{\mathcal{C}}\{1 + I\}_{I \in \mathcal{F}(\mathcal{C})} = 0$, and $I_{\mathcal{C}}$ is a smallest element of $\mathcal{I}(\mathcal{C})$. Since $I : r \in \mathcal{I}(\mathcal{C})$ for all $r \in R$, $I_{\mathcal{C}}$ is a two-sided ideal of R. The ideal $I_{\mathcal{C}}$ determines $\mathcal{I}(\mathcal{C})$, for $\mathcal{I}(\mathcal{C}) = \{I \mid I$ is a left ideal of R and $I \supset I_{\mathcal{C}}\}$. If I is any two-sided ideal of R, then the class $\mathcal{C}_I$ of all $A \in \mathcal{A}$ such that $IA = 0$ is a strongly complete multiplicative class of $\mathcal{A}$. Given a two-sided ideal I, $I_{\mathcal{C}_I} = I$, and given a strongly complete multiplicative class $\mathcal{C}$, $\mathcal{C}_{I_{\mathcal{C}}} = \mathcal{C}$. From these remarks one has

5.15. Proposition. *Let R be a ring. Then $I \to \mathcal{C}_I$ is a natural one-one correspondence between the set of two-sided ideals of R and the strongly complete multiplicative classes of $\mathcal{R}$. The inverse of this correspondence is $\mathcal{C} \to I_{\mathcal{C}}$.*

5.16. Theorem. *Let R be a ring. Then $Q \to I_{\mathcal{C}_Q}$ is a natural one-one correspondence between the harmonic functors on $\mathcal{R}$ and the two-sided ideals of R. Furthermore, $I_{\mathcal{C}_Q} = Q^*(R)$, and for $A \in \mathcal{R}$, $Q(A) = A/Q^*(R)A$.*

Proof. The first assertion follows from 5.14 and 5.15. Since $Q^2 = Q$, $Q(R) = R/Q^*(R) \in \mathcal{C}_Q$ and $I_{\mathcal{C}_Q}(R/Q^*(R)) = 0$. Thus $I_{\mathcal{C}_Q} \subset Q^*(R)$. But $R/I_{\mathcal{C}_Q} \in \mathcal{C}_Q$ so that $Q^*(R/I_{\mathcal{C}_Q}) = (Q^*(R) + I_{\mathcal{C}_Q})/I_{\mathcal{C}_Q} = 0$. Thus $Q^*(R) = I_{\mathcal{C}_Q}$. For $A \in \mathcal{R}$, $Q^*(A/Q^*(A)) = 0$. Thus $A/Q^*(A) \in \mathcal{C}_Q$ and $Q^*(R)(A/Q^*(A)) = 0$. Thus $Q^*(R)A \subset Q^*(A)$. Further, $A/(Q^*(R)A) \in \mathcal{C}_Q$ being annihilated by $Q^*(R)$, so

$$Q^*(A/Q^*(R)A) = (Q^*(A) + Q^*(R)A)/Q^*(R)A = 0.$$

Hence $Q^*(A) \subset Q^*(R)A$. Thus $Q^*(A) = Q^*(R)A$ and $Q(A) = A/Q^*(A) = A/Q^*(R)A$. This concludes the proof.

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