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RELATIVE COHEN-MACAULAYNESS AND RELATIVE UNMIXEDNESS OF BIGRADED MODULES

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ABSTRACT. In this paper we study the finitely generated bigraded modules over a standard bigraded polynomial ring that are relative Cohen-Macaulay or relatively unmixed with respect to one of the irrelevant bigraded ideals. A generalization of Reisner's criterion for Cohen-Macaulay simplicial complexes is considered.

Introduction. Let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over a field K. We set $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Let M be a finitely generated bigraded S-module. In [**11**] we call M relative Cohen-Macaulay with respect to Q if we have only one non-vanishing local cohomology with respect to Q. In other words, $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M)$ where $\operatorname{cd}(Q, M)$ denotes the cohomological dimension of M with respect to Q.

In [**11**], it is shown that if M is a finitely generated bigraded Cohen-Macaulay S-module, then M is relative Cohen-Macaulay with respect to P" if and only if "M is relative Cohen–Macaulay with respect to Q." In Section 1, inspired by this result, we raise the following question: if M is relative Cohen-Macaulay with respect to P and Q, is M Cohen-Macaulay? We have an example in dimension 2 which shows that this is not true in general. The question has a positive answer in some special cases.

Next we show M to be relatively unmixed with respect to Q if $\operatorname{cd}(Q, M) = \operatorname{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass} M$. We prove that relative Cohen-Macaulay modules with respect to Q are relatively unmixed with respect to Q but the converse is not true in general. The converse is true under some additional assumptions.

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Replacing, in the previous question, "M is relative Cohen-Macaulay with respect to Q" by the weaker assumption "M is relatively unmixed with respect to Q," one obtains the following question: if M is relative Cohen-Macaulay with respect to P and relatively unmixed with respect to Q, is M unmixed? The question has a positive answer if M satisfies one of the following conditions:

(a) $\operatorname{cd}(P, M) \leq 1$,

(b) $M = M_1 \otimes_K M_2$ where M_1 is a graded $K[x]$-module and M_2 is a graded $K[y]$-module and where $S/(\mathfrak{p}_1 + \mathfrak{p}_2)S$ is an integral domain for all $\mathfrak{p}_1 \in \operatorname{Ass} M_1$ and $\mathfrak{p}_2 \in \operatorname{Ass} M_2$,

(c) $M = S/I$ where I is a monomial ideal and

(d) every cyclic submodule of M is pure.

This question has a negative answer, if R is a Noetherian local ring and M a finitely generated R-module of dimension 2. For the bigraded case we believe that the question has a negative answer for dimension 4. We include this question at the end of this section.

In Section 2, we describe explicitly the Krull-dimension of the graded components of local cohomology of relative Cohen-Macaulay modules. We show that if M is relative Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q, M) = q$, then $\dim_S H^q_Q(M) = p$ where $p = \operatorname{cd}(P, M)$. Something more general is true for its graded components, namely, if $f_Q(M) = \operatorname{cd}(Q, M) = q$ with $p+q = \dim M$, then $\dim_{K[x]} H^q_Q(M)_j = p$ for $j \ll 0$. Here, $f_Q(M)$ is the finiteness dimension of M with relative to Q. As a consequence, if M is relative Cohen-Macaulay with respect to Q, then $H^q_Q(M)$ is an Artinian S-module if and only if $q = \dim(M)$.

In the following section we consider the hypersurface ring $R = S/fS$ where f is a bihomogeneous element of S of degree (a, b) with $a, b > 0$. Note that $H^i_Q(R) = 0$ for $i \neq n, n-1$. It is a well-known fact that $H^n_Q(R)$ is not finitely generated for $n \geq 1$. For $n \geq 2$, we prove that $H^{n-1}_Q(R)$ is not finitely generated too. Furthermore, for the Artinianness of local cohomology of R, we show that $H^n_Q(R)$ is an Artinian S-module for $m \leq 1$. Also, $H^{n-1}_Q(R)$ is an Artinian S-module if and only if $m = 0$.

In the final section, we let Δ be a simplicial complex on $[n+m]$ and $K[\Delta] = S/I_\Delta$ its Stanley-Reisner ring. We say that Δ is relative Cohen-Macaulay with respect to Q over K if $K[\Delta]$ is relative Cohen-

Macaulay with respect to Q. We first observe that $\operatorname{cd}(Q, K[\Delta]) = \dim \Delta_W + 1$ where Δ_W is the subcomplex of Δ whose faces are subsets of W. This generalizes the known fact that for every simplicial complex Δ one has $\dim K[\Delta] = \dim \Delta + 1$. Using this result and the generalization of Hochster's formula [**10**] we prove the following: Δ is relative Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q, K[\Delta]) = q$ if and only if $\widetilde{H}_i((\operatorname{link} F \cup G)_W; K) = 0$ for all $F \in \Delta_W$, $G \subseteq V$ and all $i < \dim \operatorname{link}_{\Delta_W} F$. This in particular implies the Reisner's criterion for Cohen-Macaulay simplicial complexes. A general version of this statement for the monomial case is obtained.

1. Cohen-Macaulayness and unmixedness with respect to P, Q and $P+Q$. Let K be a field and $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ the standard bigraded polynomial ring over K. We set $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. We denote by $\operatorname{cd}(Q, M)$ the cohomological dimension of M with respect to Q, which is the largest integer i for which $H^i_Q(M) \neq 0$. We recall the following [**11**, Definition 1.3].

Definition 1.1. Let M be a finitely generated bigraded S-module and $q \in \mathbf{Z}$. We call M relative Cohen-Macaulay with respect to Q if $H^i_Q(M) = 0$ for all $i \neq q$. This is equivalent to saying that $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M) = q$, see [**11**, Proposition 1.2].

We recall the following facts from [**11**] which will be used in the sequel.

$$(1) \quad \operatorname{cd}(P, M) = \dim M/QM \qquad \text{and} \qquad \operatorname{cd}(Q, M) = \dim M/PM.$$

In [**11**, Proposition 3.1] we have shown that if M is a finitely generated bigraded Cohen-Macaulay S-module, then "M is relative Cohen-Macaulay with respect to P" if and only if "M is relative Cohen-Macaulay with respect to Q." The following question is inspired by this result.

Question 1.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, I and J two ideals of R such that $I + J = \mathfrak{m}$ and M a finitely generated R-module. Let M be relative Cohen-Macaulay with respect to I, i.e., $\operatorname{grade}(I, M) = \operatorname{cd}(I, M)$ and relative Cohen-Macaulay with respect to J, i.e., $\operatorname{grade}(J, M) = \operatorname{cd}(J, M)$. Is M Cohen-Macaulay?

We give several examples to show that Question 1.2 does not hold in general. The examples are given for graded, local and bigraded cases.

Example 1.3. Consider the standard graded polynomial ring $S = K[x_1, \dots, x_{2n}]$ with $n \geq 1$ and $\deg x_i = 1$ for all i. We set $P = (x_1, \dots, x_n)$, $Q = (x_{n+1}, \dots, x_{2n})$, $\mathfrak{m} = (x_1, \dots, x_{2n})$ the unique graded maximal ideal of S and $R = S \oplus S/\mathfrak{p}$ where $\mathfrak{p} = (x_1 + x_{n+1}, x_2 + x_{n+2}, \dots, x_n + x_{2n})$. We first claim that $S/\mathfrak{p}$ is a Cohen-Macaulay S-module of dimension n. In fact, if we set $B_i = K[x_i, x_{n+i}]/(x_i + x_{n+i})$ for $i = 1, \dots, n$, then $S/\mathfrak{p} = B_1 \otimes_K B_2 \otimes_K \cdots \otimes_K B_n$. Thus, by [**12**, Corollary 2.3], $S/\mathfrak{p}$ is Cohen-Macaulay of dimension n. Hence, $\operatorname{depth} R = n$ and $\dim R = 2n$. On the other hand, $\operatorname{grade}(P, R) = \operatorname{cd}(P, R) = \operatorname{grade}(Q, R) = \operatorname{cd}(Q, R) = n$. Therefore, R is relative Cohen-Macaulay with respect to P and Q but not Cohen-Macaulay. Localizing R at the maximal ideal $\mathfrak{m}$ and noting that, for any graded ideal $I \subseteq S$, we have $\operatorname{grade}(I, R) = \operatorname{grade}(I_{\mathfrak{m}}, R_{\mathfrak{m}})$, $\operatorname{cd}(I, R) = \operatorname{cd}(I_{\mathfrak{m}}, R_{\mathfrak{m}})$, $\operatorname{depth}_S R = \operatorname{depth}_{S_{\mathfrak{m}}} R_{\mathfrak{m}}$ and $\dim_S R = \dim_{S_{\mathfrak{m}}} R_{\mathfrak{m}}$. Now one easily obtains that the question is not the case in the local case too.

Example 1.4. Let $S = K[x_1, x_2, y_1, y_2]$ be the standard bigraded polynomial ring with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$ for $i = 1, 2$. Set $R = S/I$ where $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$ with $\mathfrak{p}_i = (x_i, y_i)$ for $i = 1, 2$. Let $\mathfrak{m}$ be the unique graded maximal ideal of S. From the exact sequence $0 \to R \to S/\mathfrak{p}_1 \oplus S/\mathfrak{p}_2 \to S/\mathfrak{m} \to 0$, we have the exact sequence

$$\begin{aligned} \longrightarrow H^j_Q(R) \longrightarrow H^j_Q(S/\mathfrak{p}_1) \oplus H^j_Q(S/\mathfrak{p}_2) \\ \longrightarrow H^j_Q(S/\mathfrak{m}) \longrightarrow H^{j+1}_Q(R) \longrightarrow . \end{aligned}$$

Since $H^0_Q(S/\mathfrak{m}) = S/\mathfrak{m}$ and $H^j_Q(S/\mathfrak{p}_i) = 0$ for $j \neq 1$ and, for $i = 1, 2$, it follows that $\operatorname{grade}(Q, R) = \operatorname{cd}(Q, R) = 1$. By a similar argument, applying the functor $H^i_P(-)$ to the above short exact sequence, one obtains $\operatorname{grade}(P, R) = \operatorname{cd}(Q, R) = 1$. Thus, R is relative Cohen-Macaulay with respect to P and Q. On the other hand, one has $\operatorname{depth} R = 1$ and $\dim R = 2$. Therefore, R is relative Cohen-Macaulay with respect to P and Q but not Cohen-Macaulay.

In the following we show that Question 1.2 has a positive answer in some cases. We first recall [**11**, Theorem 3.6].

Theorem 1.5. *Let M be a finitely generated bigraded S-module and $|K| = \infty$. If M is relative Cohen-Macaulay with respect to Q, then we have* $\operatorname{cd}(Q, M) + \operatorname{cd}(P, M) = \dim M$.

Proposition 1.6. *Let M be a finitely generated bigraded S-module with* $\operatorname{cd}(P, M) = p$, $\operatorname{cd}(Q, M) = q \geq 0$ *and $|K| = \infty$. Then the following statements hold*:

(a) *If M is relative Cohen-Macaulay with respect to P and Q with $p \in \{0, \dim M\}$, then M is Cohen-Macaulay.*

(b) *Suppose $M = M_1 \otimes_K M_2$ where M_1 is a finitely generated graded $K[x]$-module and M_2 a finitely generated graded $K[y]$-module. If M is relative Cohen-Macaulay with respect to P and Q, then M is Cohen-Macaulay.*

Proof. In order to prove (a) we first let $p = 0$. We consider the spectral sequence $H^i_Q(H^j_P(M)) \underset{i}{\Rightarrow} H^{i+j}_{\mathfrak{m}}(M)$ where $\mathfrak{m} = P + Q$. As $H^j_P(M) = 0$ for all $j \neq 0$, then the above spectral sequence degenerates and one obtains for all i the following isomorphism of bigraded S-modules, $H^i_Q(H^0_P(M)) \cong H^i_{\mathfrak{m}}(M)$. Using the fact that $\operatorname{cd}(P, M) = 0$ if and only if $H^0_P(M) = M$, we therefore have $H^i_Q(M) \cong H^i_{\mathfrak{m}}(M)$. Since $H^i_Q(M) = 0$ for all $i \neq q$, it follows that $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq q$, and hence M is Cohen-Macaulay. Now let $p = \dim M$. Theorem 1.5 implies that $q = 0$, and then by a similar argument as above, M is Cohen-Macaulay. In order to prove (b), we suppose that M is relative Cohen-Macaulay with respect to P and Q. By [**11**, Proposition 1.5], M_1 is a Cohen-Macaulay $K[x]$-module of dimension p and M_2 a Cohen-Macaulay $K[y]$-module of dimension q. Therefore, $\operatorname{depth} M = \dim M = p + q$ by [**12**, Corollary 2.3]. $\square$

We recall the following known facts which will be used in the rest of the paper, see [**3**, Proposition 4.4, Corollary 4.6]. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of S-modules with M finitely generated, then

$$\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, M'), \operatorname{cd}(Q, M'')\}. \tag{2}$$

Let $\operatorname{Min} M$ denote the minimal elements of $\operatorname{Supp} M$. Then

$$\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Ass}(M)\}. \tag{3}$$

Also,

$$\begin{aligned}\operatorname{cd}(Q, M) &= \max\{\operatorname{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}(M)\}\\ &= \max\{\operatorname{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Min}(M)\}.\end{aligned}$$

Proposition 1.7. *Let M be a finitely generated bigraded S-module with $|K| = \infty$. Then we have*

$$\operatorname{grade}(Q, M) \leq \operatorname{cd}(Q, S/\mathfrak{p}) \quad \textit{for all } \mathfrak{p} \in \operatorname{Ass}(M).$$

Proof. Here we follow the proof of [**2**, Proposition 1.2.13]. Let $\mathfrak{p} \in \operatorname{Ass} M$. We proceed by induction on $\operatorname{grade}(Q, M)$. The inequality is clear if $\operatorname{grade}(Q, M) = 0$. Now let $\operatorname{grade}(Q, M) = k > 0$ and suppose inductively that the result has been proved for all finitely generated bigraded S-modules N such that $\operatorname{grade}(Q, N) < k$. We want to prove it for M. As $\operatorname{grade}(Q, M) > 0$, by [**11**, Lemma 3.4], there exists a bihomogeneous M-regular element $y \in Q$ such that $\operatorname{cd}(Q, M/yM) = \operatorname{cd}(Q, M) - 1$ and of course $\operatorname{grade}(Q, M/yM) = \operatorname{grade}(Q, M) - 1$. In the proof of [**2**, Proposition 1.2.13] we observe that $\mathfrak{p}$ consists of zero divisors of M/yM. Thus, $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass}(M/yM)$. Since y is M-regular, it follows that $y \notin \mathfrak{p}$, and hence $\mathfrak{p} \neq \mathfrak{q}$ because $y \in \mathfrak{q}$. Also, as y is M-regular and $\mathfrak{p} \in \operatorname{Ass}(M)$, one observes that y is $S/\mathfrak{p}$-regular, and hence $\operatorname{grade}(Q, S/\mathfrak{p}) > 0$. Thus, $\operatorname{cd}(Q, S/\mathfrak{p}) = \dim S/(P + \mathfrak{p}) > 0$ by (1). We claim that element y may be chosen to avoid all the minimal prime ideals of $\operatorname{Supp}(S/(P + \mathfrak{p}))$ too. Let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$ be the minimal prime ideals of $\operatorname{Supp}(S/(P + \mathfrak{p}))$. By [**11**, Lemma 3.3], it suffices to show that $Q \not\subseteq \mathfrak{q}_i$ for $i = 1, \ldots, r$. Suppose $Q \subseteq \mathfrak{q}_i$ for some i where $i = 1, \ldots, r$. Since $P + \mathfrak{p} \subseteq \mathfrak{q}_i$, it follows that $\mathfrak{q}_i = P + Q = \mathfrak{m}$, and hence $\dim\ S/(P + \mathfrak{p}) = \operatorname{cd}(Q, S/\mathfrak{p}) = 0$, a contradiction. Using inductive hypothesis and the above observation, we have

$$\begin{aligned}\operatorname{grade}(Q, M) - 1 &= \operatorname{grade}(Q, M/yM)\\ &\leq \operatorname{cd}(Q, S/\mathfrak{q})\\ &= \dim S/(P + \mathfrak{q})\\ &< \dim S/(P + \mathfrak{p})\\ &= \operatorname{cd}(Q, S/\mathfrak{p}),\end{aligned}$$

as desired. $\square$

This in particular generalizes the following known result:

Corollary 1.8. *If M is a finitely generated graded $K[y]$-module, then we have*

$$\operatorname{depth} M \leq \dim S/\mathfrak{p} \quad \textit{for all } \mathfrak{p} \in \operatorname{Ass}(M).$$

In particular, $\operatorname{depth} M \leq \dim M$*.*

Corollary 1.9. *If M is a finitely generated bigraded S-module, then we have*

$$\operatorname{grade}(Q, M) \leq \operatorname{cd}(Q, M).$$

Proof. The assertion follows from Proposition 1.7 and (3). □

Definition 1.10. Let M be a finitely generated bigraded S-module. We say that M is relatively unmixed with respect to Q if $\operatorname{cd}(Q, M) = \operatorname{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$.

In the following we observe that relative Cohen-Macaulay modules with respect to Q are relatively unmixed with respect to Q. In particular, all associated prime ideals of M are minimal in $\operatorname{Supp} M/PM$.

Corollary 1.11. *Let M be a finitely generated bigraded S-module. If M is relative Cohen-Macaulay with respect to Q, then M is relatively unmixed with respect to Q.*

Proof. By Proposition 1.7, we have $\operatorname{grade}(Q, M) \leq \operatorname{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. On the other hand, since $\mathfrak{p} \in \operatorname{Ass}(M)$, we have the monomorphism $S/\mathfrak{p} \to M$ which yields $\operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, M)$ by (2). Thus, the conclusion follows. □

Remark 1.12. Relatively unmixed modules with respect to Q need not be relative Cohen-Macaulay with respect to Q. We consider the hypersurface ring $R = S/fS$ where $f \in S$ is an irreducible bihomogeneous polynomial of degree (a, b) with $a, b > 0$. Note that $\operatorname{Ass}(R) = \{(f)\}$, $\operatorname{grade}(Q, R) = n - 1$ and $\operatorname{cd}(Q, R) = n$. Thus, R is relatively unmixed with respect to Q but not relative Cohen-Macaulay with respect to Q.

In the following we give some cases to show that the converse of Corollary 1.11 holds under some additional assumptions.

Proposition 1.13. *Let M be a finitely generated bigraded S-module with* $\operatorname{cd}(Q,M) > 0$*. If M is relatively unmixed with respect to Q, then* $\operatorname{grade}(Q,M) > 0$*. In particular, if M is relatively unmixed with respect to Q with* $\operatorname{cd}(Q,M) = 1$*, then M is relative Cohen-Macaulay with respect to Q.*

Proof. Suppose $\operatorname{grade}(Q,M) = 0$. Thus, there exists a $\mathfrak{q} \in \operatorname{Ass}(M)$ such that $Q \subseteq \mathfrak{q}$. Therefore, $\operatorname{cd}(Q,M) = \operatorname{cd}(Q,S/\mathfrak{q}) = \dim S/(P + \mathfrak{q}) = \dim S/(P+Q) = 0$, a contradiction. $\square$

Proposition 1.14. *Let M be a finitely generated bigraded S-module for which every quotient of M is relatively unmixed with respect to Q. Then M is relative Cohen-Macaulay with respect to Q.*

Proof. We proceed by induction on $q = \operatorname{cd}(Q,M)$. The claim is obvious for $q = 0$. Suppose $q > 0$ and the result has been proved for all finitely generated bigraded S-modules of cohomological dimension less than q. Since $q > 0$, it follows that $\operatorname{grade}(Q,M) > 0$ by Proposition 1.13. By [**11**, Lemma 3.4], there exists an M-regular bihomogeneous element $y \in Q$ such that $\operatorname{cd}(Q,M/yM) = \operatorname{cd}(Q,M) - 1$ as well as $\operatorname{grade}(Q,M/yM) = \operatorname{grade}(Q,M) - 1$. Our assumption implies that M/yM is relatively unmixed with respect to Q, and hence our induction hypothesis says that M/yM is relative Cohen-Macaulay with respect to Q. Therefore, M is relative Cohen-Macaulay with respect to Q, as desired. $\square$

Remark 1.15. In Remark 1.12, if one takes $n = m = 2$, $f = x_1y_1 + x_2y_2$ and $J = (x_1y_1, y_2)/(f)$, then the quotient ring $R/J = S/(x_1y_1, y_2)$ is not relatively unmixed with respect to Q. In fact, one has $\operatorname{cd}(Q,R/J) = 1$ and, as $\operatorname{Ass}(R/J) = \{(x_1,y_2),(y_1,y_2)\}$ we have $\operatorname{cd}(Q,S/(y_1,y_2)) = 0$. Of course, R is not relative Cohen-Macaulay with respect to Q.

In Question 1.2, we replace "M is relative Cohen-Macaulay with respect to J" by the weaker assumption "M is relatively unmixed with respect to J" and raise the following question:

Question 1.16. Let $(R, \mathfrak{m})$ be a Noetherian local ring, I and J two ideals of R such that $I + J = \mathfrak{m}$ and M a finitely generated R-module. If M is relative Cohen-Macaulay with respect to I and relatively unmixed with respect to J, is M unmixed?

Remark 1.17. In Example 1.3, the ring R is relative Cohen-Macaulay with respect to P and Q. Observe that $\operatorname{Ass}(R) = \{\mathfrak{p}, (0)\}$ for $n \geq 1$ and $\dim S/\mathfrak{p} = n < \dim R = 2n$. Thus, R is not unmixed, and hence Question 1.16 is not the case for dimension 2.

In the following we show that Question 1.16 has a positive answer in some cases.

Proposition 1.18. *Suppose* $\operatorname{cd}(P, M) = p$ *where* $p \in \{0, 1, \dim M\}$, $\operatorname{cd}(Q, M) = q \geq 0$ *and* $|K| = \infty$. *If M is relative Cohen-Macaulay with respect to P and relatively unmixed with respect to Q, then M is unmixed.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass} M$. We first assume that $p = \operatorname{cd}(P, M) = \operatorname{cd}(P, S/\mathfrak{p}) = 0$. Hence, $\operatorname{cd}(Q, M) = \dim M$ and $\operatorname{cd}(Q, S/\mathfrak{p}) = \dim S/\mathfrak{p}$ by Theorem 1.5. As M is relatively unmixed with respect to Q, it follows that M is unmixed. Now let $p = \operatorname{cd}(P, M) = \operatorname{cd}(P, S/\mathfrak{p}) = 1$. We claim that $S/\mathfrak{p}$ is relative Cohen-Macaulay with respect to P. Assume $\operatorname{grade}(P, S/\mathfrak{p}) = 0$. The exact sequence $0 \to S/\mathfrak{p} \to M$ yields the exact sequence $0 \to H^0_P(S/\mathfrak{p}) \to H^0_P(M)$, and hence $\operatorname{grade}(P, M) = 0$, a contradiction. Therefore, we have

$$\dim M = \operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \operatorname{cd}(P, S/\mathfrak{p}) + \operatorname{cd}(Q, S/\mathfrak{p}) = \dim S/\mathfrak{p}.$$

The first and the last equality follow from Theorem 1.5. Finally, we assume that $p = \dim M$. By Theorem 1.5 we have $q = 0$, and hence by a similar argument as the first part M is unmixed. $\square$

Corollary 1.19. *Let* $\dim M \leq 3$ *and* $|K| = \infty$. *If M is relative Cohen-Macaulay with respect to P with* $\operatorname{cd}(P, M) = p$ *and relatively unmixed with respect to Q with* $\operatorname{cd}(Q, M) = q$, *then M is unmixed.*

Proof. The assertion is clear for $\dim M \leq 2$, by Proposition 1.18. Suppose $\dim M = 3$ with $p = 1$ and $q = 2$. The assertion is also clear in this case by Proposition 1.18. Finally, we assume $\dim M = 3$ with $p = 2$ and $q = 1$. By Proposition 1.13, M is relative Cohen-Macaulay with respect to Q, and hence the assertion follows again from Proposition 1.18. $\square$

Proposition 1.20. *Let M_1 and M_2 be two non-zero finitely generated graded modules over $K[x]$ and $K[y]$, respectively, and let $|K| = \infty$. Set $M = M_1 \otimes_K M_2$, and assume that $K[x]/\mathfrak{p}_1 \otimes_K K[y]/\mathfrak{p}_2$ is an integral domain for all $\mathfrak{p}_1 \in \operatorname{Ass} M_1$ and $\mathfrak{p}_2 \in \operatorname{Ass} M_2$. If M is relative Cohen-Macaulay with respect to P and relatively unmixed with respect to Q, then M is unmixed.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(M)$. Note that

$$\operatorname{Ass}_S(M) = \bigcup_{\mathfrak{p}_1 \in \operatorname{Ass}_{K[x]}(M_1)} \bigcup_{\mathfrak{p}_2 \in \operatorname{Ass}_{K[y]}(M_2)} \operatorname{Ass}_S(K[x]/\mathfrak{p}_1 \otimes_K K[y]/\mathfrak{p}_2),$$

see [**12**, Corollary 3.7]. Thus, there exist $\mathfrak{p}_1 \in \operatorname{Ass}_{K[x]}(M_1)$ and $\mathfrak{p}_2 \in \operatorname{Ass}_{K[y]}(M_2)$ such that $\mathfrak{p} \in \operatorname{Ass}_S(K[x]/\mathfrak{p}_1 \otimes_K K[y]/\mathfrak{p}_2) = \operatorname{Ass}(S/\mathfrak{p}_1 S + \mathfrak{p}_2 S)$. Since $S/(\mathfrak{p}_1 S + \mathfrak{p}_2 S)$ is an integral domain, it follows that $\operatorname{Ass}(S/\mathfrak{p}_1 S + \mathfrak{p}_2 S) = \{\mathfrak{p}_1 S + \mathfrak{p}_2 S\}$, and hence $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$. By our assumption M is relative Cohen-Macaulay with respect to P. Thus, M is relatively unmixed with respect to P, and we have

$$\operatorname{cd}(P, M) = \operatorname{cd}(P, S/\mathfrak{p}) = \dim S/(Q+\mathfrak{p}) = \dim S/(Q+\mathfrak{p}_1) = \dim K[x]/\mathfrak{p}_1.$$

On the other hand, since M is relatively unmixed with respect to Q, we have

$$\operatorname{cd}(Q, M) = \operatorname{cd}(Q, S/\mathfrak{p}) = \dim S/(P+\mathfrak{p}) = \dim S/(P+\mathfrak{p}_2) = \dim K[y]/\mathfrak{p}_2.$$

Thus by Theorem 1.5 and [**12**, Corollary 2.3], we have

$$\dim M = \operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim K[x]/\mathfrak{p}_1 + \dim K[y]/\mathfrak{p}_2 = \dim S/\mathfrak{p},$$

as desired. $\square$

Corollary 1.21. *Let I be a monomial ideal in $K[x]$, J a monomial ideal in $K[y]$ and $|K| = \infty$. We set $M = K[x]/I \otimes_K K[y]/J$. If M is relative Cohen-Macaulay with respect to P and relatively unmixed with respect to Q, then M is unmixed.*

Proof. Note that the associated prime ideals of a monomial ideal are monomial prime ideals, see [**6**, Corollary 1.3.9]. Now the assertion follows from Proposition 1.20. □

Proposition 1.22. *Let $I \subseteq S$ be a monomial ideal, and set $R = S/I$ with $|K| = \infty$. If R is relative Cohen-Macaulay with respect to P and relatively unmixed with respect to Q, then R is unmixed.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(R)$. Since R is relative Cohen-Macaulay with respect to P, it follows that R is relatively unmixed with respect to P and we have $\operatorname{cd}(P,R) = \operatorname{cd}(P, S/\mathfrak{p})$. Our assumption also says that $\operatorname{cd}(Q,R) = \operatorname{cd}(Q, S/\mathfrak{p})$. As we mentioned above that the associated prime ideals of a monomial ideal are monomial prime ideals, we may write $\mathfrak{p} = \mathfrak{p}_x + \mathfrak{p}_y$ where $\mathfrak{p}_x$ is the monomial prime ideal in $K[x]$ and $\mathfrak{p}_y$ is the monomial prime ideal in $K[y]$. Hence,

$$\operatorname{cd}(P,R) = \operatorname{cd}(P, S/\mathfrak{p}) = \dim S/(Q+\mathfrak{p}) = \dim S/(Q+\mathfrak{p}_x) = \dim K[x]/\mathfrak{p}_x$$

and

$$\operatorname{cd}(Q,R) = \operatorname{cd}(Q, S/\mathfrak{p}) = \dim S/(P+\mathfrak{p}) = \dim S/(P+\mathfrak{p}_y) = \dim K[y]/\mathfrak{p}_y.$$

Thus, by Theorem 1.5 and [**12**, Corollary 2.3], we have

$$\dim R = \operatorname{cd}(P,R) + \operatorname{cd}(Q,R) = \dim K[x]/\mathfrak{p}_x + \dim K[y]/\mathfrak{p}_y = \dim S/\mathfrak{p},$$

as desired. □

Proposition 1.23. *Let $|K| = \infty$, and let M be a finitely generated bigraded S-module such that every cyclic submodule of M is pure. If M is relative Cohen-Macaulay with respect to P with* $\operatorname{cd}(P,M) = p$ *and relatively unmixed with respect to Q with* $\operatorname{cd}(Q,M) = q$, *then M is unmixed.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(M)$. We first claim that $S/\mathfrak{p}$ is relative Cohen-Macaulay with respect to P. Let $f_1, \dots, f_p$ be a maximal M-sequence in P. As $S/\mathfrak{p}$ is a cyclic submodule of M, we have the exact sequence $0 \to S/\mathfrak{p} \to M$ which yields the exact sequence $0 \to S/(\mathfrak{p} + (f_1, \dots, f_p)) \to M/(f_1, \dots, f_p)M$. Since $f_i \notin Z(M/(f_1, \dots, f_{i-1})M)$ for $i = 1, \dots, p$, it follows that $f_i \notin Z(S/(\mathfrak{p} + (f_1, \dots, f_{i-1})))$ for $i = 1, \dots, p$. Thus, $f_1, \dots, f_p$ is an $S/\mathfrak{p}$-sequence in P which may not be maximal. Hence, $\operatorname{grade}(P, S/\mathfrak{p}) \geq p$. On the other hand, by our assumption M is relative Cohen-Macaulay with respect to P. Thus, M is relatively unmixed with respect to P, and we have $\operatorname{cd}(P, M) = \operatorname{cd}(P, S/\mathfrak{p}) = p \leq \operatorname{grade}(P, S/\mathfrak{p})$. We conclude that $\operatorname{grade}(P, S/\mathfrak{p}) = \operatorname{cd}(P, S/\mathfrak{p}) = p$, and hence $S/\mathfrak{p}$ is relative Cohen-Macaulay with respect to P. Now, by Theorem 1.5, we have

$$\dim M = \operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \operatorname{cd}(P, S/\mathfrak{p}) + \operatorname{cd}(Q, S/\mathfrak{p}) = \dim S/\mathfrak{p},$$

as desired. $\square$

We remark that in the above proposition such a class of modules that "every cyclic submodule of M is pure" exists: Let R be a domain and $f \in R$ a non-zero element. Hence, $(f) \cong R/\operatorname{Ann}_R(f) \cong R$ and so (f) is pure. More is true in general, if M is a torsion-free R-module where R is a domain.

Remark 1.24. Let M be a finitely generated bigraded unmixed S-module. Then all the associated prime ideals of M have the same height. This number is $n + m - (p + q)$ when M satisfies the conditions of Question 1.16.

In Corollary 1.19, we observed that Question 1.16 has a positive answer for $\dim M \leq 3$. We end this section with the following question:

Question 1.25. Let M be a finitely generated bigraded S-module of dimension 4 that is relative Cohen-Macaulay with respect to P and Q. Is the module M unmixed?

2. The Krull-dimension of the graded components of local cohomology of relative Cohen-Macaulay modules. In this section we describe explicitly the Krull-dimension of the graded components of local cohomology of relative Cohen-Macaulay modules. As a first result we have the following:

Proposition 2.1. *Let M be a finitely generated bigraded S-module with* $\mathrm{cd}\,(P, M) = p$, $\mathrm{cd}\,(Q, M) = q$ *and* $|K| = \infty$. *If M is relative Cohen-Macaulay with respect to Q, then* $\dim_S H^q_Q(M) = p$.

Proof. We first note that $\mathrm{Supp}\, H^q_Q(M) \subseteq \mathrm{Supp}\, M/QM$. Thus, we have $\dim H^q_Q(M) \leq \dim M/QM = \mathrm{cd}\,(P, M) = p$. As M is relative Cohen-Macaulay with respect to Q, from the spectral sequence $H^i_P(H^j_Q(M)) \underset{i}{\Rightarrow} H^{i+j}_{\mathfrak{m}}(M)$, we get the following isomorphisms of bigraded S-modules $H^i_P(H^q_Q(M)) \cong H^{i+q}_{\mathfrak{m}}(M)$ for all i. By Theorem 1.5, we have $p + q = \dim M$ which yields $H^p_P(H^q_Q(M)) \neq 0$ and $H^i_P(H^q_Q(M)) = 0$ for $i > p$. Consequently, $\mathrm{cd}\,(P, H^q_Q(M)) = p$. As we always have $p \leq \dim H^q_Q(M)$, the desired equality follows. $\square$

Let M be a finitely generated bigraded S-module. Recall the finiteness dimension of M with respect to Q by:

$$f_Q(M) = \inf\{i \in \mathbf{N} : H^i_Q(M) \text{ is not finitely generated}\}.$$

For all integers j and k, we set

$$H^k_Q(M)_j = H^k_Q(M)_{(*,j)} = \oplus_i H^k_Q(M)_{(i,j)}.$$

Notice that $H^k_Q(M)_j$ is a finitely generated graded $K[x]$-module.

Proposition 2.2. *Let M be a finitely generated bigraded S-module with* $\mathrm{cd}\,(P, M) = p$, $\mathrm{cd}\,(Q, M) = q$ *and* $p + q = \dim M$. *If* $f_Q(M) = \mathrm{cd}\,(Q, M) = q$, *then* $\dim_{K[x]} H^q_Q(M)_j = p$ *for* $j \ll 0$.

Proof. We consider the spectral sequence $H^i_P(H^k_Q(M))_{(*,j)} \underset{i}{\Rightarrow} H^{i+k}_{\mathfrak{m}}(M)_{(*,j)}$. Observe that $H^i_P(H^k_Q(M))_{(*,j)} = H^i_{P_0}(H^k_Q(M)_{(*,j)})$ where P_0 is the graded maximal ideal of $K[x]$. This equality follows from the definition of local cohomology using the Čech complex. Note that $H^k_Q(M)_j = 0$ for all $k < \mathrm{cd}\,(Q, M) = q$ and $j \ll 0$. Thus the spectral sequence degenerates and one obtains for all i and $j \ll 0$ the following isomorphisms of bigraded $K[x]$-modules $H^i_{P_0}(H^q_Q(M)_{(*,j)}) \cong H^{i+q}_{\mathfrak{m}}(M)_{(*,j)}$. Our assumption says that $p + q = \dim M \geq 1$. Thus, as $H^{p+q}_{\mathfrak{m}}(M)$ is a non zero Artinian S-module which is not finitely generated, we have $H^{p+q}_{\mathfrak{m}}(M)_j \neq 0$ for $j \ll 0$. Hence, $H^p_{P_0}(H^q_Q(M)_j) \neq 0$ for $j \ll 0$. Since $H^i_{P_0}(H^q_Q(M)_j) = 0$ for $i > p$, it therefore follows that $\dim_{K[x]} H^q_Q(M)_j = p$ for $j \ll 0$, as desired. $\square$

Corollary 2.3. *Let M be a finitely generated bigraded S-module with $\operatorname{cd}(P, M) = p$, $\operatorname{cd}(Q, M) = q$, $p + q = \dim M$, $f_Q(M) = \operatorname{cd}(Q, M) = q$ and $|K| = \infty$. Then $H^q_Q(M)$ is an Artinian S-module if and only if $q = \dim(M)$.*

Proof. Suppose $H^q_Q(M)$ is an Artinian S-module. One has that $H^q_Q(M)_j$ is an Artinian $K[x]$-module for all j. Thus, $\dim_{K[x]} H^q_Q(M)_j = 0$ for all j. On the other hand, $\dim_{K[x]} H^q_Q(M)_j = p$ for $j \ll 0$ by Proposition 2.2. Thus, we conclude that $p = 0$, and hence $q = \dim M$ by Theorem 1.5. The converse is a well-known fact. $\square$

Corollary 2.4. *Let M be a finitely generated bigraded S-module with $|K| = \infty$. If M is relative Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q, M) = q > 0$, then $\dim_{K[x]} H^q_Q(M)_j = p$ for $j \ll 0$. Moreover, $H^q_Q(M)$ is an Artinian S-module if and only if $q = \dim(M)$.*

Proof. The assertion follows from Proposition 2.2, Theorem 1.5 and Corollary 2.3. $\square$

Recall the Q-finiteness dimension $f^Q_{\mathfrak{m}}(M)$ of M with respect to $\mathfrak{m}$ by

$$f^Q_{\mathfrak{m}}(M) = \inf\{i \in \mathbf{N}_0 : Q \nsubseteq \operatorname{rad}(0 : H^i_{\mathfrak{m}}(M))\}.$$

In view of [**8**, Proposition 2.3], one has

$$f^Q_{\mathfrak{m}}(M) = \sup\{i \in \mathbf{N}_0 : H^k_{\mathfrak{m}}(M)_j = 0 \text{ for all } k < i \text{ and all } j \ll 0\}.$$

Proposition 2.5. *Let M be a finitely generated bigraded S-module with $\operatorname{cd}(P, M) = p$, $\operatorname{cd}(Q, M) = q$ and $p + q = \dim M$. If M is generalized Cohen-Macaulay with $f_Q(M) = \operatorname{cd}(Q, M) = q$, then $\operatorname{depth}_{K[x]} H^q_Q(M)_j = p$ for $j \ll 0$.*

Proof. Since M is generalized Cohen-Macaulay, it follows that $f_{\mathfrak{m}}^{Q}(M) = \dim(M) = p+q$. By [**5**, Theorem 2.3] we have $\operatorname{grade}(P_0, H_Q^q(M)_j) = f_{\mathfrak{m}}^{Q}(M) - \operatorname{cd}(Q,M)$ for $j \ll 0$. This yields the desired conclusion. □

Corollary 2.6. *Let M be a finitely generated bigraded generalized Cohen-Macaulay S-module with $f_Q(M) = \operatorname{cd}(Q,M) = q$ and $p+q = \dim M$. Then the following statements hold*:

(a) $H_Q^q(M)_j$ *is a Cohen-Macaulay $K[x]$-module of dimension p for $j \ll 0$.*

(b) $\operatorname{proj\,dim}_{K[x]} H_Q^q(M)_j = n-p$ *for $j \ll 0$.*

3. Finiteness properties of local cohomology of hypersurface rings. It is a well-known fact that the top local cohomology modules are almost never finitely generated. Let M be relative Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q,M) = q$. Thus $H_Q^q(M)$ is not finitely generated for $q > 0$. Also $H_Q^q(M)$ is not Artinian for $q \neq \dim M$ by Corollary 2.4. We consider the hypersurface ring $R = S/fS$ where $f \in S$ is a bihomogeneous form of degree (a,b) with $a,b > 0$. We note that $H_Q^i(R) = 0$ for $i \neq n, n-1$. Thus, R is not far from being relative Cohen-Macaulay with respect to Q. In the following, for $n \geq 2$, we observe that $H_Q^{n-1}(R)$ is not finitely generated. We also obtain some results on Artinianness of $H_Q^n(R)$ and $H_Q^{n-1}(R)$.

Proposition 3.1. *Let $R = S/fS$ be the hypersurface ring. Then $H_Q^{n-1}(R)$ is not finitely generated for $n \geq 2$.*

Proof. The exact sequence $0 \to S(-a,-b) \xrightarrow{f} S \to S/fS \to 0$, induces the following exact sequence of S-modules

$$0 \longrightarrow H_Q^{n-1}(R) \longrightarrow H_Q^n(S)(-a,-b) \xrightarrow{f} H_Q^n(S) \longrightarrow H_Q^n(R) \longrightarrow 0.$$

Moreover, $H_Q^i(R) = 0$ for all $i < n-1$. Let F be the quotient field of $K[x]$. Then

$$F \otimes_{K[x]} S = F[y_1, \ldots, y_n] =: T.$$

Let T_+ be the graded maximal ideal of T. By the graded flat base change theorem, we have

$$F \otimes_{K[x]} H^i_Q(R) \cong H^i_{T_+}(F \otimes_{K[x]} R) \quad \text{for all } i.$$

Since $F \otimes_{K[x]} R = T/fT$ and $\dim T/fT = n-1$, it follows that

$$H^i_{T_+}(T/fT) = 0 \quad \text{for all } i \neq n-1.$$

Note that $H^{n-1}_{T_+}(T/fT)$ is an Artinian T-module which is not finitely generated. Thus, $H^{n-1}_{T_+}(T/fT)_j \neq 0$ for $n \geq 2$ and $j \ll 0$. Hence, $H^{n-1}_Q(R)_j \neq 0$ for $n \geq 2$ and $j \ll 0$. Therefore, $H^{n-1}_Q(R)$ is not finitely generated for $n \geq 2$. $\square$

For a bihomogeneous element $f \in S$, we denote by $c(f)$ the ideal of $K[x]$ generated by all the coefficients of f and $P_0 = (x_1, \ldots, x_m)$ the graded maximal ideal of $K[x]$. A dual version of the above observation can be discussed as Artinianness of local cohomology of hypersurface rings.

Proposition 3.2. *Let $R = S/fS$ be the hypersurface ring. Then the following statements hold*:

(a) *If $m \leq 1$, then $H^n_Q(R)$ is an Artinian S-module.*

(b) *Let $m \geq 2$. If $H^n_Q(R)$ is an Artinian S-module, then $c(f)$ is an P_0-primary ideal for which the non-zero coefficients of f do not form a system of parameters for $K[x]$.*

Proof. For the proof of (a), if $m = 0$, then Q is the graded maximal ideal of $K[y]$ and we may write $f = \sum_{|\beta|=b} c_\beta y^\beta$ where $c_\beta \in K$. Hence, R is Cohen-Macaulay of dimension $n-1$ and so $H^n_Q(R) = 0$ is Artinian. Let $m = 1$. We need to show $\operatorname{Supp} H^n_Q(R) \subseteq \{\mathfrak{m}\}$ and $\operatorname{Hom}(S/\mathfrak{m}, H^n_Q(R))$ is finitely generated S-module where $\mathfrak{m} = P + Q$ is the unique graded maximal ideal of S. We may write $f = x^a \sum_{|\beta|=b} c_\beta y^\beta$ where $c_\beta \in K$. Thus, $c(f) = (x^a)$ is an (x)-primary ideal and hence $\operatorname{Supp} H^n_Q(R) = \{\mathfrak{m}\}$ by [**9**, Corollary 2.6]. As $m = 1$, $H^n_Q(R)$ is Q-cofinite by [**4**, Theorem 1]. Thus, $\operatorname{Hom}(S/Q, H^n_Q(R))$ is finitely generated. Consequently, $\operatorname{Hom}(S/\mathfrak{m}, H^n_Q(R))$ is finitely generated.

For the proof of (b), as $H^n_Q(R)$ is an Artinian S-module, we have $\operatorname{Supp} H^n_Q(R) \subseteq \{\mathfrak{m}\}$. On the other hand, $\operatorname{Supp} H^n_Q(R) = \{\mathfrak{q} \in \operatorname{Spec} S : c(f)+Q \subseteq \mathfrak{q}\}$ by [**9**, Lemma 2.5]. Thus, the maximal ideal $\mathfrak{m}$ is the only minimal prime ideal $c(f)+Q$. Hence, P_0 is the only minimal prime ideal $c(f)$. Therefore, $c(f)$ is a P_0-primary ideal. Since $\operatorname{Hom}(S/\mathfrak{m}, H^n_Q(R))$ is finitely generated, the non-zero coefficients of f do not form a system of parameters for $K[x]$ by [**9**, Theorem 2.3]. $\square$

Proposition 3.3. *Let $R = S/fS$ be the hypersurface ring with $n \geq 1$. Then $H^{n-1}_Q(R)$ is an Artinian S-module if and only if $m = 0$.*

Proof. The exact sequence $0 \to S(-a,-b) \xrightarrow{f} S \to S/fS \to 0$ induces the following exact sequence of S-modules:

$$0 \longrightarrow H^{n-1}_Q(R) \longrightarrow H^n_Q(S)(-a,-b) \xrightarrow{f} H^n_Q(S) \longrightarrow H^n_Q(R) \longrightarrow 0.$$

Note that $H^{n-1}_Q(R) = 0 \underset{H^n_Q(S)}{:} f \supseteq 0 \underset{H^n_Q(S)}{:} Q$ and $H^n_Q(S)$ is a Q-torsion S-module. By [**1**, Theorem 7.1.2], we have $H^{n-1}_Q(R)$ is an Artinian S-module if and only if $H^n_Q(S)$ is an Artinian S-module. Hence, by Corollary 2.4, this is equivalent to saying that $m = 0$. $\square$

4. Generalization of Reisner's criterion for Cohen-Macaulay simplicial complexes. As before, let $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be the standard bigraded polynomial ring in $n + m$ variables over a field K and Δ a simplicial complex on $[n + m]$. We assume that Δ has vertices $\{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ where vertices $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_n\}$ correspond to the variables of $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, respectively. We denote by Δ_W the restriction of Δ on W which is the subcomplex

$$\Delta_W = \{F \in \Delta : F \subseteq W\}.$$

Let F be a facet simplicial complex of Δ on $[n+m]$. We denote by $\mathfrak{p}_F$ the prime ideal generated by all x_i and y_j such that $v_i, w_j \notin F$.

Proposition 4.1. *Let Δ be a simplicial complex on $[n + m]$ and $K[\Delta] = S/I_\Delta$ the Stanley-Reisner ring of Δ. Then*

$$\operatorname{cd}(Q, K[\Delta]) = \dim \Delta_W + 1.$$

Proof. Using primary decomposition of $I_\Delta = \cap_F \mathfrak{p}_F$ where the intersection is taken over all facets of Δ, together with (1) and (3) we have

$$\begin{aligned}
\operatorname{cd}(Q, K[\Delta]) &= \max\{\operatorname{cd}(Q, S/\mathfrak{q}) : \mathfrak{q} \in \operatorname{Min}(K[\Delta])\}\\
&= \max\{\operatorname{cd}(Q, S/\mathfrak{p}_F) : F \text{ is a facet of } \Delta\}\\
&= \max\{\dim S/(P + \mathfrak{p}_F) : F \text{ is a facet of } \Delta\}\\
&= \max\{\dim K[y_1, \dots, y_n]/\mathfrak{p}_G : G \text{ is a facet of } \Delta_W\}\\
&= \max\{|G| : G \text{ is a facet of } \Delta_W\}\\
&= \dim \Delta_W + 1,
\end{aligned}$$

as required. $\square$

Corollary 4.2. *If Δ is a simplicial complex on $[n]$, then $\dim K[\Delta] = \dim \Delta + 1$.*

We say that Δ is relative Cohen-Macaulay with respect to Q over K if $K[\Delta]$ is relative Cohen-Macaulay with respect to Q. A simplicial complex Δ is pure if all facets have the same cardinality.

Corollary 4.3. *If Δ is relative Cohen-Macaulay with respect to Q, then Δ_W is a pure simplicial complex.*

Proof. The assertion is immediate from Corollary 1.11 and Proposition 4.1. $\square$

Corollary 4.4. *If $\dim \Delta_W = 0$, then Δ is relative Cohen-Macaulay with respect to Q.*

Proof. By Proposition 4.1, we have $\operatorname{cd}(Q, K[\Delta]) = 1$. Since $\dim \Delta_W = 0$, it follows that the facets of Δ_W are the forms $F_i = (w_i)$ for $i = 1, \dots, n$, and hence $\mathfrak{p}_{F_i} = (x_1, \dots, x_m, y_1, \dots, \widehat{y_i}, \dots, y_n)$ where $y_i \notin \mathfrak{p}_{F_i}$. Thus $Q \nsubseteq \mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Ass}(K[\Delta])$. Therefore, $\operatorname{grade}(Q, K[\Delta]) = 1$. $\square$

Let Δ be a simplicial complex on $[n]$. For a face F of Δ, the link of F in Δ is the subcomplex

$$\operatorname{link}_\Delta F = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \varnothing\},$$

and the star of F in Δ is the subcomplex

$$\mathrm{star}_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}.$$

Note that, if Δ is a pure simplicial complex, then for any $F \in \Delta$ we have $\dim \mathrm{link}_\Delta F = \dim \Delta - |F|$. We denote by $\widetilde{H}_i(\Delta; K)$ the ith reduced homology group of Δ with coefficient in K, see [**2**, Chapter 5] for details. We say that a simplicial complex Δ is connected if there exists a sequence of facets $F = F_0, \dots , F_t = G$ such that $F_i \cap F_{i+1} \neq 0$ for $i = 0, \dots , t - 1$. One has that Δ is connected if and only if $\widetilde{H}_0(\Delta; K) = 0$. We set $\mathbf{Z}^m_- = \{a \in \mathbf{Z}^m : a_i \leq 0 \text{ for } i = 1, \dots , m\}$ and $\mathbf{Z}^n_+ = \{b \in \mathbf{Z}^n : b_i \geq 0 \text{ for } i = 1, \dots , n\}$. We recall the following theorem from [**10**, Theorem 1.3].

Theorem 4.5. *Let $I \subseteq S$ be a squarefree monomial ideal. Then the bigraded Hilbert series of the local cohomology modules of $K[\Delta] = S/I$ with respect to the $\mathbf{Z}^m \times \mathbf{Z}^n$-bigrading is given by*

$$\begin{aligned} H_{H^i_Q(K[\Delta])}(\mathbf{s}, \mathbf{t}) &= \sum_{\substack{a \in \mathbf{Z}^m_+ \\ b \in \mathbf{Z}^n_-}} \dim_K H^i_P(K[\Delta])_{(a,b)} \mathbf{s}^a \mathbf{t}^b \\ &= \sum_{F \in \Delta_W} \sum_{G \subset V} \dim_K \widetilde{H}_{i-|F|-1}((\mathrm{link}\, F \cup G)_W; K) \\ &\quad \times \prod_{v_i \in G} \frac{s_i}{1 - s_i} \prod_{w_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}} \end{aligned}$$

where $\mathbf{s} = (s_1, \dots , s_m)$, $\mathbf{t} = (t_1, \dots , t_n)$, $G = \mathrm{Supp}\, a$, $F = \mathrm{Supp}\, b$ and Δ is the simplicial complex corresponding to the Stanley-Reisner ring $K[\Delta]$.

Here we note that $(\mathrm{link}\, F \cup G)_W = \mathrm{link}_{\Delta_W} F$. As an immediate consequence we obtain:

Corollary 4.6. *The following statements hold*:

(a) *We have $H^i_Q(K[\Delta])_{(a,b)} = 0$ for all i and all $(a, b) \in \mathbf{Z}^m \times \mathbf{Z}^n$ for which $a_i < 0$ for some i or $b_j > 0$ for some j.*

(b) $H^i_Q(K[\Delta])_{(a,b)} \cong \widetilde{H}_{i-|F|-1}((\operatorname{link} F \cup G)_W; K)$ *for all* $(a,b) \in \mathbf{Z}^m_+ \times \mathbf{Z}^n_-$ *with* $G = \operatorname{Supp} a$ *and* $F = \operatorname{Supp} b$.

As a main result of this section, we have the following. Here we follow the proof [**6**, Theorem 8.1.6].

Theorem 4.7. *Let* Δ *be a simplicial complex over a field* K. *The following conditions are equivalent.*

(a) Δ *is relative Cohen-Macaulay with respect to* Q *with* $\operatorname{cd}(Q, K[\Delta]) = q$,

(b) $\widetilde{H}_i((\operatorname{link} F \cup G)_W; K) = 0$ *for all* $F \in \Delta_W$, $G \subseteq V$ *and all* $i < \dim \operatorname{link}_{\Delta_W} F$.

Proof. Note that $\dim \Delta_W = q - 1$ by Proposition 4.1. Let Δ be relative Cohen-Macaulay with respect to Q. This is equivalent to saying that $H^i_Q(K[\Delta]) = 0$ for all $i \neq q$. Hence, by Corollary 4.6, this is equivalent to saying that

(4)
$\widetilde{H}_{i-|F|-1}((\operatorname{link} F \cup G)_W; K) = 0$ for all $F \in \Delta_W, G \subseteq V$ and all $i < q$.

$(a) \Rightarrow (b)$. Since Δ is relative Cohen-Macaulay with respect to Q, it follows from Corollary 4.3 that Δ_W is pure and hence $\dim \operatorname{link}_{\Delta_W} F = \dim \Delta_W - |F| = q - |F| - 1$. Therefore, (4) implies that $\widetilde{H}_i((\operatorname{link} F \cup G)_W; K) = 0$ for all $F \in \Delta_W$, $G \subseteq V$ and all $i < \dim \operatorname{link}_{\Delta_W} F$.

$(b) \Rightarrow (a)$. Let $F \in \Delta_W$, $G \subseteq V$ and $H \in \operatorname{link}_{\Delta_W} F$. Set $\Gamma = \operatorname{link}_{\Delta_W} F$. One has

$$\operatorname{link}_\Gamma H = \operatorname{link}_{\Delta_W}(H \cup F) = (\operatorname{link}(H \cup F \cup G))_W.$$

Hence, our assumption yields

$$\widetilde{H}_i(\operatorname{link}_\Gamma H; K) = \widetilde{H}_i((\operatorname{link}(H \cup F \cup G))_W; K) = 0 \quad \text{for all } i < \dim \operatorname{link}_\Gamma H.$$

Thus, by induction on the $\dim \Delta_W$, we may assume that all proper links of Δ_W are Cohen-Macaulay over K. In particular, the link of each vertex of Δ_W is pure. Thus, all facets containing a given vertex have the same dimension. Now let $\dim \Delta_W = 0$; by Corollary 4.4, Δ is relative Cohen-Macaulay with respect to Q. Thus, we may assume that $\dim \Delta_W \geq 1$. Since $\widetilde{H}_0(\Delta_W; K) = \widetilde{H}_0(\mathrm{link}_{\Delta_W} \varnothing; K) = 0$, it follows that Δ_W is connected. Thus, Δ_W is a pure simplicial complex, and hence for any $F \in \Delta_W$, we have $\dim \mathrm{link}_{\Delta_W} F = q - |F| - 1$. Thus, our hypothesis implies (4), and so Δ is relative Cohen-Macaulay with respect to Q. $\square$

As an immediate consequence we obtain Reisner's criterion for Cohen-Macaulay simplicial complexes:

Corollary 4.8. *Let Δ be a simplicial complex and K a field. Then, Δ is Cohen-Macaulay over K if and only if $\widetilde{H}_i(\mathrm{link}\, F; K) = 0$ for all $F \in \Delta$ and all $i < \dim \mathrm{link}\, F$.*

Proof. In Theorem 4.7 we assume that $m = 0$. Then $G = \varnothing$, $(\mathrm{link}\, F \cup G)_W = \mathrm{link}\, F$, $\Delta_W = \Delta$, Q is the unique maximal ideal $\mathfrak{m}$ and $\mathrm{cd}\,(Q, K[\Delta]) = \dim K[\Delta]$. $\square$

In the proof of the theorem, we showed

Corollary 4.9. *Let Δ be relative Cohen-Macaulay with respect to Q. Then Δ_W is connected.*

Corollary 4.10. *Let Δ be relative Cohen-Macaulay complex with respect to Q and F a face of Δ_W. Then $\mathrm{link}_{\Delta_W} F$ is Cohen-Macaulay.*

Proof. The assertion follows from the beginning of the proof of Theorem 4.7 $(b) \Rightarrow (a)$ and Corollary 4.8. $\square$

Let $I \subseteq S$ be a monomial ideal and $G(I)$ the unique minimal monomial system of generators of I. For a monomial $u \in S$, we may write $u = u_1 u_2$ where $u_1 = x_1^{c_1} \cdots x_m^{c_m}$ and $u_2 = y_1^{d_1} \cdots y_n^{d_n}$. We set $\nu_i(u_1) = c_i$ for $i = 1, \ldots, m$ and $\nu_j(u_2) = d_j$ for $j = 1, \ldots, n$. We also set $\sigma_i = \max\{\nu_i(u_1) : u \in G(I)\}$ for $i = 1, \ldots, m$ and $\rho_j = \max\{\nu_j(u_2) \colon u \in G(I)\}$ for $j = 1, \ldots, n$. For $b = (b_1, \ldots, b_n) \in \mathbf{Z}^n$, we set $G_b = \{j : 1 \leq j \leq n, b_j < 0\}$ and let $a \in \mathbf{Z}_+^m$. We define the simplicial complex $\Delta_{(a,b)}(I)$ whose faces are the set $L - G_b$ with $G_b \subseteq L$ and such that L satisfies the following conditions: for all $u \in G(I)$ there exists a $j \notin L$ such that $\nu_j(u_2) > b_j \geq 0$, or for at least one i, $\nu_i(u_1) > a_i \geq 0$. We recall the following theorem from [**10**, Theorem 2.4].

Theorem 4.11. *Let $I \subseteq S$ be a monomial ideal. Then the Hilbert series of the local cohomology modules of S/I with respect to the $\mathbf{Z}^m \times \mathbf{Z}^n$-bigrading is given by*

$$H_{H_Q^i(S/I)}(\mathbf{s}, \mathbf{t}) = \sum \sum \dim_K \widetilde{H}_{i-|F|-1}(\Delta_{(a,b)}(I); K)\mathbf{s}^a \mathbf{t}^b,$$

where the first sum runs over all $F \in \Delta_W$, $b \in \mathbf{Z}^n$ for which $G_b = F$ and $b_j \leq \rho_j - 1$ for $j = 1, \ldots, n$, and the second sum runs over all $a \in \mathbf{Z}^m$ for which $N_a = G$ and $a_i \geq \sigma_i - 1$ for $i = 1, \ldots, m$. Here $N_a = \operatorname{Supp} a$ and Δ is the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$.

The precise expression of the Hilbert series is given in [**10**]. As a first consequence, we have

Corollary 4.12. *The following statements hold*:

(a) *we have $H_Q^i(S/I)_{(a,b)} = 0$ for all i and all $(a, b) \in \mathbf{Z}^m \times \mathbf{Z}^n$ for which $a_i < \sigma_i - 1$ for some i or $b_j > \rho_j - 1$ for some j.*

(b) *$H_Q^i(S/I)_{(a,b)} \cong \widetilde{H}_{i-|F|-1}(\Delta_{(a,b)}(I); K)$ for all $(a, b) \in \mathbf{Z}^m \times \mathbf{Z}^n$ with $N_a = G$, $G_b = F$, $a_i \geq \sigma_i - 1$ for $i = 1, \ldots, m$ and $b_j \leq \rho_j - 1$ for $j = 1, \ldots, n$.*

For a bigraded S-module M, we recall that the a-invariant of M is defined by

$$a_Q^i(M) = \sup\{\mu : H_Q^i(M)_{(*,\mu)} \neq 0\},$$

and so $\operatorname{reg}(M) = \max_i\{a_Q^i(M) + i : i \geq 0\}$.

Corollary 4.13. *Suppose $I \subseteq S$ is a monomial ideal such that S/I is relative Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q, S/I) = q$. Then*

$$\operatorname{reg}(S/I) \leq \sum_{j=1}^{n} \rho_j - n + q.$$

Proof. Note that, for all $k, j \in \mathbf{Z}$, we have

$$H_Q^q(S/I)_{(k,j)} = \bigoplus_{\substack{a\in\mathbf{Z}^m, |a|=k \\ b\in\mathbf{Z}^n, |b|=j}} H_Q^q(S/I)_{(a,b)},$$

where $|a| = \sum_{i=1}^m a_i$ for $a = (a_1, \dots, a_m)$ and $|b| = \sum_{i=1}^n b_i$ for $b = (b_1, \dots, b_n)$. By Corollary 4.12, $H_Q^q(S/I)_{(k,j)} = 0$ for $k < \sum_{i=1}^m \sigma_i - m$ or $j > \sum_{j=1}^n \rho_j - n$. Thus, we have

$$H_Q^q(S/I)_j = \bigoplus_k H_Q^q(S/I)_{(k,j)} = 0 \text{ for } j > \sum_{j=1}^n \rho_j - n.$$

Hence, $a_Q^q(S/I) \le \sum_{j=1}^n \rho_j - n$, and so the conclusion follows. □

As a generalization of [**7**, Corollary 2.3], we have

Corollary 4.14. *Let $I \subseteq S$ be a monomial ideal. Then, for all i, we have the following isomorphisms of K-vector spaces*

$$H_Q^i(S/I)_{(a,b)} \cong H_Q^i(S/\sqrt{I})_{(a,b)},$$

for all $a \in \mathbf{Z}_+^m$ and $b \in \mathbf{Z}_-^n$. In particular, $\mathrm{cd}\,(Q, S/I) = \mathrm{cd}\,(Q, S/\sqrt{I})$.

Proof. By a similar proof as [**7**, Corollary 2.3], one has $\Delta_{(a,b)}(I) = \Delta_{(a,b)}(\sqrt{I})$. Thus, Corollary 4.12 yields the desired isomorphisms. □

Now we come to a general version of Theorem 4.7 as follows:

Corollary 4.15. *Let $I \subseteq S$ be a monomial ideal and Δ the simplicial complex corresponding to $\sqrt{I}$. The following conditions are equivalent.*

(a) *S/I is relative Cohen-Macaulay with respect to Q with* $\mathrm{cd}\,(Q, S/I) = q$,

(b) *$\widetilde{H}_i((\mathrm{link}\, F \cup G)_W; K) = 0$ for all $F \in \Delta_W$, $G \subseteq V$ and all $i < \dim \mathrm{link}_{\Delta_W} F$.*

Proof. Note that

$$\begin{aligned}\Delta_{(a,b)}(I) = \Delta_{(a,b)}(\sqrt{I}) &= \mathrm{link}_{\mathrm{star}\, N_a \cup H_b} G_b \\ &= \mathrm{link}_{\mathrm{star}\, N_a} G_b \\ &= (\mathrm{link}\, F \cup G)_W,\end{aligned}$$

see the remark after [**10**, Theorem 2.4] and also the proof [**13**, Corollary 1]. Now the assertion follows by applying Corollary 4.12 and Corollary 4.14 to Theorem 4.7. □

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