JOURNAL OF COMMUTATIVE ALGEBRA
Volume 4, Number 4, Winter 2012

MAXIMAL DENUMERANT OF A NUMERICAL SEMIGROUP WITH EMBEDDING DIMENSION LESS THAN FOUR

LANCE BRYANT, JAMES HAMBLIN, AND LENNY JONES

ABSTRACT. Given a numerical semigroup $S = \langle a_1, a_2, \ldots, a_t\rangle$ and $s \in S$, we consider the factorization $s = c_1a_1 + c_2a_2 + \cdots + c_ta_t$ where $c_i \geq 0$. Such a factorization is maximal if $c_1 + c_2 + \cdots + c_t$ is a maximum over all such factorizations of s. We show that the number of maximal factorizations, varying over the elements in S, is always bounded. Thus, we define $\mathrm{d}_{\max}(S)$ to be the maximum number of maximal factorizations of elements in S. We study maximal factorizations in depth when S has embedding dimension less than four, and establish formulas for $\mathrm{d}_{\max}(S)$ in this case.

1. Introduction. Let $\mathbf{N}$ denote the nonnegative integers. A *numerical semigroup* S is a subsemigroup of $\mathbf{N}$ that contains 0 and has a finite complement in $\mathbf{N}$. For two elements u and u' in S, $u \preceq u'$ if there exists an $s \in S$ such that $u + s = u'$. This defines a partial ordering on S. The minimal elements in $S \setminus \{0\}$ with respect to this ordering form the unique *minimal set of generators of* S, which is denoted by $\{a_1, a_2, \ldots, a_t\rangle$ where $a_1 < a_2 < \cdots < a_t$. The numerical semigroup $S = \{\sum_{i=1}^{t} c_ia_i \mid c_i \geq 0\}$ is represented using the notation $S = \langle a_1, \ldots, a_t\rangle$. Since the minimal generators of S are distinct modulo a_1, the set of minimal generators is finite. Furthermore, S having a finite complement in N is equivalent to $\gcd(a_1, a_2, \ldots, a_t) = 1$. The cardinality, t, of the set of minimal generators of a semigroup S is called the *embedding dimension of* S. The element a_1 is called the *multiplicity* of S. When $S \neq \mathbf{N}$, we have $2 \leq t \leq a_1$.

By definition, if $s \in S$, then there exists a t-tuple of nonnegative integers $(c_1, c_2, \ldots, c_t)$ such that $s = c_1a_1 + c_2a_2 + \cdots + c_ta_t$. We call $(c_1, c_2, \ldots, c_t)$ a *factorization* of s. For two factorizations $(c_1, c_2, \ldots, c_t)$ and $(d_1, d_2, \ldots, d_t)$ of s, we say they are different if $c_i \neq d_i$ for some $1 \leq i \leq t$. The *length* of a factorization $(c_1, c_2, \ldots, c_t)$ is defined as $c_1 + c_2 + \cdots + c_t$. The set of factorizations of s, denoted by $\mathcal{F}(s)$,

Received by the editors on February 25, 2011, and in revised form on July 16, 2012.

DOI:10.1216/JCA-2012-4-4-489

is precisely the set of nonnegative integer solutions of the equation $x_1a_1 + x_2a_2 + \cdots + x_ta_t = s$ and is therefore finite.

A basic arithmetic constant that measures the behavior of factorizations in a numerical semigroup is the cardinality of $\mathcal{F}(s)$, which is called the *denumerant* of s in S. See [**13**] for an exhaustive view of results related to the denumerant. Recently, there has been interest in the factorization theory of numerical semigroups and the insight it provides into the general theory of commutative monoids; for example, see [**1**–4, 7, 8, 10]. Here we consider a variation of the denumerant.

Definition 1.1. The maximal denumerant of s in S is the number of factorizations of s that have maximal length and is denoted by $\mathrm{d}_{\max}(s;S)$.

Certainly, the maximal denumerant of s in S is less than or equal to its denumerant in S and thus also finite. On the other hand, unlike the denumerant, as we vary over the elements in S, the maximal denumerant is always bounded. This is not difficult to see and will be proven in Theorem 2.3. Thus, we have the well-defined quantity given in the next definition.

Definition 1.2. The maximal denumerant of S is

$$\mathrm{d}_{\max}(S) = \max_{s\in S}\{\mathrm{d}_{\max}(s;S)\}.$$

We will focus on computing $\mathrm{d}_{\max}(S)$ when S can be generated by three elements; in particular, when S has embedding dimension 3. When S is (perhaps non-minimally) generated by a_1, a_2 and a_3, by letting $g = \gcd(a_2 - a_1, a_3 - a_1)$, $m = (a_2 - a_1)/g$ and $n = (a_3 - a_1) = g$, we can write

$$S = \langle a_1, a_1 + gm, a_1 + gn\rangle,$$

which leads to unexpectedly nice formulas. Theorems 3.5 and 3.6 will be proven in Section 3.

Theorem 3.5. *Let $0 \le \alpha < mn$ be such that $\alpha \equiv -a_1 \bmod mn$. We have the following formula*:

$$\mathrm{d}_{\max}(S) = \begin{cases} \left\lceil \frac{a_1}{mn} \right\rceil & \text{if } \alpha \in \langle m, n\rangle \\ \left\lceil \frac{a_1}{mn} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

Theorem 3.6. *If x and y are integers such that $mx+ny=a_1$, then we have the following formula*:

$$\mathrm{d}_{\max}(S) = \left\lceil \frac{x}{n} \right\rceil + \left\lceil \frac{y}{m} \right\rceil .$$

The motivation for such a variation of the denumerant is the consideration of length-preserving restrictions. For example, perhaps we are interested in factorizations of an element that have either maximal or minimal length. This might happen when working with the numerical semigroup ring $R = k[[t^{a_1}, t^{a_2}, \dots t^{a_d}]]$, where k is a field and $\mathbf{m} = (t^{a_1}, t^{a_2}, \dots, t^{a_d})R$ is the unique maximal ideal. In this case, the maximal length of the factorizations of $s \in S$ is the $\mathbf{m}$-adic order of $t^s \in R$, i.e., the largest power of m that contains t^s, see [**5**]. Another instance occurs in money-changing problems where the minimal length of the factorizations of $s \in S$ is the fewest number of coins needed to make change for s using the denominations $a_1, a_2, \dots, a_t$, see [**6**]. Of course, the overarching concern is changing from one factorization to another in a numerical semigroup. The minimal presentation of a numerical semigroup (see [**12**]) is helpful when studying all factorizations; however, we note that it is not as useful for our current endeavor because these "basic trades" do not generally preserve length. With the appropriate modifications, an approach via a minimal presentation may be fruitful, and we leave this as an avenue for further research.

In the next section, we show that the maximal denumerant is always finite and that, for semigroups with embedding dimension less than 3, $\mathrm{d}_{\max}(S) = 1$. In Section 3, we focus on numerical semigroups with embedding dimension exactly equal to 3. In the last section, we demonstrate the utility of our results by explicitly computing the maximal denumerant of semigroups with multiplicity 7 and embedding dimension 3.

2. The finiteness of $\mathrm{d}_{\max}(S)$. For a given numerical semigroup $S = \langle a_1, a_2, \dots, a_t \rangle$, we need only find a finite set $U \subset S$ such that $\mathrm{d}_{\max}(S) = \max_{s \in U}\{\mathrm{d}_{\max}(s; S)\}$ to establish the finiteness of $\mathrm{d}_{\max}(S)$. To this end, we make the following definition.

Definition 2.1. An element $u \in S$ is called *maximally reduced* if, for each i, with $1 \le i \le t$, there exists a factorization $(c_1, c_2, \dots, c_t)$ of u with maximal length such that $c_i = 0$.

We do not need t distinct factorizations with maximal length to satisfy the definition of maximally reduced, as the next example shows.

Example 2.2. In $S = \langle 7, 8, 13 \rangle$, the element 48 has the following factorizations:

- $(0, 6, 0)$
- $(5, 0, 1)$
- $(2, 1, 2)$.

Notice that only the first two have maximal length. The first factorization with maximal length has a 0 in the first and third entries, and the second factorization with maximal length has a 0 in the second entry. Thus, 48 is a maximally reduced element.

Theorem 2.3. *Let U be the set of maximally reduced elements in S. Then we have the following*:

1. *U is a finite set*,
2. $\mathrm{d}_{\max}(S) = \max_{s \in U}\{\mathrm{d}_{\max}(s; S)\}$.

Thus, $\mathrm{d}_{\max}(S)$ *is finite.*

Proof. To show that U is a finite set, it suffices to prove that the maximally reduced elements are bounded above. To see this, set $N = (a_1 - 1)\sum_{i=2}^{t} a_i$. Suppose that $s > N$ and that $C = (0, c_2, \ldots, c_t)$ is a representation of s. Then there exists a j, with $2 \leq j \leq t$, such that $c_j \geq a_1$, and so $(a_j, c_2, \ldots, c_j - a_1, \ldots, c_t)$ is a representation of s with greater length than C. Therefore, every maximal representation has a first component that is nonzero and s is not maximally reduced.

Now we need to show that $\mathrm{d}_{\max}(S) = \max_{s \in U}\{\mathrm{d}_{\max}(s; S)\}$. For $s \in S$, with maximal representations $\{C_j = (c_{j,1}, c_{j,2}, \ldots, c_{j,t})\}$, let $c_i = \min_j\{c_j, i\}$, and consider the element $s^* = s - \sum_{i=1}^{t} c_i a_i \in S$. Then it is not difficult to see that s^* is maximally reduced and that $\mathrm{d}_{\max}(s; S) = \mathrm{d}_{\max}(s^*; S)$. ◻

Theorem 2.3 outlines an algorithm for computing $\mathrm{d}_{\max}(S)$: We check to see which elements up to $N = (a_1 - 1)\sum_{i=2}^{t} a_i$ are maximally reduced, and then take the maximum of the $\mathrm{d}_{\max}(s; S)$ where s is a maximally reduced element of S.

Example 2.4. Let $S = \langle 7, 11, 13, 15 \rangle$. Checking up to 234, the maximally reduced elements along with their maximal factorizations are

- 0; $(0, 0, 0, 0)$
- 22; $(0, 2, 0, 0)$, $(1, 0, 0, 1)$
- 26; $(0, 0, 2, 0)$, $(0, 1, 0, 1)$
- 33; $(0, 3, 0, 0)$, $(1, 0, 2, 0)$, $(1, 1, 0, 1)$
- 37; $(0, 1, 2, 0)$, $(0, 2, 0, 1)$, $(1, 0, 0, 2)$
- 44; $(1, 2, 0, 1)$, $(1, 1, 2, 0)$, $(0, 4, 0, 0)$, $(2, 0, 0, 2)$.

Therefore, $\mathrm{d}_{\max}(S) = 4$.

We see from the example that we can potentially improve this algorithm since we only need to check up to 44 to find the maximally reduced elements. We leave this improvement as an open question.

Question 2.5. *Can we improve the algorithm described in Theorem* 2.3?

In the next section we will focus on numerical semigroups with embedding dimension less than 4, but first we consider the case when S has embedding dimension strictly less than 3. When $S = \mathbf{N}$, then every element $s \in S$ has a unique factorization, namely, $s = s \cdot 1$. Thus, $\mathrm{d}_{\max}(\mathbf{N}) = 1$. We show that, when S has embedding dimension 2, we also have that $\mathrm{d}_{\max}(S) = 1$.

Proposition 2.6. *If* $S = \langle a_1, a_2 \rangle$, *then* $\mathrm{d}_{\max}(S) = 1$.

Proof. We will show that every element of S has only one maximal factorization. Suppose that

$$s = c_1 a_1 + c_2 a_2 \quad \text{and} \quad s = d_1 a_1 + d_2 a_2,$$

where $c_1 + c_2 = d_1 + d_2$. If $c_1 = d_1$ or $c_2 = d_2$, then it follows that both $c_1 = d_1$ and $c_2 = d_2$. If this is not the case, then we may assume without loss of generality that $c_1 > d_1$. But then we have $(c_1 - d_1)a_1 + c_2a_2 = d_2a_2$ and

$$\begin{aligned}(c_1 - d_1)a_1 + c_2a_2 &< (c_1 - d_1)a_2 + c_2a_2 \\ &= (c_1 - d_1 + c_2)a_2 \\ &= d_2a_2.\end{aligned}$$

This is a contradiction. Since we cannot have two factorizations of an element of S with the same length, we certainly cannot have two with maximal length. $\square$

3. The maximal denumerant of a semigroup with embedding dimension less than four. Throughout this section, unless otherwise stated, we assume that $S = \langle a_1, a_2, a_3\rangle$ is a numerical semigroup with embedding dimension 3. Set $g = \gcd(a_2 - a_1, a_3 - a_1)$, $m = (a_2 - a_1)/g$ and $n = (a_3 - a_1)/g$. Then

$$S = \langle a_1, a_1 + m_g, a_1 + ng\rangle, \tag{1}$$

where $\gcd(m, n) = \gcd(a_1, g) = 1$. In the following lemma, we determine the maximally reduced elements of S and their maximal factorizations.

Lemma 3.1. *Let s be a maximally reduced element of S. Then s is a multiple of na_2. Moreover, if $s = kna_2$, then $\{pU + qV \mid p, q \geq 0$ and $p + q = k\}$ is the set of maximal factorizations of s where $U = (0, n, 0)$ and $V = (n - m, 0, m)$ (using the standard addition and scalar multiplication of vectors).*

Proof. The element $s = 0$, which is always maximally reduced, has the unique (maximal) factorization $(0, 0, 0)$. Certainly, s is a multiple of na_2. It is also easy to verify that the rest of the theorem is satisfied in this case. Now we assume that $s > 0$. Since s is maximally reduced, there exists a maximal representation of s with the first component equal to 0, say $D = (0, d_2, d_3)$. Suppose that $d_3 \neq 0$. Since there exists another maximal factorization $C = (c_1, c_2, 0)$ of s, we have

$$s = d_2a_2 + d_3a_3 > (d_2 + d_3)a_2 = (c_1 + c_2)a_2 \geq c_1a_1 + c_2a_2 = s,$$

which is impossible. Thus, $d_3 = 0$ and $D = (0, d_2, 0)$.

The element $s > 0$ also has another maximal factorization $E = (e_1, 0, e_3)$ distinct from D. We now have

$$d_2 a_2 = e_1 a_1 + e_3 a_3, \tag{2}$$

and

$$d_2 a_1 = e_1 a_1 + e_3 a_1. \tag{3}$$

Subtracting equation (3) from equation (2) and dividing by g yields $md_2 = ne_3$. Since m and n are relatively prime, we have that $d_2 = kn$ for some $k > 0$, and so $s = d_2 a_2 = kna_2$. Therefore, s is a multiple of na_2.

Next we show that $\{pU + qV \mid p, q \geq 0 \text{ and } p + q = k\}$ is the set of maximal representations of the maximally reduced element $s = kna_2$. Notice that our proof of the first statement of the theorem shows that $kU = (0, kn, 0)$ is a maximal factorization of s. It is not difficult to see that $pU + qV$, where $p, q \geq 0$ and $p + q = k$ is also a factorization of s having the same length as kU. Thus, all of these factorizations are maximal. We still need to show that no other maximal factorizations exist.

Let $C = (c_1, c_2, c_3)$ be a maximal factorization of s. Similar to before, using that $(0, kn, 0)$ is a maximal factorization, we have

$$(kn - c_2)a_2 = c_1 a_1 + c_3 a_3, \tag{4}$$

and

$$(kn - c_2)a_1 = c_1 a_1 + c_3 a_1. \tag{5}$$

Subtracting equation (4) from equation (5) and dividing by g yields $m(kn - c_2) = nc_3$. Since m and n are relatively prime, we have that $kn - c_2 = k'n$ and $c_3 = k'm$ for some $0 \leq k' \leq k$. It follows that $c_2 = (k - k')n$ and $c_1 = k'(n - m)$. Therefore, we have that $C = (k - k')U + k'V$.

From Lemma 3.1, we can precisely describe the set of maximally reduced elements. This is the content of the next theorem.

Theorem 3.2. *There exists an integer* $k \geq 0$ *such that* $\{0, na_2, 2na_2, \ldots, kna_2\}$ *is the set of maximally reduced elements in* S. *Furthermore,* $\mathrm{d}_{\max}(ina_2; S) = i + 1$ *for* $0 \leq i \leq k$.

Proof. We already know that every maximally reduced element in S is a multiple of na_2. Thus, for the first statement it suffices to assume that $s = hna_2$ is maximally reduced and show that this implies that $s' = (h-1)na_2$ is also maximally reduced.

If $(0, (h-1)n, 0)$ is not a maximal factorization of s', then s' has a factorization $C = (c_1, c_2, c_3)$ such that $c_1 + c_2 + c_3 > (h-1)n$. It follows that $(c_1, c_2 + n, c_3)$ would be a factorization of s with length greater than hn. Therefore, $(0, hn, 0)$ is not a maximal factorization of s, and s is not maximally reduced by Theorem 3.1. From this contradiction, we conclude that indeed $(0, (h-)n, 0)$ is a maximal factorization of s'. Clearly, we also have that $((h-1)(n-m), 0, (h-1)m)$ is a maximal factorization of s' since it is a factorization with length $(h-1)n$. This shows that s' is maximally reduced.

Now, if $0 \leq i \leq k$, then ina_2 is maximally reduced and, by Theorem 3.1, its maximal factorizations are $\{p(0, n, 0) + q(n-m, 0, m) \mid p, q \geq 0 \text{ and } p + q = i\}$. Since p can range from 0 to i (with q depending on p), it follows that $\mathrm{d}_{\max}(ina_2; S) = i + 1$. ◻

Our main results, the formulas provided in Theorems 3.5 and 3.6, are both dependent upon Lemma 3.3. Notice that, since m and n are relatively prime, $U = \langle m, n \rangle$ is a semigroup with embedding dimension less than three. It is well known that U is a *symmetric* semigroup, i.e., for every $z \in \mathbf{Z}$, exactly one of z or $f - z$ is in U, where f is the Frobenius number of U. The Frobenius number of U is the largest integer not in U, and we have $f = mn - m - n$. See [**11, 12**] for more information concerning symmetric semigroups.

Lemma 3.3. *The following are equivalent*:

(a) hna_2 *is not a maximally reduced element of* S,

(b) $(0, hn, 0)$ *is not a maximal factorization of* hna_2,

(c) $hmn - a_1 \in \langle m, n \rangle$.

Moreover, $\mathrm{d}_{\max}(S) = \min\{h \mid hmn - a_1 \in \langle m, n \rangle\}$.

Proof. For (a) implies (b), if $(0, hn, 0)$ is a maximal factorization, then so is $(h(n-m), 0, hm)$. Thus, hna_2 is maximally reduced. For (b) implies (a), if hna_2 is maximally reduced, then by Lemma 3.1, $(0, hn, 0)$ is a maximal factorization of hna_2.

For (b) implies (c), we may assume that $a_1 \in \langle m, n \rangle$, since otherwise we would have $a_1 - m - n(h-1)mn \notin \langle m, n \rangle$. By the symmetry of $\langle m, n \rangle$, it follows that $hmn - a_1 \in \langle m, n \rangle$. By assumption we have that

$$hna_2 = c_1 a_1 + c_2 a_2 + c_3 a_3, \tag{6}$$

where $hn < c_1 + c_2 + c_3$. Write $k = hn - (c_1 + c_2 + c_3)$. Subtracting $(c_1 + c_2 + c_3)a_1$ from both sides of (6), we have

$$hngm - ka_1 = c_2 gm + c_3 gn.$$

Since $\gcd(a_1, g) = 1$, it follows that k is divisible by g and so

$$c_2 m + c_3 n = hmn - k' a_1,$$

for some $k' > 0$. Thus, $hmn - k' a_1 \in \langle m, n \rangle$, and since a_1 is as well, we have $hmn - a_1 \in \langle m, n \rangle$.

For (c) implies (b), we essentially reverse these steps. Since $hmn - a_1 \in \langle m, n \rangle$, we have $hmn - a_1 = c_2 m + c_3 n$ where $c_2, c_3 \geq 0$. Multiplying both sides by g, adding $c_2 a_1 + c_3 a_1$ to both sides and rearranging gives

$$hna_2 = (hn + g - c_1 - c_2)a_1 + c_2 a_2 + c_3 a_3.$$

If we can verify that $hn + g - c_1 - c_2 \geq 0$, then $(0, hn, 0)$ is not a maximal factorization of hna_2. To do this, suppose that $hn + g - c_1 - c_2 < 0$. Then, using the fact that $m < n$, we get $mhn + mg < mc_2 + mc_3 < mc_2 + nc_3 = hmn - a_1$. It follows that $a_2 = a_1 + mg < 0$, which is a contradiction. $\square$

Before we prove the main results, we consider when S is generated by three elements, a_1, a_2 and a_3, that do *not* form the minimal generating set. In this case, S has embedding dimension less than three, and we have seen that $\mathrm{d}_{\max}(S) = 1$. The next lemma shows that we still have $\mathrm{d}_{\max}(S) = \min\{h \mid hmn - a_1 \in \langle m, n \rangle\}$.

Lemma 3.4. *Let S be a numerical semigroup generated by a_1, a_2 and a_3 such that these elements do not form the minimal generating set of S. Then* $\min\{h \mid hmn - a_1 \in \langle m, n\rangle\} = 1$*, or equivalently,* $mn - a_1 \in \langle m, n\rangle$.

Proof. Since a_1, a_2 and a_3 do not form the minimal generating set, we have either $a_2 = ka_1$ for $k \geq 2$ or $a_3 = pa_1 + qa_2$ where $p + q \geq 2$. In both case we can show that $a_1 < m + n$.

For the former case, we have $a_1 + gm = ka_1$. Thus, $gm = (k-1)a_1$, and since $(a_1, g) = 1$, it follows that $m = k'a_1$, where $1 \leq k' = (k-1)/g \in \mathbf{Z}$. Therefore, $a_1 < m + n$. For the latter case, we have $a_1 + gn = pa_1 + q(a_1 + gm)$. Thus, $g(n - qm) = (p + q - 1)a_1$ and, since $(a_1, g) = 1$, it follows that $(n - qm) = k'a_1$, where $1 \leq k' = (p + q - 1)/g \in \mathbf{Z}$. Therefore, $a_1 < m + n$.

Notice that, if $a_1 < m+n$, then $a_1 - m - n \notin \langle m, n\rangle$. By the symmetry of $\langle m, n\rangle$, we have $mn - m - n - (a_1 - m - n) = mn - a_1 \in \langle m, n\rangle$. □

Next, we prove our main results with the following setting: Numerical semigroup S is (perhaps non-minimally) generated by a_1, a_2 and a_3. Moreover, we set $g = \gcd(a_2 - a_1, a_3 - a_1)$, $m = (a_2 - a - 1)/g$ and $n = (a_3 - a_1)/g$ such that

$$S = \langle a_1, a_1 + mg, a_1 + ng\rangle, \tag{7}$$

where $\gcd(m, n) = \gcd(a_1, g) = 1$.

Theorem 3.5. *Let* $0 \leq \alpha < mn$ *be such that* $\alpha \equiv -a_1 \bmod mn$. *We have the following formula*:

$$\mathrm{d}_{\max}(S) = \begin{cases} \lceil \frac{a_1}{mn} \rceil & \text{if } \alpha \in \langle m, n\rangle \\ \lceil \frac{a_1}{mn} \rceil + 1 & \text{otherwise.} \end{cases}$$

Proof. First note that $\alpha = \lceil a_1/mn\rceil mn - a_1$ and $0 \leq \lceil a_1/mn\rceil mn - a_1 < mn$. Thus, if $h < \lceil a_1/mn\rceil$, then $hmn - a_1 \notin \langle m, n\rangle$. On the other hand, if $h > \lceil a_1/mn\rceil$, then $hmn - a_1 > mn$ and thus $hmn - a_1$ is an element of $\langle m, n\rangle$.

The result now follows from Lemmas 3.3 and 3.4. If $\lceil a_1/mn \rceil mn - a_1 \in \langle m, n \rangle$, then $\mathrm{d}_{\max}(S) = \min\{h \mid hmn - a_1 \in \langle m, n \rangle\} = \lceil a_1/mn \rceil$. Otherwise, $\mathrm{d}_{\max}(S) = \min\{h \mid hmn - a_1 \in \langle m, n \rangle\} = \lceil a_1/mn \rceil + 1$. □

Theorem 3.6. *If x and y are integers such that $mx + ny = a_1$, then*

$$\mathrm{d}_{\max}(S) = \left\lceil \frac{x}{n} \right\rceil + \left\lceil \frac{y}{m} \right\rceil .$$

Proof. Note that $a_1 = mx + ny = mu + nv$ implies that $m(x - u) = n(v - y)$. Since $\gcd(m, n) = 1$, we have that $x - u = kn$ and $v - y = km$ for some integer k.

Thus, we see that

$$\begin{aligned} \left\lceil \frac{u}{n} \right\rceil + \left\lceil \frac{v}{m} \right\rceil &= \left\lceil \frac{x + kn}{n} \right\rceil + \left\lceil \frac{y - km}{m} \right\rceil \\ &= \left\lceil \frac{x}{n} + k \right\rceil + \left\lceil \frac{y}{m} - k \right\rceil \\ &= \left\lceil \frac{x}{n} \right\rceil + k + \left\lceil \frac{y}{m} \right\rceil - k \\ &= \left\lceil \frac{x}{n} \right\rceil + \left\lceil \frac{y}{m} \right\rceil . \end{aligned}$$

In other words, the formula is independent of the linear combination that we choose.

Now let $k = \mathrm{d}_{\max}(S)$, so that, by Lemmas 3.3 and 3.4, we have $k = \min\{h \mid hmn - a_1 \in \langle m, n \rangle\}$. Thus, $kmn - a_1 = c_1 m + c_2 n$ for some $c_1, c_2 \geq 0$. Furthermore, $c_1 < n$ and $c_2 < m$ since, otherwise, $(k - 1)mn - a_1 \in \langle m, n \rangle$. We now have that

$$\begin{aligned} \left\lceil \frac{x}{n} \right\rceil + \left\lceil \frac{y}{m} \right\rceil &= \left\lceil \frac{kn - c_1}{n} \right\rceil + \left\lceil \frac{-c_2}{m} \right\rceil \\ &= k + \left\lceil \frac{-c_1}{n} \right\rceil + \left\lceil \frac{-c_2}{m} \right\rceil \\ &= k \\ &= \mathrm{d}_{\max}(S). \qquad \square \end{aligned}$$

Of course, the formulas presented in Theorems 3.5 and 3.6 are most interesting when S has embedding dimension 3 since, otherwise, we know that $\mathrm{d}_{\max}(S) = 1$. However, the fact that these formulas work for all numerical semigroups with embedding dimension less than 4 naturally raises the following question:

Question 3.7. *When S has embedding dimension less than $t + 1$, where $t \geq 4$, do there exist formulas dependent upon a set of t generators analogous to those in Theorems 3.5 and 3.6 that yield the maximal denumerant of S?*

4. The maximal denumerant of basic semigroups. We begin with the following proposition that will aid in some computations.

Proposition 4.1. *Let $S = \langle a_1, a_2, a_3 \rangle$ be a semigroup. If either m or n divides a_1, then $\mathrm{d}_{\max}(S) = \lceil a_1/mn \rceil$.*

Proof. Assume that m divides a_1. Then $a_1 = km$ for some $k > 0$ and, by Theorem 3.6, we have $\mathrm{d}_{\max}(S) = \lceil k/n \rceil = \lceil km/nm \rceil = \lceil a_1/mn \rceil$. We have a similar proof whenever n divides a_1. □

Recall our setting: $S = \langle a_1, a_1 + mg, a_1 + ng \rangle$, where $g = \gcd(a_2 - a_1, a_3 - a_1)$, $m = (a_2 - a_1)/g$ and $n = (a_3 - a_1)/g$. From Theorems 3.5 and 3.6, we see that if $T = \langle a_1, a_1 + m, a_1 + n \rangle$, then $\mathrm{d}_{\max}(S) = \mathrm{d}_{\max}(T)$. Thus, we will restrict our attention to the following class of numerical semigroups.

Definition 4.2. The semigroup $S = \langle a_1, a_2, a_3 \rangle$ is called *basic* if $\gcd(a_2 - a_1, a_3 - a_1) = 1$.

Proposition 4.3. *Let $S = \langle a_1, a_2, a_3 \rangle$ be a basic semigroup. Then*:

1. *If $4a_1 = 2a_2 + a_3$, then $\mathrm{d}_{\max}(S) = 2$.*
2. *If $3a_1 = a_2 + a_3$, then $\mathrm{d}_{\max}(S) = 2$.*
3. *If $4a_1 < 2a_2 + a_3$ and $3a_1 \neq a_2 + a_3$, then $\mathrm{d}_{\max}(S) = 1$.*

Proof. For the first case, by subtracting $3a_1$, we have $a_1 = 2m + n$. Thus, $\mathrm{d}_{\max}(S) = \lceil 2/n \rceil + \lceil 1/m \rceil$. Since $1 \leq m < n$, we have $\mathrm{d}_{\max}(S) = 2$. Similarly, for the second case, we obtain $a_1 = m + n$. Thus, $\mathrm{d}_{\max}(S) = \lceil 1/n \rceil + \lceil 1/m \rceil = 2$.

For the third case, consider when $a_1 < m + n$. Then $a_1 - m - n < 0$, and hence, is not in $\langle m, n \rangle$. By the symmetry of S, we have $mn - m - n - (a_1 - m - n) = mn - a_1 \in \langle m, n \rangle$. By Lemma 3.3, $\mathrm{d}_{\max}(S) = 1$. On the other hand, if we have $m + n < a_1 < 2m + n$, then $0 < a_1 - m - n < m$. Again, we have that $a_1 - m - n \notin \langle m, n \rangle$, and it follows that $\mathrm{d}_{\max}(S) = 1$.

Remark 4.4. Note that Proposition 4.3 is not exhaustive in the sense that, when $4a_1 > 2a_2 + a_3$, there are more possibilities to consider. Nevertheless, for a given a_1, Proposition 4.3 does address all but finitely many situations.

The next example demonstrates how we can easily compute the maximal denumerant of all basic semigroups with a fixed multiplicity.

Example 4.5. Let $S = \langle 7, a_2, a_3 \rangle$ be a basic semigroup with multiplicity $a_1 = 7$. Using Proposition 4.3, we carry out the following steps:

1. Solve $28 = 2a_2 + a_3$ to get the pairs of solutions $(8, 12)$ and $(9, 10)$.
2. Solve $21 = a_2 + a_3$ to get the pairs of solutions $(8, 13)$, $(9, 12)$ and $(10, 11)$.
3. Solve $4a_1 > 2a_2 + a_3$ to get the pairs of solutions $(8, 9)$, $(8, 10)$ and $(8, 11)$.

The semigroups from the first two steps will have maximal denumerant equal to 2. The maximal denumerant of the semigroups in the third step can be computed using Proposition 4.1. All other basic semigroups with multiplicity 7 have maximal denumerant equal to 1. The list below summarizes our computations.

1. $\mathrm{d}_{\max}(\langle 7, 8, 9 \rangle) = 4$,
2. $\mathrm{d}_{\max}(\langle 7, 8, 10 \rangle) = 3$,
3. $\mathrm{d}_{\max}(S) = 2$ if S is one of the following:

(a) $\langle 7, 8, 11\rangle$,

(b) $\langle 7, 8, 12\rangle$,

(c) $\langle 7, 9, 10\rangle$,

(d) $\langle 7, 8, 13\rangle$,

(e) $\langle 7, 9, 12\rangle$,

(f) $\langle 7, 10, 11\rangle$,

4. $d_{\max}(S) = 1$ otherwise.

This example raises a natural question: for which values of m and n is $d_{\max}(\langle a_1, a_1 + m, a_1 + n\rangle)$ maximized? Considering Theorem 3.5, if $m = 1$, then $\langle m, n\rangle = \mathbf{N}$ and $d_{\max}(S) = \lceil a_1/n\rceil$. This is maximized when $n = 2$. If $m > 1$, then since $a_1 \geq 3$, we have $\lceil a_1/mn\rceil + 1 \leq \lceil a_1/2\rceil$. Hence, $d_{\max}(S)$ is largest when $m = 1$ and $n = 2$.

REFERENCES

1. F. Aguiló-Gost and P.A. García-Sánchez, *Factorization and catenary degree in* 3*-generated numerical semigroups*, Europ. Conf. Combin. Graph Theory Appl. (EuroComb 2009), 157–161, Electron. Notes Discrete Math. **34**, Elsevier Science, Amsterdam.

2. ———, *Factoring in embedding dimension three numerical semigroups*, Electron. J. Combin. **17** (2010).

3. J. Amos, S.T. Chapman, N. Hine and J. Paixão, *Sets of lengths do not characterize numerical monoids*, Integers **7** (2007), A50.

4. C. Bowles, S.T. Chapman, N. Kaplan and D. Reiser, *On delta sets of numerical monoids*, J. Alg. Appl. **5** (2006), 695–718.

5. L. Bryant, *Goto numbers of a numerical semigroup ring and the Gorensteiness of associated graded rings*, J. Comm. Algebra. **38** (2010), 2092–2128.

6. L. Bryant, J. Hamblin and L. Jones, *A variation on the money-changing problem*, Amer. Math. Month. **109** (2012), 406–414.

7. S.T. Chapman, J. Daigle, R. Hoyer and N. Kaplan, *Delta sets of numerical monoids using nonminimal sets of generators*, Comm. Algebra **38** (2010), 2622–2634.

8. S.T. Chapman, P.A. García-Sánchez, D. Llena and J. Marshall, *Elements in a numerical semigroup with factorizations of the same length*, Canad. Math. Bull. **54** (2011), 39–43.

9. S.T. Chapman, P.A. García-Sánchez, D. Llena, V. Ponomarenko and J.C. Rosales, *The catenary and tame degree in finitely generated commutative cancellative monoids*, Manuscr. Math. **120** (2006), 253–264.

10. S.T. Chapman, R. Hoyer and N. Kaplan, *Delta sets of numerical monoids are eventually periodic*, Aequat. Math. **77** (2009), 273–279.

11. R. Fröberg, C. Gottlieb and R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35** (1987), 63–83.

12. P.A. García-Sánchez and J.C. Rosales, *Numerical semigroups*, Developments Math. **20**, Springer, New York, 2009.

13. J.L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lect. Ser. Math. Appl. **30**, Oxford University Press, Oxford, 2005.

Department of Mathematics, Shippensburg University, 1871 Old Main Drive, Shippensburg, PA 17257
Email address: lebryant@ship.edu

Department of Mathematics, Shippensburg University, 1871 Old Main Drive, Shippensburg, PA 17257
Email address: jehamb@ship.edu

Department of Mathematics, Shippensburg University, 1871 Old Main Drive, Shippensburg, PA 17257
Email address: lkjone@ship.edu