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CONSTRUCTING MODULES WITH PRESCRIBED COHOMOLOGICAL SUPPORT

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To Phil Griffith, algebraist and friend.

Abstract. A cohomological support, $\mathrm{Supp}^*_{\mathcal{A}}(M)$, is defined for finitely generated modules M over a left noetherian ring R, with respect to a ring $\mathcal{A}$ of central cohomology operations on the derived category of R-modules. It is proved that if the $\mathcal{A}$-module $\mathrm{Ext}^*_R(M,M)$ is noetherian and $\mathrm{Ext}^*_R(M,R)=0$ for $i \gg 0$, then every closed subset of $\mathrm{Supp}^*_{\mathcal{A}}(M)$ is the support of some finitely generated R-module. This theorem specializes to known realizability results for varieties of modules over group algebras, over local complete intersections, and over finite dimensional algebras over a field. The theorem is also used to produce large families of finitely generated modules of finite projective dimension over commutative local noetherian rings.

Introduction

Quillen introduced methods from algebraic geometry to the study of cohomology rings of finite groups in a seminal paper, [21]. His ideas and techniques have led to the appearance of a number of highly developed theories, which provide insight into the structure of an algebraic object through some geometric 'variety' attached to it. Use of such geometric invariants has been crucial to progress on a number of difficult problems.

Variety theories share certain formal properties needed in applications. Some of them guarantee that homologically similar modules, such as all syzygy modules of a given module, have the same variety. Modules with distinct varieties are therefore expected to exhibit quantifiable differences in homological behavior. For this reason, a description of all the varieties produced by a given theory is a useful tool for classifying homological patterns.

The prototype theory applies to all finite dimensional representations of a finite group; see [8] for a detailed exposition. It has been extended to

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representations of finite dimensional cocommutative Hopf algebras, [13], [25]. Parallel theories have been constructed for finitely generated modules over finite dimensional self-injective algebras, [12], [22], and over local complete intersection rings, [1], [2]. Historically, in each concrete case the proofs of the formal properties of a theory and of the relevant realizability theorem have involved delicate arguments specific to that context.

We are interested in modules over a fixed associative ring R.

The vehicle for passing from algebra to geometry is provided by a choice of commutative graded ring $\mathcal{A}$ of central cohomology operations on the derived category of R. In the examples above there are natural candidates for $\mathcal{A}$: the even cohomology ring of a group (or a Hopf algebra); the even subalgebra of the Hochschild cohomology of an associative algebras; the polynomial ring of Gulliksen operators over a complete intersection. However, other choices are possible and sometimes are desirable.

For each pair (M,N) of R-modules the graded group $\operatorname{Ext}^*_R(M,N)$ has a natural structure of graded $\mathcal{A}$-module. The set

$$\operatorname{Supp}^*_{\mathcal{A}}(M,N) = \{\mathfrak{p} \in \operatorname{Proj}\mathcal{A} \mid \operatorname{Ext}^*_R(M,N)_{\mathfrak{p}} \neq 0\},$$

where $\operatorname{Proj}\mathcal{A}$ is the space of all essential homogeneous prime ideals in $\mathcal{A}$ with the Zariski topology, is called the *cohomological support* of (M,N). The cohomological support of M is the set $\operatorname{Supp}^*_{\mathcal{A}}(M,M)$.

The principal contribution of this work is a method for constructing modules with prescribed cohomological supports. Part of our main result reads:

Theorem 1. *Let R be a noetherian ring and let M and N be finite R-modules, such that the graded $\mathcal{A}$-module $\operatorname{Ext}^*_R(M,N)$ is noetherian.*

If $\operatorname{Ext}^i_R(M,R)=0$ holds for all $i \gg 0$, then for every closed subset X of $\operatorname{Supp}^*_{\mathcal{A}}(M,N)$ there exist finite R-modules M_X and N_X such that

$$\operatorname{Supp}^*_{\mathcal{A}}(M_X,N) = X = \operatorname{Supp}^*_{\mathcal{A}}(M,N_X)\,.$$

Moreover, when $N=M$ one can choose $N_X=M_X$.

Suitable specializations of Theorem 1 yield several known realizibility results: See Section 5 for Hopf algebras and Section 6 for associative algebras. Their earlier proofs were modeled on Carlson's Tensor Product Theorem [11] for varieties over group algebras; they rely heavily on the nature of R (in the first case) or on that of $\mathcal{A}$ (in the second).

Theorem 1 is proved in Section 4, based on work in Sections 1 and 3. Our argument requires few structural restrictions on R and none on $\mathcal{A}$ itself. The crucial input is the noetherian property of $\operatorname{Ext}^*_R(M,N)$ as a module over $\mathcal{A}$.

Another application of Theorem 1 goes into a completely different direction:

Theorem 2. *Let $(Q,\mathfrak{q},k)$ be a commutative noetherian local ring and $\boldsymbol{f}$ a Q-regular sequence of length c contained in $\mathfrak{q}^2$.*

For $R = Q/Q\boldsymbol{f}$ and $\bar{k}$ an algebraic closure of k there exists a map

$$V\colon \left\{ \begin{array}{c} \text{isomorphism classes } [M] \\ \text{of finite } R\text{-modules with} \\ \operatorname{proj\,dim}_Q M < \infty \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{closed algebraic} \\ \text{sets } \ X \subseteq \mathbb{P}^{c-1}_{\bar{k}} \\ \text{defined over } k \end{array} \right\}$$

with the following properties:

(1) *V is surjective.*
(2) *$V([M]) = \varnothing$ if and only if* $\operatorname{proj\,dim}_R M < \infty$.
(3) *$V([M]) = V([\Omega^R_n(M)])$ for every syzygy module $\Omega^R_n(M)$.*
(4) *$V([M]) = V([M/\boldsymbol{x}M])$ for every M-regular sequence $\boldsymbol{x}$ in R.*

This result is surprising. Indeed, it exhibits large families of modules of finite projective dimension over any ring Q with $\operatorname{depth} Q \geq 2$, contrary to a commonly held perception that finite projective dimension is 'rare' over singular commutative rings. Furthermore, the remaining statements ascertain that modules mapping to distinct closed cones in $\bar{k}^c$ cannot be linked by any sequence of standard operations known to preserve finite projective dimension.

In Section 7 we prove Theorem 2, and deduce from it a recent theorem on the existence of cohomological varieties for modules over complete intersection local rings. For the latter we establish a descent result of independent interest.

In this paper varieties of modules are discussed in the broader context of varieties of complexes. The resulting marginal technical complications are easily offset by a gain in flexibility: We first realize a given set as the cohomological support of a bounded complex by using constructions whose effect is easy to track. To show that this set is also the support of a module we use 'syzygy complexes', a notion introduced and discussed in Section 1.

This paper is part of an ongoing study of cohomological supports of modules over general associative rings. In [5] we focus on proving existence of variety theories with desirable properties under a small set of conditions on a ring, its module(s), and a ring of central cohomological operators. The properties that have to be established are clarified in [9] by Benson, Iyengar, and Krause, who investigate a notion of support for triangulated categories equipped with an action by a central ring of operators. On the other hand, the methods of this paper can be adapted to prove realizability results in that context. Of particular interest is the case of certain monoidal categories, where work of Suarez-Alvarez, [24], provides natural candidates for rings of operators.

1. Syzygy complexes

In this section we recall a few basic concepts of DG homological algebra, following [4], and extend the notion of syzygy from modules to complexes.

Let R be an associative ring and $\mathsf{D}(R)$ the full derived category of left R-modules. We write $\simeq$ to indicate a quasi-isomorphism of complexes; these are the isomorphisms in $\mathsf{D}(R)$. The symbol $\cong$ is reserved for isomorphisms of complexes, and $\equiv$ is used to denote homotopy equivalences. Given a complex M of $\mathsf{D}(R)$, we write $\mathsf{Thick}_R(M)$ for its *thick closure*, that is to say, the intersection of the thick subcategories of $\mathsf{D}(R)$ containing M.

1.1. *Semiprojective complexes.* A complex P of R-modules is called *semiprojective* if $\operatorname{Hom}_R(P,-)$ preserves surjective quasi-isomorphisms; equivalently, if P is a complex of projective modules and $\operatorname{Hom}_R(P,-)$ preserves quasi-isomorphisms. The following properties are used in the proofs below.

Every quasi-isomorphism of semiprojective complexes is a homotopy equivalence. Every surjective quasi-isomorphism to a semiprojective complex has a left inverse. Every semiprojective complex C with $\mathrm{H}(C)=0$ is equal to $\mathsf{cone}(\mathrm{id}^B)$ for some complex B of projective modules with zero differential.

Lemma 1.2. *If $\pi\colon P\to Q$ is a quasi-isomorphism of semiprojective complexes of R-modules and n is an integer, then there is a homotopy equivalence*

$$P_{\geqslant n}\oplus\Sigma^n Q'\equiv Q_{\geqslant n}\oplus\Sigma^n P',$$

where P' and Q' are projective R-modules.

Proof. Assume first that π is surjective. It then has a left inverse, hence one gets $P\cong Q\oplus E$ with $E=\operatorname{Ker}(\pi)$. This implies that E is semiprojective with $\mathrm{H}(E)=0$, and hence $E=\mathsf{cone}(\mathrm{id}^F)$ for some complex F of projective R-modules with $\partial^F=0$. Hence one gets a quasi-isomorphism

$$P_{\geqslant n}\cong Q_{\geqslant n}\oplus\mathsf{cone}(\mathrm{id}^{F_{\geqslant n}})\oplus\Sigma^n F_{n-1}\,.$$

The canonical map $P_{\geqslant n}\to Q_{\geqslant n}\oplus\Sigma^n F_{n-1}$ is thus a homotopy equivalence, as $\mathsf{cone}(\mathrm{id}^{F_{\geqslant n}})$ is homotopy equivalent to 0. This settles the surjective case.

In general, π factors as $P\to\widetilde{P}\xrightarrow{\psi}Q$, where $\widetilde{P}$ is equal to $P\oplus\Sigma^{-1}\mathsf{cone}(\mathrm{id}^Q)$ and ψ is the sum of π and the canonical surjection $\Sigma^{-1}\mathsf{cone}(\mathrm{id}^Q)\to Q$. Thus, ψ is a surjective quasi-isomorphism of semi-projective complexes. So is the canonical map $\widetilde{P}\to P$. The already settled case yields homotopy equivalences

$$P_{\geqslant n}\oplus\Sigma^n Q'\leftarrow\widetilde{P}\to Q_{\geqslant n}\oplus\Sigma^n P'\,.$$

for appropriate projective modules P' and Q'. □

1.3. *Syzygy complexes.* Let M be a complex of R-modules.

A *semiprojective resolution* of M is a quasi-isomorphism $P\to M$ from a semiprojective complex P. Every complex M has one, and it is unique up to homotopy equivalence. Thus, the preceding result may be viewed as a homotopical version of Shanuel's Lemma. Based on it, we introduce a homotopical version of the notion of syzygy module.

For each $n \in \mathbb{Z}$ let $\Omega_n^R(M)$ stand for any complex $\Sigma^{-n}(P_{\geqslant n})$, where P is a semiprojective resolution of M, and call it an nth *syzygy complex* of M over R. Its dependence on the choice of P is made precise by the preceding lemma.

Any complex P of projective modules with $P_i = 0$ for $i \ll 0$ is semiprojective. Thus, when M is an R-module and P is its projective resolution the complex $\Sigma^{-n}(P_{\geqslant n})$ is isomorphic in $\mathsf{D}(R)$ to an nth syzygy module of M.

The next lemma expands upon the last observation.

LEMMA 1.4. *If M is a complex of R-modules, $s = \sup\{i \mid \mathrm{H}_i(M) \neq 0\}$, and n is an integer with $n \geq s$, then $\Omega_n^R(M)$ is quasi-isomorphic to $\mathrm{H}_0(\Omega_n^R(M))$.*

Proof. Let $P \to N$ be a semiprojective resolution with $\Omega_n^R(M) = \Sigma^{-n}(P_{\geqslant n})$. For $i \geq n+1$ one has isomorphisms $\mathrm{H}_i(P_{\geqslant n}) \cong \mathrm{H}_i(P) \cong \mathrm{H}_i(M) = 0$, the first one of which comes from the exact sequence of complexes

$$0 \to P_{<n} \to P \to P_{\geqslant n} \to 0\,. \tag{1.4.1}$$ □

1.5. *Cohomology.* Let M be a complex of R-modules and $P \to M$ a semiprojective resolution. For every complex N and each $i \in \mathbb{Z}$ the abelian group

$$\mathrm{Ext}^i_R(M,N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(P,N)) = \mathrm{H}^i(\mathrm{Hom}_R(P,N))$$

is independent of the choice of resolution P; see 1.1. It is a module over R^{c}, the center of the ring R. For modules M, N this is the usual gadget; see 1.3.

Over noetherian rings syzygy modules inherit finiteness properties of the original module. We show that syzygy complexes behave similarly.

LEMMA 1.6. *If R is a noetherian ring and M is a complex with $\mathrm{H}(M)$ a finite R-module, then one can find a syzygy complex $\Omega_n^R(M)$ in $\mathsf{Thick}_R(M \oplus R)$.*

Furthermore, for every complex $C \in \mathsf{Thick}_R(M \oplus R)$ the following hold.

1. *The R-module $\mathrm{H}(C)$ is noetherian.*
2. $\mathrm{Ext}_R^{\gg 0}(M,N) = 0$ *for a bounded complex N implies* $\mathrm{Ext}_R^{\gg 0}(C,N) = 0$.
3. $\mathrm{Ext}_R^{\gg 0}(M,R) = 0$ *implies* $\mathrm{Ext}_R^{\gg 0}(C,F) = 0$ *for every projective R-module F.*

Proof. Under the hypotheses on R and M, one can choose a semiprojective resolution $P \simeq M$ with each P_i finite and $P_i = 0$ for $i \ll 0$. It follows that $P_{<n}$ is in $\mathsf{Thick}_R(R)$, so the exact sequence (1.4.1) yields $P_{\geqslant n} \in \mathsf{Thick}_R(M \oplus R)$.

The complexes L with $\mathrm{H}(L)$ finite form a thick subcategory of $\mathsf{D}(R)$. As it contains M and R, it contains $\mathsf{Thick}_R(M \oplus R)$ as well. This proves (1). A similar argument settles (2). For M and F as in (3) there is an isomorphism

$$\mathrm{Ext}^*_R(M,F) \cong \mathrm{Ext}^*_R(M,R) \otimes_R F\,,$$

which one can get by using the resolution P above. Thus, $\mathrm{Ext}_R^{\gg 0}(M,R) = 0$ implies $\mathrm{Ext}_R^{\gg 0}(M,F) = 0$. Now (2) yields $\mathrm{Ext}_R^{\gg 0}(C,F) = 0$, as desired. □

2. Graded rings

Here we describe notation and terminology for dealing with graded objects.

2.1. *Graded modules.* Let $\mathcal{A}$ be a commutative ring that is *non-negatively graded*: $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i$ with $\mathcal{A}^i\mathcal{A}^j \subseteq \mathcal{A}^{i+j}$ and $\mathcal{A}^i = 0$ for $i < 0$.

Modules over $\mathcal{A}$ are $\mathbb{Z}$-graded: $\mathcal{M} = \bigoplus_{j\in\mathbb{Z}} \mathcal{M}^j$ with $\mathcal{A}^i\mathcal{M}^j \subseteq \mathcal{M}^{i+j}$. For such an $\mathcal{M}$ *finite* means finitely generated, *eventually noetherian* means $\mathcal{M}^{\geqslant j}$ is noetherian for $j \gg 0$, and *eventually zero* means $\mathcal{M}^{\geqslant j} = 0$ for $j \gg 0$.

The *annihilator* of $\mathcal{M}$ is the set $\operatorname{ann}_{\mathcal{A}} \mathcal{M} = \{a \in \mathcal{A} \mid a\mathcal{M} = 0\}$. It is a homogeneous ideal in $\mathcal{A}$, so $\mathcal{A}/\operatorname{ann}_{\mathcal{A}} \mathcal{M}$ is a graded ring and $\mathcal{M}$ is a graded module over it. When $\mathcal{M}$ is noetherian so is $\mathcal{A}/\operatorname{ann}_{\mathcal{A}} \mathcal{M}$, so modulo $\operatorname{ann}_{\mathcal{A}} \mathcal{M}$ every ideal in $\mathcal{A}$ is generated by finitely many homogeneous elements.

2.2. *Supports.* Let $\operatorname{Spec} \mathcal{A}$ be the space of prime ideals of $\mathcal{A}$, with the Zariski topology. For an $\mathcal{A}$-module $\mathcal{M}$, set

$$\begin{aligned}\operatorname{Supp}_{\mathcal{A}} \mathcal{M} &= \{\mathfrak{p} \in \operatorname{Spec} \mathcal{A} \mid \mathcal{M}_{\mathfrak{p}} \neq 0\}\,;\\ \operatorname{Proj} \mathcal{A} &= \{\mathfrak{p} \in \operatorname{Spec} \mathcal{A} \mid \mathfrak{p} \text{ homogeneous and } \mathfrak{p} \not\supseteq \mathcal{A}^{\geqslant 1}\}\,;\\ \operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{M} &= \operatorname{Supp}_{\mathcal{A}} \mathcal{M} \cap \operatorname{Proj} \mathcal{A}\,.\end{aligned}$$

The following properties of graded $\mathcal{A}$-modules $\mathcal{L}, \mathcal{M}$, and $\mathcal{N}$ follow from the definition of support and the exactness of localization.

(1) If $\mathcal{L} \xrightarrow{\iota} \mathcal{M} \xrightarrow{\varepsilon} \mathcal{N}$ is an exact sequence, then
$$\operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{M} \subseteq \operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{L} \cup \operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{N}\,;$$
equality holds when ι is injective and ε is surjective.

(2) For each $i \in \mathbb{Z}$, one has $\operatorname{Supp}^{+}_{\mathcal{A}} (\mathcal{M}^{\geqslant i}) = \operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{M}$.

(3) If some $\mathcal{M}^{\geqslant n}$ is finite, say, if $\mathcal{M}$ is eventually noetherian, then
$$\operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{M} = \{\mathfrak{p} \in \operatorname{Proj} \mathcal{A} \mid \mathfrak{p} \supseteq \operatorname{ann}_{\mathcal{A}}(\mathcal{M}^{\geqslant i})\}$$
holds for every $i \geq n$; thus, $\operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{M}$ is a closed subset of $\operatorname{Proj} \mathcal{A}$.

(4) If the $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$ are finite, then
$$\operatorname{Supp}^{+}_{\mathcal{A}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) = \operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{M} \cap \operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{N}\,.$$

(5) If $\mathcal{M}$ is eventually zero, then $\operatorname{Supp}^{+}_{\mathcal{A}} \mathcal{M} = \varnothing$. The converse holds when $\mathcal{M}$ is eventually noetherian over $\mathcal{A}$.

In some cases, supports have a natural geometric interpretation.

2.3. *Varieties.* Let k be a field and $\bar{k}$ an algebraic closure of k. Assume that the graded ring $\mathcal{A}$ has $\mathcal{A}^0 = k$ and is generated over k by finitely many homogeneous elements of positive degree. For each graded $\mathcal{A}$-module $\mathcal{M}$ set

$$V_{\mathcal{A}}(\mathcal{M}) = \big(\operatorname{Supp}_{\bar{\mathcal{A}}} (\mathcal{M} \otimes_k \bar{k}) \cap \operatorname{Max} \bar{\mathcal{A}}\big) \cup \{\bar{\mathcal{A}}^{\geqslant 1}\},$$

where $\bar{\mathcal{A}}$ denotes the ring $\mathcal{A} \otimes_k \bar{k}$ and $\operatorname{Max} \bar{\mathcal{A}}$ the set of its maximal ideals.

Let $\mathcal{M}$ be a finite graded $\mathcal{A}$-module. The subset $V_{\mathcal{A}}(\mathcal{M})$ of $\operatorname{Max}\bar{A}$ then is closed in the Zariski topology; it is also k-rational and conical, in the sense that it can be defined by homogeneous elements in $\mathcal{A}$. The Nullstellensatz implies that each one of the sets $V_{\mathcal{A}}(\mathcal{M})$ and $\operatorname{Supp}^{+}_{\mathcal{A}}\mathcal{M}$ determines the other.

The graded rings and modules of interest in this paper are generated by cohomological constructions, which we recall below.

2.4. *Products in cohomology.* Let M and N be complexes of R-modules, and let $P \to M$ and $Q \to N$ be semiprojective resolutions. For each $i \in \mathbb{Z}$ one has

$$\mathrm{H}_{-i}(\operatorname{Hom}_R(P,Q)) = \mathrm{H}^i(\operatorname{Hom}_R(P,Q)) \cong \mathrm{H}^i(\operatorname{Hom}_R(P,N)) = \operatorname{Ext}^i_R(M,N)$$

in view of properties discussed in 1.1 and 1.5. We set

$$\operatorname{Ext}^*_R(M,N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_R(M,N)\,.$$

This is a graded module over R^{c}, the center of the ring R.

Composition of homomorphisms turns $\operatorname{Hom}_R(Q,Q)$ and $\operatorname{Hom}_R(P,P)$ into DG algebras over the center R^{c} of R, and $\operatorname{Hom}_R(P,Q)$ into a left DG module over the first and a right DG module over the second. The actions are compatible, so $\operatorname{Ext}^*_R(N,N)$ and $\operatorname{Ext}^*_R(M,M)$ become graded R^{c}-algebras and $\operatorname{Ext}^*_R(M,N)$ a left-right graded bimodule over them.

These structures do not depend on choices of resolutions.

3. Cohomological supports

In this section R denotes an associative ring.

3.1. *Cohomology operations.* A *ring of central cohomology operations* is a commutative graded ring $\mathcal{A}$ equipped with a homomorphism of graded rings

$$\zeta_M \colon \mathcal{A} \longrightarrow \operatorname{Ext}^*_R(M,M)$$

for each $M \in \mathsf{D}(R)$, such that for all $N \in \mathsf{D}(R)$ and $\xi \in \operatorname{Ext}^*_R(M,N)$ one has

$$\xi \cdot \zeta_M(a) = \zeta_N(a) \cdot \xi \quad \text{for every} \quad a \in \mathcal{A}\,. \tag{3.1.1}$$

For $N = M$ this formula implies that $\zeta_M(\mathcal{A})$ is in the center of $\operatorname{Ext}^*_R(M,M)$.

We assume that $\mathcal{A}$ is non-negatively graded and that $\mathcal{A}^i = 0$ for i odd or $2\mathcal{A} = 0$; this hypothesis covers existing examples and avoids sign trouble.

3.2. *Scalars.* Using the standard identifications of rings

$$\operatorname{Ext}^*_R(R,R) = \operatorname{Hom}_R(R,R) = R^{\mathsf{o}}\,,$$

where R^{o} denotes the opposite ring of R, one sees from (3.1.1) that the homomorphism of rings $\zeta_R \colon \mathcal{A} \to R^{\mathsf{o}}$ maps every element $a \in \mathcal{A}^0$ to the center of R^{o}. We identify the centers of R^{o} and R. Formula (3.1.1) then shows that the action of a on $\mathrm{Ext}_R^*(M,N)$ coincides with the maps induced by left multiplication with $\zeta_R(a)$ on M or on N.

For the next definition we use the notion of support introduced in 2.2.

3.3. *Cohomological supports.* Let $\mathcal{A}$ be a graded ring of central cohomology operations, as above. For each pair (M,N) of complexes we call the subset

$$\mathrm{Supp}_{\mathcal{A}}^*(M,N) = \mathrm{Supp}_{\mathcal{A}}^+\left(\mathrm{Ext}_R^*(M,N)\right) \subseteq \mathrm{Proj}\,\mathcal{A}$$

the *cohomological support* of (M,N). The cohomological support of M is

$$\mathrm{Supp}_{\mathcal{A}}^*(M) = \mathrm{Supp}_{\mathcal{A}}^*(M,M)\,.$$

The theorem below is the main result of this section.

THEOREM 3.4. *Let M and N be complexes of R-modules.*

If the graded $\mathcal{A}$-module $\mathrm{Ext}_R^*(M,N)$ is noetherian, then for every closed subset X of $\mathrm{Supp}_{\mathcal{A}}^*(M,N)$ there exist complexes M_X in $\mathsf{Thick}_R(M)$ and N_X in $\mathsf{Thick}_R(N)$, such that the following equalities hold:

$$X = \mathrm{Supp}_{\mathcal{A}}^*(M_X,N) = \mathrm{Supp}_{\mathcal{A}}^*(M_X,N_X) = \mathrm{Supp}_{\mathcal{A}}^*(M,N_X)\,.$$

Moreover, when $N = M$ one can take $N_X = M_X$.

The proof appears at the end of the section. Some of the preparatory material is used repeatedly throughout the paper.

Let d be an integer. The dth *shift* of a complex M is the complex ΣM with $(\Sigma^d M)_n = M_{n-d}$ for all n and $\partial^{\Sigma^d M} = (-1)^d \partial^M$. The dth *twist* of a graded $\mathcal{A}$-module $\mathcal{M}$ is the graded module $\mathcal{M}(d)$ with $\mathcal{M}(d)^j = \mathcal{M}^{d+j}$ for all j.

3.5. *Functoriality.* Let M, M', M'' and N, N', N'' be complexes of R-modules.

There exist canonical isomorphisms of graded $\mathcal{A}$-modules:

$$\mathrm{Ext}_R^*(\Sigma M,N)(1) \cong \mathrm{Ext}_R^*(M,N) \cong \mathrm{Ext}_R^*(M,\Sigma N)(-1)\,; \tag{3.5.1}$$

$$\begin{aligned} \mathrm{Ext}_R^*(M'\oplus M'',N) &\cong \mathrm{Ext}_R^*(M',N) \oplus \mathrm{Ext}_R^*(M'',N)\,;\\ \mathrm{Ext}_R^*(M,N'\oplus N'') &\cong \mathrm{Ext}_R^*(M,N') \oplus \mathrm{Ext}_R^*(M,N'')\,. \end{aligned} \tag{3.5.2}$$

Indeed, basic properties of the functor $\mathrm{Hom}_{\mathsf{D}(R)}(-,-)$ show that for a fixed N (respectively, M) the canonical isomorphisms of graded R^{c}-modules are linear for the action of $\mathrm{Ext}_R^*(N,N)$ on the left (respectively, of $\mathrm{Ext}_R^*(M,M)$ on the right). They are $\mathcal{A}$-linear because of the centrality of $\mathcal{A}$; see (3.1.1).

Similarly, exact triangles $M' \to M \to M'' \to$ and $N' \to N \to N'' \to$ in $\mathsf{D}(R)$ induce exact sequences of graded $\mathcal{A}$-modules

$$\begin{gathered} \operatorname{Ext}^*_R(M'',N) \longrightarrow \operatorname{Ext}^*_R(M,N) \longrightarrow \operatorname{Ext}^*_R(M',N) \longrightarrow \\ \operatorname{Ext}^*_R(M'',N)(1) \longrightarrow \operatorname{Ext}^*_R(M,N)(1) \\ \operatorname{Ext}^*_R(M,N') \longrightarrow \operatorname{Ext}^*_R(M,N) \longrightarrow \operatorname{Ext}^*_R(M,N'') \longrightarrow \\ \operatorname{Ext}^*_R(M,N')(1) \longrightarrow \operatorname{Ext}^*_R(M,N)(1) \end{gathered} \tag{3.5.3}$$

Putting together the remarks in 2.2 and 3.5, one gets:

Lemma 3.6. *In the notation of* 3.5 *the following statements hold.*

$$\operatorname{Supp}^*_{\mathcal{A}}(\Sigma M,N) = \operatorname{Supp}^*_{\mathcal{A}}(M,N) = \operatorname{Supp}^*_{\mathcal{A}}(M,\Sigma N)\,. \tag{3.6.1}$$

$$\begin{gathered} \operatorname{Supp}^*_{\mathcal{A}}(M'\oplus M'',N) = \operatorname{Supp}^*_{\mathcal{A}}(M',N) \cup \operatorname{Supp}^*_{\mathcal{A}}(M'',N)\,. \\ \operatorname{Supp}^*_{\mathcal{A}}(M,N'\oplus N'') = \operatorname{Supp}^*_{\mathcal{A}}(M,N') \cup \operatorname{Supp}^*_{\mathcal{A}}(M,N'')\,. \end{gathered} \tag{3.6.2}$$

$$\begin{gathered} \operatorname{Supp}^*_{\mathcal{A}}(M,N) \subseteq \operatorname{Supp}^*_{\mathcal{A}}(M',N) \cup \operatorname{Supp}^*_{\mathcal{A}}(M'',N)\,. \\ \operatorname{Supp}^*_{\mathcal{A}}(M,N) \subseteq \operatorname{Supp}^*_{\mathcal{A}}(M,N') \cup \operatorname{Supp}^*_{\mathcal{A}}(M,N'')\,. \end{gathered} \tag{3.6.3}$$

If $\operatorname{Ext}^*_R(M,N)$ *is eventually zero, then* $\operatorname{Supp}^*_{\mathcal{A}}(M,N)=\varnothing$. *The converse holds when* $\operatorname{Ext}^*_R(M,N)$ *is eventually noetherian over* $\mathcal{A}$. □

The exact sequences (3.5.3) imply the following statement:

Lemma 3.7. *Let* M *be a complex of* R*-modules.*

The full subcategory of $\mathsf{D}(R)$ *consisting of complexes* L *with* $\operatorname{Ext}^*_R(M,L)$ *(respectively,* $\operatorname{Ext}^*_R(L,M)$*) eventually noetherian over* $\mathcal{A}$ *is thick.* □

3.8. *Mapping cone.* Let M be a complex of R-modules.

For each $\varphi \in \mathcal{A}^d$ the morphism $\zeta_M(\varphi)\colon M \to \Sigma^d M$ defines an exact triangle

$$M \xrightarrow{\zeta_M(\varphi)} \Sigma^d M \longrightarrow M/\!\!/\varphi \longrightarrow\,, \tag{3.8.1}$$

which is unique up to isomorphism.

Let N be a complex of R-modules and set $\mathcal{M} = \operatorname{Ext}^*_R(M,N)$. By (3.5.3) and (3.5.1), the triangle above yields an exact sequence of graded $\mathcal{A}$-modules

$$\mathcal{M}(-d-1) \longrightarrow \mathcal{M}(-1) \longrightarrow \operatorname{Ext}^*_R(M/\!\!/\varphi,N) \longrightarrow \mathcal{M}(-d) \longrightarrow \mathcal{M}\,. \tag{3.8.2}$$

The maps at both ends are given by multiplication with φ, so from (3.8.2) one can extract an exact sequence of graded $\mathcal{A}$-modules

$$0 \longrightarrow \big(\mathcal{M}/\mathcal{M}\varphi\big)(-1) \longrightarrow \operatorname{Ext}^*_R(M/\!\!/\varphi,N) \longrightarrow (0:_{\mathcal{M}}\varphi)(-d) \longrightarrow 0\,. \tag{3.8.3}$$

Let $\boldsymbol{\varphi} = \varphi_1,\dots,\varphi_n$ be homogeneous elements in $\mathcal{A}$. Set $\boldsymbol{\varphi}' = \varphi_1,\dots,\varphi_{n-1}$. A complex $(M/\!\!/\boldsymbol{\varphi}')/\!\!/\varphi_n$ is defined uniquely up to isomorphism in $\mathsf{D}(R)$; we let $M/\!\!/\boldsymbol{\varphi}$ denote any such complex. Iterated references to (3.8.1) yield

$$M/\!\!/\boldsymbol{\varphi} \in \mathsf{Thick}_R(M)\,. \tag{3.8.4}$$

EXAMPLE 3.9. If $\varphi_1, \dots, \varphi_n$ are in $\mathcal{A}^0$, then $\zeta_M(\varphi_i)$ is the homothety $M \to M$ defined by the central element $z_i = \zeta_R(\varphi_i) \in R$; see 3.2. Thus, in $\mathsf{D}(R)$ one has $M/\!\!/\boldsymbol{\varphi} \simeq M \otimes_{R^c} K(\boldsymbol{z})$, where $K(\boldsymbol{z})$ is the Koszul complex on $\boldsymbol{z} = z_1, \dots, z_n$.

PROPOSITION 3.10. *Let M, N be complexes of R-modules and $\boldsymbol{\varphi} = \varphi_1, \dots, \varphi_n$ a sequence of homogeneous elements in $\mathcal{A}$.*

If the $\mathcal{A}$-module $\operatorname{Ext}^*_R(M,N)$ is eventually noetherian, then so are the $\mathcal{A}$-modules $\operatorname{Ext}^*_R(M/\!\!/\boldsymbol{\varphi}, N)$, $\operatorname{Ext}^*_R(M, N/\!\!/\boldsymbol{\varphi})$, and $\operatorname{Ext}^*_R(M/\!\!/\boldsymbol{\varphi}, N/\!\!/\boldsymbol{\varphi})$, and

$$\begin{aligned}\operatorname{Supp}^*_{\mathcal{A}}(M/\!\!/\boldsymbol{\varphi}, N) = \operatorname{Supp}^*_{\mathcal{A}}&(M/\!\!/\varphi, N/\!\!/\boldsymbol{\varphi}) = \operatorname{Supp}^*_{\mathcal{A}}(M, N/\!\!/\boldsymbol{\varphi})\\ &= \operatorname{Supp}^*_{\mathcal{A}}(M,N) \cap \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\boldsymbol{\varphi})\,.\end{aligned}$$

Proof. It suffices to treat the case when $\boldsymbol{\varphi}$ has a single element, φ. From the exact sequence (3.8.3) one sees that $\operatorname{Ext}^*_R(M/\!\!/\varphi, N)$ is eventually noetherian.

Set $\mathcal{M} = \operatorname{Ext}^*_R(M,N)$. The inclusion below holds because $(0 :_{\mathcal{M}} \varphi)$ is a submodule of $\mathcal{M}$ and a module over $\mathcal{A}/\mathcal{A}\varphi$; the equality comes from 2.2(5):

$$\operatorname{Supp}^+_{\mathcal{A}}(0 :_{\mathcal{M}} \varphi) \subseteq \operatorname{Supp}^+_{\mathcal{A}} \mathcal{M} \cap \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\varphi)\,;$$

$$\operatorname{Supp}^+_{\mathcal{A}}\big(\mathcal{M}/\mathcal{M}\varphi\big) = \operatorname{Supp}^+_{\mathcal{A}} \mathcal{M} \cap \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\varphi)\,.$$

The exact sequence (3.8.3) and 2.2(2) now imply an equality

$$\operatorname{Supp}^*_{\mathcal{A}}(M/\!\!/\varphi, N) = \operatorname{Supp}^+_{\mathcal{A}} \mathcal{M} \cap \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\varphi)\,.$$

By a similar argument, $\operatorname{Ext}^*_R(M, N/\!\!/\varphi)$ is eventually noetherian and one has

$$\operatorname{Supp}^*_{\mathcal{A}}(M, N/\!\!/\varphi) = \operatorname{Supp}^+_{\mathcal{A}} \mathcal{M} \cap \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\varphi)\,.$$

The remaining equality is a formal consequence of those already available. □

Proof of Theorem 3.4. Set $\mathcal{M} = \operatorname{Ext}^*_R(M,N)$ and $\mathcal{I} = \operatorname{ann}_{\mathcal{A}} \mathcal{M}$.

From 2.2(3) and 2.2(4) we get

$$\operatorname{Supp}^*_{\mathcal{A}}(M,N) = \operatorname{Supp}^+_{\mathcal{A}} \mathcal{M} = \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{I}).$$

As $\mathcal{M}$ is noetherian the closed subset X of $\operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{I})$ has the form

$$X = \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{I}) \cap \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\boldsymbol{\varphi}),$$

where $\boldsymbol{\varphi}$ is a finite set of homogeneous elements of $\mathcal{A}$; see 2.1. Thus, one gets

$$X = \operatorname{Supp}^*_{\mathcal{A}}(M,N) \cap \operatorname{Supp}^+_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\boldsymbol{\varphi})\,.$$

Choose complexes M_X and N_X representing $M/\!\!/\boldsymbol{\varphi}$ and $N/\!\!/\boldsymbol{\varphi}$, respectively. One has $M_X \in \mathsf{Thick}_R(M)$ and $N_X \in \mathsf{Thick}_R(N)$; see (3.8.4). Also, one gets

$$X = \operatorname{Supp}^*_{\mathcal{A}}(M_X, N) = \operatorname{Supp}^*_{\mathcal{A}}(M_X, N_X) = \operatorname{Supp}^*_{\mathcal{A}}(M, N_X)$$

from Proposition 3.10. Clearly, when $M = N$ one may choose $N_X = M_X$. □

4. Realizability by modules

In this section R is an associative ring and $\mathcal{A}$ is a ring of central cohomology operations on $\mathsf{D}(R)$; see 3.1. The principal result here is a partial enhancement of Theorem 3.4. It contains Theorem 1 from the introduction.

Existence Theorem 4.1. *Let R be a left noetherian ring.*

When M and N are complexes of R-modules with $\mathrm{H}(M)$ and $\mathrm{H}(N)$ finite, and X is a closed subset of $\operatorname{Supp}^*_{\mathcal{A}}(M,N)$ the following hold.

(1) *If $\operatorname{Ext}^*_R(M,N)$ is eventually noetherian over $\mathcal{A}$ (and X is irreducible), then there exists a finite (and indecomposable) R-module M_X with*

$$X = \operatorname{Supp}^*_{\mathcal{A}}(M_X, N) \quad \text{and} \quad M_X \in \mathsf{Thick}_R(M \oplus R)\,.$$

(2) *If, furthermore, $\operatorname{Ext}^*_R(M,R)$ is eventually zero (and X is irreducible), then there exists a finite (and indecomposable) R-module N_X with*

$$X = \operatorname{Supp}^*_{\mathcal{A}}(M, N_X) \quad \text{and} \quad N_X \in \mathsf{Thick}_R(N \oplus R)\,.$$

When $N = M$ one may choose $N_X = M_X$.

Proof. Using Theorem 3.4, choose complexes C in $\mathsf{Thick}_R(M)$ and D in $\mathsf{Thick}_R(N)$, satisfying $\operatorname{Supp}^*_{\mathcal{A}}(C,N) = X = \operatorname{Supp}^*_{\mathcal{A}}(M,D)$.

By Lemma 1.6(1), the R-modules $\mathrm{H}(C)$ and $\mathrm{H}(D)$ are noetherian, so one has $\mathrm{H}_{\geqslant n}(C) = 0 = \mathrm{H}_{\geqslant n}(D)$ for some n. Lemma 1.6 provides syzygy complexes $\Omega^R_n(C)$ in $\mathsf{Thick}_R(M \oplus R)$ and $\Omega^R_n(D)$ in $\mathsf{Thick}_R(N \oplus R)$. Another application of Lemma 1.6(1) shows that the following R-modules are finite:

$$M_X = \mathrm{H}_0(\Omega^R_n(C)) \quad \text{and} \quad N_X = \mathrm{H}_0(\Omega^R_n(D))\,.$$

Lemma 1.4 yields $\Omega^R_n(C) \simeq M_X$ and $\Omega^R_n(D) \simeq N_X$.

(1) comes from the equalities below, the second one given by Lemma 4.2(1):

$$X = \operatorname{Supp}^*_{\mathcal{A}}(C,N) = \operatorname{Supp}^*_{\mathcal{A}}(\Omega^R_n(C),N) = \operatorname{Supp}^*_{\mathcal{A}}(M_X,N)\,.$$

(2) As $\operatorname{Ext}^*_R(M,R)$ is eventually zero, so is $\operatorname{Ext}^*_R(M,F)$; see Lemma 1.6(3). Thus, referring to Lemma 4.2(2) for the second equality, one obtains

$$X = \operatorname{Supp}^*_{\mathcal{A}}(M,D) = \operatorname{Supp}^*_{\mathcal{A}}(M,\Omega^R_n(D)) = \operatorname{Supp}^*_{\mathcal{A}}(M,N_X)\,.$$

When $N = M$ one can choose $D = C$ by Theorem 3.4, and hence get

$$N_X = M_X \simeq \Omega^R_n(C) \in \mathsf{Thick}_R(M \oplus R)\,.$$

Lemma 1.6(3) now shows that $\operatorname{Ext}^*_R(M_X,F)$ is eventually zero when F is free. Thus, the already established assertion of the theorem apply to M_X and give

$$X = \operatorname{Supp}^*_{\mathcal{A}}(M_X,M_X)\,.$$

It remains to establish the additional property when X is irreducible. Being a noetherian module, M_X is a finite direct sum of indecomposables. It follows from (3.6.2) that one can replace M_X with such a summand, without changing $\operatorname{Supp}^*_{\mathcal{A}}(M_X,N)$. A similar argument works for N_X. □

The following general property of syzygy complexes was used above.

LEMMA 4.2. *Let M, N be complexes of R-modules with bounded homology. For every integer n the following hold.*

(1) *There is an equality* $\operatorname{Supp}^*_{\mathcal{A}}(M,N) = \operatorname{Supp}^*_{\mathcal{A}}(\Omega^R_n(M),N)$.

(2) *If* $\operatorname{Ext}^*_R(M,F)$ *is eventually zero for every free module F, then also*
$$\operatorname{Supp}^*_{\mathcal{A}}(M,N) = \operatorname{Supp}^*_{\mathcal{A}}(M,\Omega^R_n(N))\,.$$

Proof. (2) Replacing N with a semiprojective resolution, we may assume that each N_j is projective and $N_j = 0$ for all $j \ll 0$. An elementary argument using the formulas in 3.5 shows that the complexes C of R-modules, for which $\operatorname{Ext}^*_R(M,C)$ is eventually zero, form a thick subcategory of $\mathsf{D}(R)$. It contains the free modules by hypothesis, and hence it contains all bounded complexes of projective modules. Therefore, $\operatorname{Ext}^{\geqslant i}_R(M,N_{<n}) = 0$ holds for all $i \gg 0$.

The inclusion $N_{<n} \subseteq N$ gives rise to an exact triangle $N_{<n} \to N \to N_{\geq n} \to$ in $\mathsf{D}(R)$. Again by (3.5.3), it induces an exact sequence of graded $\mathcal{A}$-modules
$$\operatorname{Ext}^*_R(M,N_{<n}) \longrightarrow \operatorname{Ext}^*_R(M,N) \longrightarrow \operatorname{Ext}^*_R(M,N_{\geq n}) \to$$
$$\operatorname{Ext}^*_R(M,N_{<n})(1) \to \operatorname{Ext}^*_R(M,N)(1)$$
In view of the preceding discussion it yields $\operatorname{Ext}^{\geqslant i}_R(M,N) \cong \operatorname{Ext}^{\geqslant i}_R(M,N_{\geqslant n})$ for all $i \gg 0$. On the other hand, one has $N_{\geqslant n} = \Sigma^n\Omega^R_n(N)$ because N is semiprojective. The desired equality now follows from 2.2(2).

(1) This follows from a similar, and simpler, argument. □

5. Bialgebras and Hopf algebras

In this section k denotes a field. We recall some notions concerning bialgebras and Hopf algebras, referring to [19] for details.

A *bialgebra* over k is a k-algebra R with structure map $\eta\colon k \to R$ and product $\mu\colon R\otimes_k R \to k$, equipped with homomorphisms of rings $\varepsilon\colon R \to k$, the *augmentation* and $\Delta\colon R \to R\otimes_k R$, the *co-product*, satisfying equalities
$$\varepsilon\eta = \mathrm{id}^k\,, \qquad (\Delta\otimes\mathrm{id}^R)\Delta = \Delta(\Delta\otimes\mathrm{id}^R)\,,$$
$$\mu(\mathrm{id}^R\otimes\eta\varepsilon)\Delta = \mathrm{id}^R = \mu(\eta\varepsilon\otimes\mathrm{id}^R)\Delta\,.$$
Given R-modules M, N over a bialgebra R, the natural $R\otimes_k R$-module structure on $M\otimes_k N$ restricts along Δ to produce a canonical R-module structure:
$$r\cdot(m\otimes n) = \sum_{i=1}^{n}(r'_i m\otimes_k r''_i n) \quad \text{when} \quad \Delta(r) = \sum_{i=1}^{n}(r'_i\otimes_k r''_i)\,.$$
This extends to tensor products of complexes of R-modules. Let M be such a complex. The canonical isomorphisms below are easily seen to be R-linear:

(5.0.1) $$k\otimes_k M \cong M \quad\text{and}\quad M\otimes_k k \cong M\,.$$

5.1. *Cohomology operations.* Let R be a bialgebra over k, and view k as an R-module via the augmentation ε. The ring $\operatorname{Ext}^*_R(k,k)$ has $\operatorname{Ext}^0_R(k,k)=k$ and is *graded-commutative*: for all $\alpha\in\operatorname{Ext}^i_R(k,k)$ and $\beta\in\operatorname{Ext}^j_R(k,k)$ one has

$$\alpha\cdot\beta=(-1)^{ij}\beta\cdot\alpha\,;$$

see [17, (VIII.4.7), (VIII.4.3)] or [16, (5.5)]. Thus, every graded subring

$$\text{(a)}\qquad \mathcal{A}\subseteq\operatorname{Ext}^\bullet_R(k,k)=\begin{cases}\bigoplus_{i\geqslant 0}\operatorname{Ext}^{2i}_R(k,k) & \text{if } \operatorname{char}(k)\neq 2\,;\\ \bigoplus_{i\geqslant 0}\operatorname{Ext}^{i}_R(k,k) & \text{if } \operatorname{char}(k)=2\,.\end{cases}$$

is commutative. The functor $-\otimes_k M$ preserves quasi-isomorphisms of complexes of R-modules, so it induces a functor $-\otimes_k M\colon \mathsf{D}(R)\to\mathsf{D}(R)$. In view of the isomorphism $k\otimes_k M\cong M$; see (5.0.1), for each M one gets a map

$$\text{(b)}\qquad \zeta_M\colon\ \operatorname{Ext}^\bullet_R(k,k)\to\operatorname{Ext}^*_R(M,M)\,.$$

It is readily verified to be a central homomorphism of graded k-algebras.

The results in Section 3 apply to any algebra $\mathcal{A}$ as above. More comprehensive information is available for special classes of bialgebras.

A *Hopf algebra* is a bialgebra R with a k-linear map $\sigma\colon R\to R$, the *antipode*, satisfying $\varepsilon\sigma=\varepsilon$ and $\mu(1\otimes\sigma)\Delta=\mu(\sigma\otimes 1)\Delta$. Quantum groups offer prime examples. A Hopf algebra is *cocommutative* if $\tau\Delta=\Delta$ holds, where $\tau(r\otimes s)=s\otimes r$. For instance, for a group G the k-linear maps defined by

$$\varepsilon(g)=1\,,\quad \Delta(g)=g\otimes g\,,\quad\text{and}\quad \sigma(g)=g^{-1}\quad\text{for}\quad g\in G$$

turn the group algebra kG into a cocommutative Hopf algebra. Other classical examples are universal enveloping algebras of Lie algebras and restricted universal enveloping algebras of p-Lie algebras, where $p=\operatorname{char}(k)>0$.

5.2. *Finiteness.* Let R be a Hopf algebra such that $\operatorname{rank}_k R$ finite.

If R is cocommutative, then $\operatorname{Ext}^\bullet_R(k,k)$ is finitely generated as a k-algebra, and $\operatorname{Ext}^*_R(M,N)$ is a finite $\operatorname{Ext}^\bullet_R(k,k)$-module for all R-modules M,N of finite k-rank: This is a celebrated theorem of Friedlander and Suslin [13, (1.5.2)], which extends earlier results for group algebras (Evens, Golod, Venkov) and for restricted Lie algebras (Friedlander and Parshall).

It is not known whether cohomology has similar finiteness properties when R is not cocommutative; for positive solutions in interesting classes of such Hopf algebras see Pevtsova and Witherspoon [20], and the bibliography there.

5.3. *Cohomological varieties.* Let R be a Hopf algebra with $\operatorname{rank}_k R$ finite, set $\mathcal{A}=\operatorname{Ext}^\bullet_R(k,k)$ (see 5.1(a)), and let $\bar{k}$ be an algebraic closure of k.

For a complex M with $\operatorname{Ext}^*_R(M,M)$ eventually noetherian over $\mathcal{A}$ define (with notation as in 2.3) the *cohomological variety* of M to be the subset

$$V^*_R(M)=V_{\mathcal{A}}(\operatorname{Ext}^*_R(M,M))\subseteq\operatorname{Max}\,(\mathcal{A}\otimes_k\bar{k})\,.$$

Existence Theorem 5.4. *Let R be a Hopf algebra over k, such that $\operatorname{rank}_k R$ is finite; set $\mathcal{A} = \operatorname{Ext}^{\bullet}_R(k,k)$. Let M be a complex with $\operatorname{H}(M)$ finite over R.*

If $\operatorname{Ext}^*_R(M,M)$ is eventually noetherian over $\mathcal{A}$, then for each closed conical k-rational subset X of $V^*_{\mathcal{A}}(M)$ there is a finite R-module M_X, such that

$$X = V^*_{\mathcal{A}}(M_X) \quad \text{and} \quad M_X \in \mathsf{Thick}_R(M \oplus R)\,.$$

Proof. Hopf algebras of finite rank are self-injective (see [19, (2.1.3)(4)]), so one has $\operatorname{Ext}^{\geqslant 1}_R(-,R) = 0$. It remains to invoke Theorem 4.1 and refer to 2.3. □

5.5. *Applications.* In view of 5.2, for $M = k$ the theorem specializes to results of Carlson [11], Suslin, Friedlander, and Bendel [25, (7.5)], Pevtsova and Witherspoon [20, (4.5)], among others. It is clear that there are also versions dealing with supports of pairs of modules, and with complexes.

6. Associative algebras

Here k is a field and R is a k-algebra. Let R° denote the opposite algebra of R, set $R^{\mathsf{e}} = R \otimes_k R^{\circ}$, and turn R into a left R^{e}-module by $(r \otimes r') \cdot s = rsr'$.

6.1. *Cohomology operations.* The *Hochschild cohomology* of R is the k-algebra

$$\operatorname{H}^*(R\,|\,k) = \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_{R^{\mathsf{e}}}(R,R)\,.$$

Gerstenhaber [14, Cor. 1] proved that it is graded-commutative, so any subring

$$\text{(a)} \qquad \mathcal{A} \subseteq \operatorname{H}^{\bullet}(R\,|\,k) = \bigoplus_{i \geqslant 0} \operatorname{H}^{2i}(R\,|\,k)$$

is commutative. The map $r \mapsto 1 \otimes r$ is a homomorphism of rings $R^{\circ} \to R^{\mathsf{e}}$. It turns each complex of R^{e}-modules into one of right R-modules. Thus, $- \otimes_R M$ is an additive functor from complexes of R^{e}-modules to complexes of R-modules, where R acts on the target via the homomorphism of rings $R \to R^{\mathsf{e}}$ given by $r \mapsto r \otimes 1$. It induces an exact functor $- \otimes^{\mathbf{L}}_R M \colon \mathsf{D}(R^{\mathsf{e}}) \to \mathsf{D}(R)$ of derived categories, which produces a homomorphism of graded rings

$$\operatorname{Ext}^*_{R^{\mathsf{e}}}(R,R) \to \operatorname{Ext}^*_R(R \otimes^{\mathbf{L}}_R M, R \otimes^{\mathbf{L}}_R M)\,.$$

The isomorphism $R \otimes^{\mathbf{L}}_R M \simeq M$ now yields a natural homomorphism

$$\text{(b)} \qquad \zeta_M \colon \operatorname{H}^{\bullet}(R\,|\,k) \to \operatorname{Ext}^*_R(M,M)$$

of graded rings. These maps satisfy condition (3.1.1); see [23, (10.1)].

The results in Sections 3 and 4 apply to any algebra $\mathcal{A}$ as above. Once again, we focus on a special case to relate them to available literature.

6.2. *Finiteness.* Let R be a k-algebra with $\operatorname{rank}_k R$ finite, J the Jacobson radical of R, and set $K = R/J$. It is rarely the case that the $\mathrm{H}^*(R\,|\,k)$-module $\operatorname{Ext}^*_R(M,K)$ is noetherian for every finite R-module M; see [12, §1]. Examples when this property holds include the Hopf algebras in (5.2), exterior algebras, and commutative complete intersections rings; see (7.1).

6.3. *Cohomological varieties.* For R as in (6.2), let $\mathcal{A}$ be a subring of $\mathrm{H}^\bullet(R\,|\,k)$ with $\mathcal{A}^0 = k$ (see 6.1), and let $\bar{k}$ be an algebraic closure of k.

For a complex of R-modules M, such that the $\mathcal{A}$-module $\operatorname{Ext}^*_R(M,K)$ is eventually noetherian, define the *cohomological variety* of M to be the subset

$$V^*_{\mathcal{A}}(M) = V_{\mathcal{A}}(\operatorname{Ext}^*_R(M,K)) \subseteq \operatorname{Max}(\mathcal{A} \otimes_k \bar{k})\,.$$

As $\mathcal{A}$ acts on $\operatorname{Ext}^*_R(M,K)$ through $\operatorname{Ext}^*_R(K,K)$, one has $V^*_{\mathcal{A}}(M) \subseteq V^*_{\mathcal{A}}(K)$.

Existence Theorem 6.4. *Let R and $\mathcal{A}$ be as in* 6.3*, and let M be a complex of R-modules with* $\mathrm{H}(M)$ *finite over R.*

If $\operatorname{Ext}^*_R(M,K)$ is eventually noetherian over $\mathcal{A}$, then for each closed conical k-rational subset X of $V^*_{\mathcal{A}}(M)$ there is a finite R-module M_X, such that

$$X = V^*_{\mathcal{A}}(M_X) \quad \textit{and} \quad M_X \in \mathsf{Thick}_R(M \oplus R)\,.$$

Proof. The R-module R admits a finite filtration with subquotients isomorphic to direct summands of K. Thus, when the $\mathcal{A}$-module $\operatorname{Ext}^*_R(M,K)$ is noetherian, so is $\operatorname{Ext}^*_R(M,R)$. On it $\mathcal{A}$ acts through $\operatorname{Ext}^*_R(R,R) = R^{\mathsf{c}}$; see 3.2. This means $\operatorname{Ext}^{\gg 0}_R(M,R) = 0$, so we may use Theorem 4.1, then 2.3. □

6.5. *Application.* When the $\mathcal{A}$-module $\operatorname{Ext}^*_R(K,K)$ is noetherian, Theorem 6.4 with $M = K$ yields a result of Erdmann *et al.*; see [12, (3.4)].

7. Commutative local rings

We say that $(R,\mathfrak{m},k)$ is a *local ring* if R is a commutative noetherian ring with unique maximal ideal $\mathfrak{m}$ and $k = R/\mathfrak{m}$.

An *embedded deformation* of codimension c of R is a surjective homomorphism $\varkappa\colon Q \to R$ of rings with $(Q,\mathfrak{q},k)$ a local ring and $\operatorname{Ker}(\varkappa)$ an ideal generated by a Q-regular sequence in $\mathfrak{q}^2$, of length c.

7.1. *Cohomology operations.* Let $(R,\mathfrak{m},k)$ be a local ring with an embedded deformation $\varkappa\colon Q \to R$ of codimension c. Set

(a) $$\mathcal{A} = R[\chi_1,\dots,\chi_c]$$

where $\chi_1,\dots,\chi_c$ are indeterminates of degree 2. For each $M \in \mathsf{D}(R)$ Avramov and Sun [7, (2.7), p. 700] construct a natural homomorphism of graded rings

(b) $$\zeta_M : \mathcal{A} \to \operatorname{Ext}^*_R(M,M)$$

satisfying condition (3.1.1); when M and N are R-modules the resulting structure of graded $\mathcal{A}$-module on $\operatorname{Ext}_R^*(M,N)$ coincides with that defined by Gulliksen [15]. For complexes M and N, the $\mathcal{A}$-module $\operatorname{Ext}_R^*(M,N)$ is finite if and only if the Q-module $\operatorname{Ext}_Q^*(M,N)$ is finite; see [7, (5.1)] and [3, (4.2)].

The action of $\mathcal{A}$ on $\operatorname{Ext}_R^*(M,k)$ factors through the graded ring

$$\mathcal{R} = \mathcal{A} \otimes_R k = k[\chi_1, \dots, \chi_c]\,. \tag{c}$$

Recall that a complex of Q-modules is said to be *perfect* if it is isomorphic, in $\mathsf{D}(Q)$, to a bounded complex of finite free Q-modules.

Lemma 7.2. *Let Q, R, and $\mathcal{R}$ be the rings in* 7.1.

The following conditions are equivalent for each complex M of R-modules:

(i) *M is perfect over Q.*

(ii) $\mathrm{H}(M)$ *is finite over R and* $\operatorname{Ext}_R^*(M,k)$ *is finite over $\mathcal{R}$.*

Proof. (i) $\implies$ (ii): As M is perfect over Q, the Q-module $\operatorname{Ext}_Q^*(M,k)$ is finite, and hence the $\mathcal{R}$-module $\operatorname{Ext}_R^*(M,k)$ is finite; see 7.1.

(ii) $\implies$ (i): It follows from 7.1 that $\operatorname{Ext}_Q^*(M,k)$ is finite over Q, and hence is eventually zero. Since $\mathrm{H}(M)$ is finite over Q, the complex M admits a semiprojective resolution F with each F_i finite, $F_i = 0$ for $i \ll 0$, and $\partial(F) \subseteq \mathfrak{m}F$; see, for example, [4]. This yields an isomorphism $\operatorname{Ext}_Q^i(M,k) \cong \operatorname{Hom}_Q(F_i,k)$. Thus $\operatorname{Ext}_Q^i(M,k) = 0$ for $i \geq n$ implies $F_i = 0$ for $i \geq n$, so F is a perfect complex of Q-modules that is quasi-isomorphic to M. $\square$

7.3. *Cohomological varieties.* Let $(R,\mathfrak{m},k)$ be a local ring with an embedded deformation $\varkappa$ of codimension c, as in 7.1, and $\bar{k}$ an algebraic closure of k. Let M be a complex of R-modules with $\operatorname{Ext}_R^*(M,k)$ noetherian over $\mathcal{R}$. The *cohomological variety* of M is the subset $V_\varkappa^*(M)$ of $\bar{k}^c$ defined by the formula

$$V_\varkappa^*(M) = V_{\mathcal{R}}(\operatorname{Ext}_R^*(M,k)) \subseteq \operatorname{Max}(\mathcal{R} \otimes_k \bar{k}) = \bar{k}^c\,,$$

where the second equality comes from Hilbert's Nullstellensatz. When M is a module and $\boldsymbol{f}$ is a Q-regular sequence that generates $\operatorname{Ker}(\varkappa)$, the construction above yields the cone $V_R^*(\boldsymbol{f};M)$ defined in [1].

A Q-module is a perfect complex in $\mathsf{D}(Q)$ if and only if $\operatorname{proj\,dim}_Q M$ is finite. Thus, Theorem 2 from the introduction is obtained from the next result by replacing affine cones by their projectivizations.

Existence Theorem 7.4. *Let $(Q,\mathfrak{q},k)$ be a local ring, $\boldsymbol{f} \subset \mathfrak{q}^2$ a Q-regular sequence of length c, and $\varkappa\colon Q \to Q/Q\boldsymbol{f} = R$ the canonical surjection.*

The assignment $M \mapsto V_\varkappa^*(M)$, which maps complexes in $\mathsf{D}(R)$ that are perfect over Q to closed k-rational cones in $\bar{k}^c$, has the following properties:

(1) *It is surjective, even when restricted to modules.*

(2) $V_\varkappa^*(M) = \{0\}$ *if and only if M is perfect over R.*

(3) $V_{\varkappa}^*(M) = V_{\varkappa}^*(\Omega_n^R(M))$ *for every syzygy complex* $\Omega_n^R(M)$.
(4) $V_{\varkappa}^*(M) = V_{\varkappa}^*(M/\boldsymbol{x}M)$ *for a module* M *and an* M*-regular sequence* $\boldsymbol{x}$.

We import some material for the proof of part (1) of the theorem.

7.5. Under the hypotheses of the theorem Avramov, Gasharov, and Peeva [3, (3.3), (3.11), (6.2)] give a nonzero finite R-module G with

$$\operatorname{Ext}_R^*(G,k) \cong \mathcal{R} \otimes_k \operatorname{Ext}_Q^*(G,k)\,,$$

which also satisfies the conditions $\operatorname{proj\,dim}_Q G < \infty$ and $\operatorname{Ext}_R^{\geqslant 1}(G,R) = 0$.

Proof of Theorem 7.4. (1) One has $V_{\mathcal{R}}(\mathcal{R}) = \bar{k}^c$ by the Nullstellensatz. In view of 2.3 it thus suffices to find a module G with $\operatorname{Supp}_{\mathcal{R}}^*(G,k) = \operatorname{Spec}\mathcal{R}$, and to show that every closed subset X of $\operatorname{Supp}_{\mathcal{R}}^*(G,k)$ is realizable by a module M_X with $\operatorname{proj\,dim}_Q M_X < \infty$.

The R-module G from 7.5 has the necessary property, as $\operatorname{Ext}_R^*(G,k)$ is a nonzero graded free $\mathcal{R}$-module. Theorem 4.1 yields a module M_X with the desired cohomological support and is in $\mathsf{Thick}_R(G\oplus R)$. Since $G\oplus R$ is perfect over Q, the last condition implies that so does M_X.

(2) Evidently $V_{\varkappa}^*(M) = \{0\}$ if and only if $\operatorname{Supp}_{\mathcal{R}} \operatorname{Ext}_R^*(M,k) = \varnothing$. As the $\mathcal{R}$-module $\operatorname{Ext}_R^*(M,k)$ is finite, this is equivalent to $\operatorname{Ext}_R^{\gg 0}(M,k) = 0$. Lemma 7.2, applied with $Q = R$, yields the desired equivalence.

(3) This follows from Lemma 4.2.

(4) As noted in Example 3.9, the complex $M/\!\!/\boldsymbol{x}$ is quasi-isomorphic to the Koszul complex on $\boldsymbol{x}$, and hence to the R-module $M/\boldsymbol{x}M$. Thus, Proposition 3.10 implies $V_{\varkappa}^*(M/\boldsymbol{x}M) = V_{\varkappa}^*(M)$; see 2.3. □

In Theorem 7.4 the hypothesis that R has an embedded deformation can be weakened in a useful way. The main property is (1), so we focus on it.

7.6. *Completions.* The $\mathfrak{m}$-adic completion of $(R,\mathfrak{m},k)$ is a local ring, $(\widehat{R},\widehat{\mathfrak{m}},k)$. The maps $R \to \widehat{R}$ and $M \to \widehat{R}\otimes_R M = \widehat{M}$ induce isomorphisms

$$\text{(d)}\qquad \operatorname{Ext}_{\widehat{R}}^*(k,k) \longrightarrow \operatorname{Ext}_R^*(k,k) \quad \text{and} \quad \operatorname{Ext}_{\widehat{R}}^*(\widehat{M},k) \longrightarrow \operatorname{Ext}_R^*(M,k)\,,$$

the first one of graded k-algebras, the second of graded modules, equivariant over the first. Thus, when $\widehat{R}$ has an embedded deformation $\varkappa$ of codimension c the ring $\mathcal{R}$ from 7.1 acts on $\operatorname{Ext}_R^*(M,k)$ for each $M \in \mathsf{D}(R)$. As in 7.3, when $\operatorname{Ext}_R^*(M,k)$ is noetherian over $\mathcal{R}$ we define a *cohomological variety* by:

$$V_{\varkappa}^*(M) = V_{\mathcal{R}}(\operatorname{Ext}_R^*(M,k)) \subseteq \bar{k}^c\,.$$

Observe that if $\varkappa$ is an embedded deformation of R, then $\widehat{\varkappa}$, completion with respect to the maximal ideal of Q, is an embedded deformation of $\widehat{R}$, so (d) above yields $V_{\varkappa}^*(M) = V_{\widehat{\varkappa}}^*(\widehat{M})$.

The following descent result is of independent interest.

Theorem 7.7. *Let $(R,\mathfrak{m},k)$ be a local ring, $Q \to R$ an embedded deformation, and L a complex of $\widehat{R}$-modules.*

If L is perfect over Q, then there exists a finite R-module M with

$$V^*_{\varkappa}(M) = V^*_{\varkappa}(L) \quad \text{and} \quad \operatorname{proj\,dim}_Q \widehat{M} < \infty\,.$$

Proof. By 2.3, it suffices to prove $\operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_R(M,k) = \operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_R(L,k)$.

Choose a set of generators of the ideal $\mathfrak{m}$ and let $\boldsymbol{x}$ be its image under the composition $R \to \widehat{R} \to \mathcal{A}$. The equality $\boldsymbol{x} \operatorname{Ext}^*_{\widehat{R}}(L,k) = 0$ implies an inclusion

$$\operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_{\widehat{R}}(L,k) \subseteq \operatorname{Supp}^+_{\mathcal{A}} (\mathcal{A}/(\boldsymbol{x})\mathcal{A})\,.$$

This yields the second equality below; the first one holds by Lemma 3.7:

$$\begin{aligned}\operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_{\widehat{R}}(L/\!\!/\boldsymbol{x},k) &= \operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_{\widehat{R}}(L,k) \cap \operatorname{Supp}^+_{\mathcal{A}} (\mathcal{A}/(\boldsymbol{x})\mathcal{A})\\ &= \operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_{\widehat{R}}(L,k)\,.\end{aligned}$$

The complex $L/\!\!/\boldsymbol{x}$ is quasi-isomorphic to the Koszul complex on $\boldsymbol{x}$ with coefficients in L; see 3.9. Thus, $\mathrm{H}(L/\!\!/\boldsymbol{x})$ has finite length over $\widehat{R}$, hence also over R. Let $F \to L/\!\!/\boldsymbol{x}$ be a semi-projective resolution over R with F a finite free complex. One then has quasi-isomorphisms

$$\widehat{R} \otimes_R F \simeq \widehat{R} \otimes_R (L/\!\!/\boldsymbol{x}) \simeq L/\!\!/\boldsymbol{x}$$

due to the flatness of $\widehat{R}$ over R and, for the second one, also to the finiteness of the length of $\mathrm{H}(L/\!\!/\boldsymbol{x})$ over R. Fix n so that $\mathrm{H}_{\geqslant n}(L/\!\!/\boldsymbol{x}) = 0$ holds and set $M = \mathrm{H}_n(F_{\geqslant n})$. One then has quasi-isomorphisms of complexes of $\widehat{R}$-modules

$$\widehat{M} \cong \widehat{R} \otimes_R M \simeq \widehat{R} \otimes_R (F_{\geqslant n})\,.$$

They imply $\widehat{M} \cong \Omega^{\widehat{R}}_i(L/\!\!/\boldsymbol{x})$, hence the first equality below:

$$\begin{aligned}\operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_{\widehat{R}}(\widehat{M},k) &= \operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_{\widehat{R}}(\Omega^{R}_i(L/\!\!/\boldsymbol{x}),k)\\ &= \operatorname{Supp}^+_{\mathcal{A}} \operatorname{Ext}^*_{\widehat{R}}(L/\!\!/\boldsymbol{x},k)\,.\end{aligned}$$

Lemma 1.4 gives the second one. It remains to note that since L and $\widehat{R}$ are both perfect over Q, so is any complex in $\mathsf{Thick}_{\widehat{R}}(L \oplus \widehat{R})$. Thus, $L/\!\!/\boldsymbol{x}$ is perfect over Q, by (3.8.4), and hence so is $\widehat{M} = \Omega^{\widehat{R}}_i(L/\!\!/\boldsymbol{x})$, by Lemma 1.4. □

Existence Theorem 7.8. *Let $(R,\mathfrak{m},k)$ be a local ring.*

If $\varkappa\colon Q \to \widehat{R}$ is an embedded deformation of codimension c, then for each closed k-rational cone $X \subseteq \bar{k}^c$ there exists a finite R-module M with

$$X = V^*_{\varkappa}(M) \quad \text{and} \quad \operatorname{proj\,dim}_Q \widehat{M} < \infty\,.$$

Proof. Theorem 7.4(1) provides a finite $\widehat{R}$-module L with $X = V^*_{\varkappa}(L)$, and of finite projective dimension over Q. Now apply Theorem 7.7. □

7.9. *Application.* A local ring R is *complete intersection* if $\widehat{R}$ admits an embedded deformation $\varkappa\colon Q \to R$ where Q is a regular local ring; see [18, §29]. For such an R Theorem 7.8 specializes to a result proved by Bergh [10, (2.3)], who uses Tate cohomology, and by Avramov and Jorgensen [6], who establish an existence theorem for cohomology modules by using equivalences of triangulated categories and Koszul duality.

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