ON OPTIMAL STOPPING RULES FOR s_n/n

BY

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1. Introduction

Let

$$x_1, x_2, \cdots \tag{1}$$

be a sequence of independent, identically distributed random variables on a probability space $(\Omega, \mathfrak{F}, P)$ with

$$P(x_1 = 1) = P(x_1 = -1) = \tfrac{1}{2}, \tag{2}$$

and let $s_n = x_1 + \cdots + x_n$. Let $i = 0, \pm 1, \cdots$ and $j = 0, 1, \cdots$ be two fixed integers. Assume that we observe the sequence (1) term by term and can decide to stop at any point; if we stop with x_n we receive the reward $(i + s_n)/(j + n)$. What stopping rule will maximize our expected reward?

Formally, a stopping rule t of (1) is any positive integer-valued random variable such that the event $\{t = n\}$ is in $\mathfrak{F}_n$ $(n \geq 1)$ where $\mathfrak{F}_n$ is the Borel field generated by $x_1, \cdots, x_n$. Let T denote the class of all stopping rules; for any t in T, s_t is a well-defined random variable, and we set

$$v_j(i \mid t) = E\left(\frac{i + s_t}{j + t}\right), \qquad v_j(i) = \sup_{t \in T} v_j(i \mid t). \tag{3}$$

It is by no means obvious that for given i and j there exists a stopping rule $\mathfrak{J}_j(i)$ in T such that

$$v_j(i \mid \mathfrak{J}_j(i)) = v_j(i) = \max_{t \in T} v_j(i \mid t); \tag{4}$$

such a stopping rule of (1) will be called *optimal* for the reward sequence

$$\frac{i + s_1}{j + 1}, \quad \frac{i + s_2}{j + 2}, \cdots. \tag{5}$$

Theorem 1 below asserts that for every $i = 0, \pm 1, \cdots$ and $j = 0, 1, \cdots$ there exists an optimal stopping rule $\mathfrak{J}_j(i)$ for the reward sequence (5).

We remark that for any t in T and any $i = 0, \pm 1, \cdots$ and $j = 0, 1, \cdots$ the random variable

$$\begin{aligned} t' &= t && \text{if} \quad i + s_t \geq 1, \\ &= \text{first } n > t \text{ such that } i + s_n = 1 && \text{if} \quad i + s_t \leq 0 \end{aligned} \tag{6}$$

is in T and

$$i + s_{t'} \geq 1, \qquad 0 < E\left(\frac{i + s_{t'}}{j + t'}\right) \geq E\left(\frac{i + s_t}{j + t}\right). \tag{7}$$

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It follows that

$$v_j(i) = \sup_{t \in T} E\left[\frac{(i + s_t)^+}{j + t}\right], \tag{8}$$

where by definition $a^+ = \max(0, a)$.

2. Reduction of the problem to the study of bounded stopping rules

For any fixed $N = 1, 2, \cdots$ let T_N denote the class of all t in T such that $t \le N$. By the usual backward induction (see e.g. [1]) it may be shown that in T_N there exists a *minimal optimal stopping rule* of (1) for the reward sequence

$$\frac{(i + s_1)^+}{j + 1}, \quad \frac{(i + s_2)^+}{j + 2}, \cdots; \tag{1}$$

that is, an element $\mathfrak{Z}_j^N(i)$ of T_N such that, setting

$$w_j(i \mid t) = E\left[\frac{(i + s_t)^+}{j + t}\right], \tag{2}$$

we have

$$w_j(i \mid \mathfrak{Z}_j^N(i)) = \max_{t \in T_N} w_j(i \mid t), \tag{3}$$

and such that if $\bar{t}$ is any element of T_N for which

$$w_j(i \mid \bar{t}) = \max_{t \in T_N} w_j(i \mid t), \tag{4}$$

then $\mathfrak{Z}_j^N(i) \le \bar{t}$. The sequence $\mathfrak{Z}_j^1(i)$, $\mathfrak{Z}_j^2(i) \cdots$ is such that as $N \to \infty$,

$$\begin{gathered} 1 \le \mathfrak{Z}_j^1(i) \le \mathfrak{Z}_j^2(i) \le \cdots \to \mathfrak{Z}_j^*(i) \le \infty, \\ 0 \le w_j(i \mid \mathfrak{Z}_j^1(i)) \le w_j(i \mid \mathfrak{Z}_j^2(i)) \le \cdots \to \sup_{t \in T} w_j(i \mid t) = v_j(i), \end{gathered} \tag{5}$$

the last equality following from (1.8). It is shown in [1] that there exists an optimal element in T for the reward sequence (1.5) if and only if

$$\mathfrak{Z}_j^*(i) = \lim_{N \to \infty} \mathfrak{Z}_j^N(i) \tag{6}$$

is in T—that is, if and only if

$$P(\mathfrak{Z}_j^*(i) < \infty) = 1 \tag{7}$$

—and when (7) holds $\mathfrak{Z}_j^*(i)$ is the minimal element of T which satisfies (1.4). The remainder of the present paper is devoted to proving that (7) holds.

3. The constants $a_n^N(i)$ and $a_n(i)$

In order to study the nature of the optimal bounded stopping rules $\mathfrak{Z}_j^N(i)$ of Section 2 we proceed as follows. Define for $N = 1, 2, \cdots$ and $i = 0,$

$\pm 1, \cdots$ the constants

$$b_N^N(i) = \frac{i^+}{N},$$

$$b_n^N(i) = \max\left(\frac{i^+}{n}, \frac{b_{n+1}^N(i+1) + b_{n+1}^N(i-1)}{2}\right) \tag{1}$$

$$(n = 1, 2, \cdots, N-1).$$

Then

$$b_n^N(i) = \max\left(\frac{i^+}{n}, \sup_{t \in T_{N-n}} E\left[\frac{(i+s_t)^+}{n+t}\right]\right) \qquad (n = 1, 2, \cdots, N-1), \tag{2}$$

$$\mathfrak{J}_j^N(i) = \text{first } n \geq 1 \text{ such that } \ b_{j+n}^{j+N}(i+s_n) = \frac{(i+s_n)^+}{j+n}, \tag{3}$$

and

$$\sup_{t \in T_N} E\left[\frac{(i+s_t)^+}{j+t}\right] = \tfrac{1}{2}[b_{j+1}^{j+N}(i+1) + b_{j+1}^{j+N}(i-1)]. \tag{4}$$

In view of (2) and (3) it is convenient to introduce the constants $a_n^N(i)$ defined for $N = 1, 2, \cdots; i = 0, \pm 1, \cdots; n = 1, 2, \cdots, N$ by

$$a_n^N(i) = b_n^N(i) - \frac{i^+}{n}; \tag{5}$$

then (3) becomes

$$\mathfrak{J}_j^N(i) = \text{first } n \geq 1 \text{ such that } a_{j+n}^{j+N}(i+s_n) = 0. \tag{6}$$

From (5) and (1) it follows that the constants $a_n^N(i)$ satisfy the recursion relations

$$a_N^N(i) = 0 \qquad (\text{all } i),$$

$$a_n^N(i) = \left[\frac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} + \frac{(i+1)^+ + (i-1)^+}{2(n+1)} - \frac{i^+}{n}\right]^+ \qquad (n = 1, 2, \cdots, N-1) \tag{7}$$

from which they may be successively computed for $n = N, N-1, \cdots, 1$. Moreover, from (2) and (4) we have

$$a_n^N(i) = \sup_{t \in T_{N-n}} \left\{E\left[\frac{(i+s_t)^+}{n+t} - \frac{i^+}{n}\right]\right\}^+ \qquad (n = 1, 2, \cdots, N-1) \tag{8}$$

and

$$\begin{aligned}(9)\qquad &\sup_{t\in T_N} E\left[\frac{(i+s_t)^+}{j+t}\right]\\ &\quad = \tfrac{1}{2}\left[a_{j+1}^{j+N}(i+1) + a_{j+1}^{j+N}(i-1) + \frac{(i+1)^+ + (i-1)^+}{j+1}\right].\end{aligned}$$

For any $i = 0, \pm 1, \cdots$ and $n = 1, 2, \cdots$ we have

$$0 = a_n^n(i) \le a_n^{n+1}(i) \le \cdots,$$

and letting $N \to \infty$ we obtain constants $a_n(i) = \lim_{N\to\infty} a_n^N(i)$ such that

$$(10)\qquad a_n^N(i) \uparrow a_n(i) = \sup_{t\in T} E^+\left[\frac{(i+s_t)^+}{n+t} - \frac{i^+}{n}\right],$$

while for $j = 0, 1, \cdots$

$$\begin{aligned}&\sup_{t\in T} E\left[\frac{(i+s_t)^+}{j+t}\right]\\ (11)\qquad &\quad = \sup_{t\in T} E\left(\frac{i+s_t}{j+t}\right) = v_j(i)\\ &\quad = \tfrac{1}{2}\left[\frac{(i+1)^+ + (i-1)^+}{j+1} + a_{j+1}(i+1) + a_{j+1}(i-1)\right];\end{aligned}$$

moreover $\mathfrak{J}_j^N(i) \uparrow \mathfrak{J}_j^*(i)$ where

$$(12)\qquad \begin{aligned}\mathfrak{J}_j^*(i) &= \text{first } n \ge 1 \text{ such that } a_{j+n}(i+s_n) = 0,\\ &= \infty \quad \text{if no such } n \text{ exists.}\end{aligned}$$

Thus (2.7) holds if and only if

$$(13)\qquad P(a_{j+n}(i+s_n) = 0 \text{ for some } n \ge 1) = 1.$$

In the next section we shall prove (Lemma 4) that there exists a positive integer n_0 such that $n \ge n_0$ and $i > 13\sqrt{n}$ together imply that $a_n(i) = 0$. Hence

$$(14)\qquad \begin{aligned}&P(a_{j+n}(i+s_n) = 0 \text{ for some } n \ge 1)\\ &\qquad \ge P(s_n > 13\sqrt{j+n} - i \text{ for some } n \ge n_0).\end{aligned}$$

The law of the iterated logarithm implies that the latter probability is 1 and this establishes (13); hence $\mathfrak{J}_j^*(i)$ defined by (12) is in T and is optimal for the reward sequence (1.5). We thus have the following:

THEOREM 1. *For the sequence* (1.1) *with the distribution* (1.2) *and the reward sequence* (1.5) *there exists an optimal stopping rule* $\mathfrak{J}_j^*(i)$ *defined by* (12);

the expected reward in using $\mathfrak{T}_j^*(i)$ *is*

$$
\begin{aligned}
v_j(i) &= \max_{t \in T} E\left(\frac{i + s_t}{j + t}\right) \\
&= \tfrac{1}{2}\left[\frac{(i+1)^+ + (i-1)^+}{j+1} + a_{j+1}(i+1) + a_{j+1}(i-1)\right] \qquad (15)\\
&\qquad\qquad (i = 0, \pm 1, \cdots; j = 0, 1, \cdots).
\end{aligned}
$$

The constants $a_n(i) = \lim_{N\to\infty} a_n^N(i)$ *which occur in* (12) *and* (15) *are determined by* (7).

4. Lemmas

LEMMA 1. $$a_n(0) \le 1/\sqrt{n} \qquad (n = 1, 2, \cdots).$$

Proof. From (3.7) we have

$$
\begin{aligned}
a_n^N(i) &= \frac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} && (i \le -1),\\
&= \frac{a_{n+1}^N(1) + a_{n+1}^N(-1)}{2} + \frac{1}{2(n+1)} && (i = 0), \qquad (1)\\
&= \left[\frac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} - \frac{i}{n(n+1)}\right]^+ \\
&\le \frac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} && (i \ge 1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_n^N(0) &= \frac{a_{n+1}^N(1) + a_{n+1}^N(-1)}{2} + \frac{1}{2(n+1)} \\
&\le \frac{1}{2^2}\left[a_{n+2}^N(2) + 2a_{n+2}^N(0) + a_{n+2}^N(-2)\right] + \frac{1}{2(n+1)} \\
&\le \frac{1}{2^3}\left[a_{n+3}^N(3) + 3a_{n+3}^N(1) + 3a_{n+3}^N(-1) + a_{n+3}^N(-3)\right] \qquad (2)\\
&\qquad + \frac{1}{2(n+1)} + \frac{\binom{2}{1}}{2^3(n+3)} \\
&\le \cdots \le \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k+1}(n+2k+1)},
\end{aligned}
$$

since $a_N^N(i) \equiv 0$. By Stirling's formula

$$\binom{2k}{k} < \frac{2^{2k}}{\sqrt{k\pi}}, \qquad (3)$$

and

$$\sum_{k=n}^{\infty} \frac{1}{2\sqrt{k\pi}\,(n+2k+1)} \le \frac{1}{2\sqrt{\pi}} \int_{r-1/2}^{\infty} \frac{dx}{\sqrt{x}(n+2x+1)} = \frac{1}{\sqrt{2\pi(n+1)}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{2r-1}{n+1}}\right). \tag{4}$$

Hence

$$a_n(0) = \lim_{N\to\infty} a_n^N(0) \le \sum_{k=0}^{r-1} \frac{\binom{2k}{k}}{2^{2k+1}(n+2k+1)} + \frac{1}{\sqrt{2\pi(n+1)}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{2r-1}{n+1}}\right). \tag{5}$$

For $r = 1$ this gives

$$a_n(0) \le \frac{1}{2(n+1)} + \frac{1}{\sqrt{2n}} \le \frac{1}{\sqrt{n}}. \tag{6}$$

LEMMA 2. *For* $n = 1, 2, \cdots$

$$0 < \cdots \le a_n(-2) \le a_n(-1) \le a_n(0) \ge a_n(1) \ge a_n(2) \ge \cdots \ge 0, \tag{7}$$

$$a_{n+1}(i) \ge \frac{n+1}{n+2}\, a_n(i) \qquad (\textit{all } i). \tag{8}$$

Proof. For $i \le 0$ we have from (3.10) and (1.7)

$$a_n(i) = \sup_{t\in T} E\left[\frac{(i+s_t)^+}{n+t}\right] > 0; \tag{9}$$

hence

$$a_n(i) \ge \sup_{t\in T} E\left[\frac{(i-1+s_t)^+}{n+t}\right] = a_n(i-1). \tag{10}$$

For $i \ge 0$ we have

$$\begin{aligned} a_n(i) &= \sup_{t\in T} E^+\left[\frac{i+s_t}{n+t} - \frac{i}{n}\right] = \sup_{t\in T} E^+\left[\frac{ns_t - it}{n(n+t)}\right] \\ &\ge \sup_{t\in T} E^+\left[\frac{ns_t - (i+1)t}{n(n+t)}\right] = a_n(i+1) \ge 0. \end{aligned} \tag{11}$$

(7) follows from (10) and (11). To prove (8) we shall show that for $n = 1, 2, \cdots, N$,

$$\frac{n+2}{n+1}\, a_{n+1}^{N+1}(i) \ge a_n^N(i) \qquad (\text{all } i); \tag{12}$$

(8) will follow from (12) on letting $N \to \infty$. (12) is true trivially for $n = N$ since $a_N^N(i) = 0$. Assume now that (12) holds; for $i \neq 0$ we have by (1),

$$\begin{aligned} \frac{n+1}{n} a_n^{N+1}(i) &= \frac{n+1}{n}\left[\frac{a_{n+1}^{N+1}(i+1) + a_{n+1}^{N+1}(i-1)}{2} - \frac{i^+}{n(n+1)}\right]^+ \\ &\ge \frac{n+1}{n}\left[\frac{n+1}{n+2}\,\frac{a_n^N(i+1) + a_n^N(i-1)}{2} - \frac{i^+}{n(n+1)}\right]^+ \qquad (13) \\ &\ge \left[\frac{a_n^N(i+1) + a_n^N(i-1)}{2} - \frac{i^+}{(n-1)n}\right]^+ = a_{n-1}^N(i). \end{aligned}$$

The case $i = 0$ is treated similarly. Thus (12) holds with n replaced by $n - 1$, and hence (12) holds for all $n = N, N - 1, \cdots, 2, 1$.

LEMMA 3. *Let i and j be non-negative integers such that $a_n(i + j) > 0$. Let $\mathfrak{Z}_0$ denote the first integer $m \ge 1$ such that $s_m = j + 1$. Then for any given t in T there exists a $\mathfrak{Z}$ in T such that*

$$(14) \qquad \mathfrak{Z} \ge t, \qquad \mathfrak{Z} \ge \mathfrak{Z}_0, \qquad E\left(\frac{i + s_{\mathfrak{Z}}}{n+\mathfrak{Z}}\right) \ge E\left(\frac{i + s_t}{n + t}\right).$$

Proof. We have from (3.10) and (3.11) for $i \ge 0$,

$$(15) \qquad a_n(i) = \left[\sup_{t \in T} E\left(\frac{i + s_t}{n + t}\right) - \frac{i}{n}\right]^+.$$

By (7) and (8) the inequality $a_n(i + j) > 0$ implies that for every positive integer m and every integer $k \le j$,

$$(16) \qquad a_{n+m}(i + k) > 0,$$

and hence that there exists a stopping rule $t_{m,k}$ of the sequence $x_{m+1}, x_{m+2}, \cdots$ such that

$$(17) \qquad E\left(\frac{i + k + x_{m+1} + x_{m+2} + \cdots + x_{m+t_{m,k}}}{n + m + t_{m,k}}\right) > \frac{i + k}{n + m}.$$

Let A be the event $\{t < \mathfrak{Z}_0\}$, and define

$$\begin{aligned} t_1(\omega) &= t(\omega) && \text{if } \omega \notin A, \\ (18) \qquad &= t(\omega) + t_{m,k}(\omega) && \text{if } \omega \in A, t(\omega) = m, s_{t(\omega)} = k \\ &&& (m = 1, 2, \cdots; k \le j). \end{aligned}$$

Then t_1 is a stopping rule, $t_1 \ge t$, and $t_1(\omega) \ge t(\omega) + 1$ if $\omega \in A$. Moreover

$$\begin{aligned} E\left(\frac{i + s_{t_1}}{n + t_1}\right) &= \int_{\Omega - A} \frac{i + s_t}{n + t}\, dP + \sum_{m,k} \int_{\{t=m, s_t=k, t<\mathfrak{Z}_0\}} \frac{i + s_{t+t_{m,k}}}{n + t + t_{m,k}}\, dP \\ (19) \qquad &\ge \int_{\Omega - A} \frac{i + s_t}{n + t}\, dP + \sum_{m,k} \int_{\{t=m, s_t=k, t<\mathfrak{Z}_0\}} \frac{i + k}{n + m}\, dP \\ &= E\left(\frac{i + s_t}{n + t}\right). \end{aligned}$$

Set $t_0 = t$ and $A_0 = A$. By a repetition of the preceding argument we may define a sequence of stopping rules t_l,

$$t = t_0 \leq t_1 \leq t_2 \leq \cdots \tag{20}$$

and events $A_l = \{t_l < \mathfrak{z}_0\}$ with

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots \tag{21}$$

such that

$$\begin{aligned} t_{l+1}(\omega) &= t_l(\omega) && \text{if} \quad \omega \notin A_l, \\ &\geq t_l(\omega) + 1 && \text{if} \quad \omega \in A_l. \end{aligned} \tag{22}$$

Set

$$\mathfrak{z} = \lim_{l\to\infty} t_l\,; \tag{23}$$

then $\{\mathfrak{z} = \infty\} = \{\mathfrak{z}_0 = \infty\}$, so that $\mathfrak{z}$ is in T, and $\mathfrak{z} \geq \mathfrak{z}_0$, $\mathfrak{z} \geq t$. By the Lebesgue dominated convergence theorem,

$$E\left(\frac{i + s_{\mathfrak{z}}}{n + \mathfrak{z}}\right) = \lim_{l\to\infty} E\left(\frac{i + s_{t_l}}{n + t_l}\right) \geq E\left(\frac{i + s_t}{n + t}\right), \tag{24}$$

and the proof is complete.

LEMMA 4. *There exists a positive integer n_0 such that $n \geq n_0$ and $i > 13\sqrt{n}$ imply that $a_n(i) = 0$.*

Proof. Let i be a positive integer such that $a_n(2i) > 0$, and let $\mathfrak{z}$ denote the first integer $m \geq 1$ such that $s_m = i$. Then [2, p. 87] as $i \to \infty$,

$$P(\mathfrak{z} \geq i^2) \to \frac{\sqrt{2}}{\pi}\int_0^1 e^{-u^2/2}\,du > \sqrt{\frac{2}{\pi e}} > \tfrac{1}{3}. \tag{25}$$

Hence there exists $i_0 > 0$ such that

$$E\left(\frac{\mathfrak{z}}{i^2 + \mathfrak{z}}\right) > \tfrac{1}{6} \qquad (i \geq i_0), \tag{26}$$

and therefore

$$E\left(\frac{\mathfrak{z}}{n + \mathfrak{z}}\right) > \tfrac{1}{6} \qquad (i \geq i_0,\ 1 \leq n \leq i^2). \tag{27}$$

By (7), $a_n(i) > 0$, and hence by Lemma 3 (putting $j = i$) there exists a $t \in T$ such that $t \geq \mathfrak{z}$ and

$$E\left(\frac{i + s_t}{n + t}\right) > \frac{i}{n}. \tag{28}$$

Hence by Lemma 1 and (11),

$$\frac{1}{\sqrt{n}} \geq a_n(0) \geq E\left(\frac{s_t}{n+t}\right)$$

$$(29) \qquad \begin{aligned} &= E\left(\frac{i+s_t}{n+t}\right) - E\left(\frac{i}{n+t}\right) \\ &> \frac{i}{n} - E\left(\frac{i}{n+t}\right) = \frac{i}{n} E\left(\frac{t}{n+t}\right) \\ &\geq \frac{i}{n} E\left(\frac{\mathfrak{J}}{n+\mathfrak{J}}\right) > \frac{i}{6n} \qquad (i \geq i_0, 1 \leq n \leq i^2). \end{aligned}$$

Assume now that $a_n(j) > 0$ for some $j > 13\sqrt{n}$ and $n \geq n_0 = i_0^2$. Then by (7), letting square brackets denote integral part,

$$(30) \qquad a_n\left(2\left[\frac{j}{2}\right]\right) > 0, \qquad \left[\frac{j}{2}\right]^2 \geq n \geq 1, \qquad \left[\frac{j}{2}\right] \geq i_0.$$

Hence, setting

$$i = \left[\frac{j}{2}\right]$$

in (29),

$$(31) \qquad \left[\frac{j}{2}\right] < 6\sqrt{n},$$

and therefore

$$(32) \qquad j < 12\sqrt{n} + 1 \leq 13\sqrt{n},$$

a contradiction. The proof of Lemma 4, and hence of Theorem 1, is complete.

5. Remarks

1. If we define for $n = 1, 2, \cdots$

$$(1) \qquad k_n = \text{smallest integer } k \text{ such that } a_n(k) = 0,$$

then from Lemma 2 it follows that

$$(2) \qquad 0 < k_1 \leq k_2 \leq \cdots$$

and that

$$(3) \qquad a_n(i) = 0 \quad \text{if and only if} \quad i \geq k_n.$$

It is easily seen that

$$(4) \qquad \begin{aligned} \mathfrak{J}_j^*(i) &= \text{first } n \geq 1 \text{ such that } a_{j+n}(i+s_n) = 0 \\ &= \text{first } n \geq 1 \text{ such that } i + s_n = k_{j+n}. \end{aligned}$$

Hence the stopping rules $\mathfrak{I}_j^*(i)$ are completely defined by the sequence of positive integers k_n. It is difficult to obtain an explicit formula for k_n; by Lemma 4 we know that $k_n = 0(\sqrt{n})$ as $n \to \infty$. We note also that

$$\lim_{n\to\infty} k_n = \infty. \tag{5}$$

Otherwise we would have $k_n < M$ for some finite positive integer M and every $n = 1, 2, \cdots$. If so, let $t =$ first $m \geq 1$ such that $s_m = M$. Then since $a_n(M) = 0$,

$$E\left(\frac{M + s_t}{n + t}\right) \leq \frac{M}{n}, \tag{6}$$

and hence

$$E\left(\frac{2M}{n + t}\right) \leq \frac{M}{n}, \qquad E\left(\frac{n}{n + t}\right) \leq \tfrac{1}{2}. \tag{7}$$

But as $n \to \infty$,

$$E\left(\frac{n}{n + t}\right) \to 1, \tag{8}$$

which contradicts (7).

2. We have from (3.15),

$$v_0(0) = \max_{t \in T} E\left(\frac{s_t}{t}\right) = \tfrac{1}{2}[1 + a_1(1) + a_1(-1)]. \tag{9}$$

Now by (4.15), since $s_t \leq t$,

$$a_1(1) = \left[\sup_{t \in T} E\left(\frac{1 + s_t}{1 + t}\right) - 1\right]^+ = 0, \tag{10}$$

and by (4.6) and (4.7),

$$a_1(-1) \leq a_1(0) \leq \tfrac{1}{4} + 1/\sqrt{2} < .96. \tag{11}$$

Hence

$$v_0(0) < .98. \tag{12}$$

This inequality is very crude and can be greatly improved by a more detailed analysis of the term $a_1(-1)$, but it is interesting to note that even (12) is not easy to prove directly from the definition of $v_0(0)$.

3. In this connection let us define

$$v_N = \max_{t \in T_N} E\left[\frac{s_t^+}{t}\right]; \tag{13}$$

then as $N \to \infty$

$$v_N \uparrow v_0(0) = \max_{t \in T} E\left(\frac{s_t^+}{t}\right) = \max_{t \in T} E\left(\frac{s_t}{t}\right). \tag{14}$$

Now for any fixed $N = 1, 2, \cdots$ the value v_N can be computed by recursion; by (3.4) and (3.2),

$$v_N = \tfrac{1}{2}[b_1^N(1) + b_1^N(-1)] = \tfrac{1}{2}[1 + b_1^N(-1)], \tag{15}$$

where by (3.1)

$$\begin{aligned} b_N^N(i) &= \frac{i^+}{N}, \\ b_n^N(i) &= \max\left(\frac{i^+}{n}, \frac{b_{n+1}^N(i+1) + b_{n+1}^N(i-1)}{2}\right) \quad (n = 1,2,\cdots,N-1). \end{aligned} \tag{16}$$

The computation of the $b_n^N(i)$ is easily programmed for a high speed computer; the following results were kindly supplied to us by R. Bellman and S. Dreyfus:

$$\begin{aligned} v_{100} &= .5815 \\ v_{200} &= .5835 \\ v_{500} &= .5845 \\ v_{1000} &= .5850. \end{aligned} \tag{17}$$

4. *Remarks.* (i) It would be interesting to see whether the existence of an optimal stopping rule for s_n/n can be proved for sequences $x_1, x_2, \cdots$ with a more general distribution than (1.2). We have some preliminary extensions of Theorem 1 to more general cases but no definitive results as yet.

(ii) While the optimal stopping rules for s_n/n and s_n^+/n are the same, the optimal truncated rules, $1 \le n \le N$, are quite different.

(iii) The reward sequence

$$cs_1, c^2s_2, \cdots, c^ns_n, \cdots \tag{1}$$

where $0 < c < 1$ also admits an optimal stopping rule; the proof of this is quite simple compared to that for s_n/n.

Added in Proof. A. Dvoretzky has recently communicated to us the proof of the existence of an optimal stopping rule for s_n/n for any sequence x_1, $x_2, \cdots$ of independent, identically distributed random variables with a finite second moment.

References

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