THE σ-SYMBOL OF THE SINGULAR INTEGRAL OPERATORS OF CALDERÓN AND ZYGMUND

BY

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In this paper we extend the σ-symbol of [2]. Our extension is a homomorphism of a C^*-subalgebra, $\mathfrak{A}$, of bounded operators on $L^2(R^n)$ onto the bounded continuous functions on $R^n \times S^{n-1}$. The kernel of this homomorphism is the set of all T such that ψT and $T\psi$ are compact operators for each $\psi \in C_0^\infty(R^n)$. We also show that $\mathfrak{A}$ and σ are uniquely determined by these properties.

Cordes [3] and Seeley [7] have considered σ on a smaller algebra than $\mathfrak{A}$ and obtained a homomorphism onto the continuous functions on $S^n \times S^{n-1}$, whose kernel is the compact operators. Our results, which extend theirs, were obtained after reading their papers.

Our results are stated precisely in §1. The information used about singular integral operators is discussed in §2. All of it is contained in [2]. The seminar notes [1] also contain this information.

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1. The statement of the main results

R^n will always denote Euclidean n-space $(n > 1)$ and S^{n-1} will be the unit sphere in R^n. We use $\langle\ ,\ \rangle$ for the usual scalar product in R^n and $\|\quad\|$ for the corresponding norm. The word function will always mean a complex-valued function. We denote the coordinate functions on R^n by $u_1, \cdots, u_n$ and if $\alpha = (\alpha_1, \cdots, \alpha_n)$ where the α_j are nonnegative integers, we write

$$u_\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n} \quad \text{and} \quad D_\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial u_1^{\alpha_1} \cdots \partial u_n^{\alpha_n}}.$$

We use the standard notation $C^\infty(R^n)$ for the set of functions defined on R^n, whose partial derivatives of all orders exist and are continuous, and use $C_0^\infty = C_0^\infty(R^n)$ for functions in $C^\infty(R^n)$ that have compact support.

We also use the notation $BC = BC(R^n \times S^{n-1})$ for the set of all bounded continuous functions on $R^n \times (R^n - [0])$ such that $k(x, \lambda y) = k(x, y)$ for all $\lambda > 0$. Thus BC is essentially the same as the bounded continuous functions on $R^n \times S^{n-1}$. If U is any open set in R^n, we also write

$$B^\infty(U) = [f \in C^\infty(U) : D_\alpha f \text{ is bounded for every } \alpha].$$

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We write $B^\infty C = B^\infty C(R^n \times S^{n-1})$ for the set

$$(BC) \cap B^\infty(R^n \times [y \in R^n : \| y \| > 1]).$$

We consider BC as a B^*-algebra with the sup-norm, and conjugation as the $*$-operation, denoting the sup-norm by $| \quad |_0$; so $B^\infty C$ is a subalgebra of BC.

We introduce the following further notation:

$B(L^2)$ is the set of bounded operators on $L^2(R^n)$.

We wish to remark that we write ψ for the functions $\psi \in L^\infty(R^n)$ as well as for the corresponding multiplication operators of $B(L^2)$. Also, if $f \in L^2(R^n)$ then $\| f \|_0^2 = \int | f |^2$; if $T \in B(L^2)$ then $\| T \| = \sup_{\| f \|_0 = 1} \| Tf \|_0$. (Note that $\int g$ denotes the Lebesque integral of g over R^n).

$\mathcal{C}$ is the set of B^∞ singular integral operators; the inverse image of $B^\infty C$ under the σ-symbol in [2]. $\sigma_0 : \mathcal{C} \to B^\infty C$ is the restriction of the σ-symbol to $\mathcal{C}$.

$\mathcal{K}$ is the set of compact operators of $B(L^2)$.

$\mathcal{K}^{\text{loc}} = [T \in B(L^2) : \psi T$ and $T\psi \in \mathcal{K}$ for every $\psi \in C_0^\infty(R^n)]$.

If A is a set in a topological space, we write A^- for the closure of A.

THEOREM 1. *There is a C^* subalgebra of $B(L^2)$, which we shall denote by $\mathcal{A}$, and a (continuous) $*$ homomorphism σ of $\mathcal{A}$ onto $BC(R^n \times S^{n-1})$ having the following properties:*

(a) *$\mathcal{A}$ contains $\mathcal{C}$ and $\mathcal{K}^{\text{loc}}$.*

(b) *σ is an extension of σ_0.*

(c) *The kernel of σ is $\mathcal{K}^{\text{loc}}$.*

(d) *The pair $(\mathcal{A}, \sigma)$ is maximal with respect to* (a)–(c). *Precisely, if $\mathcal{A}'$ is is a C^* subalgebra of $B(L^2)$ and if σ' is a $*$ homomorphism of $\mathcal{A}'$ into $BC(R^n \times S^{n-1})$ satisfying* (a)–(c) *with respect to $(\mathcal{A}', \sigma')$, then $A' \subset \mathcal{A}$ and $\sigma' = \sigma | \mathcal{A}'$.*

This theorem, which is the main result, will be obtained by an extension procedure that has three parts:

(1) The symbol σ_0 extends to a continuous $*$ homomorphism σ_1 of the algebra $\mathcal{C} + \mathcal{K}^{\text{loc}}$ onto $B^\infty C(R^n \times S^{n-1})$ with kernel $\mathcal{K}^{\text{loc}}$.

(2) σ_1 extends by continuity to a $*$ homomorphism σ_2 of $(\mathcal{C} + \mathcal{K}^{\text{loc}})^-$ onto $(B^\infty C)^-$. The kernel of σ_2 is $\mathcal{K}^{\text{loc}}$ and the range of σ_2 is the bounded, uniformly continuous functions on $R^n \times S^{n-1}$.

(3) $\mathcal{A} = [T \in B(L^2) : \psi T$ and $T\psi \in (\mathcal{C} + \mathcal{K}^{\text{loc}})^-$ for every $\psi \in C_0^\infty]$. The symbol is extended to $\mathcal{A}$ by the defining formula $\sigma(T)(x, \xi) = \sigma_2(\psi T)(x, \xi)$ where $(x, \xi) \in R^n \times S^{n-1}$, $T \in \mathcal{A}$, and ψ is any C_0^∞ function satisfying $\psi(x) = 1$.

The C_β^∞ operators of [2], $\beta \geq 0$, are elements of $\mathcal{A}$ and σ agrees with the σ-symbol of [2] on these operators. In fact the entire extension could proceed starting with the C_β^∞ operators (provided $\beta > 1$) instead of $\mathcal{C}$. The same proofs apply. The details regarding this paragraph are in Section 7.

The algebra considered by Seeley in Section II of [7], and also by Cordes in [3], is a subalgebra of $\mathfrak{A}$ and σ agrees with the σ-symbol on these operators. In fact, $\mathfrak{A} = [T \in B(L^2) : \psi T \text{ and } T\psi \in \mathfrak{A}_1]$ where "$\mathfrak{A}_1$" denotes the above mentioned algebra. If one uses this as a definition of $\mathfrak{A}$ and applies the method of (3) together with the results of [7] or [3], one could extend the σ-symbol from $\mathfrak{A}_1$ to $\mathfrak{A}$ directly.

THEOREM 2. *There is a sequence of C_0^∞ functions ϕ_m, $m = 1, 2, \cdots$, and a sequence of real numbers $\delta_m \uparrow \infty$, $m = 1, 2, \cdots$, with the following properties*:

(1) $\| \phi_m \|_0 = 1$.

(2) support $\phi_m \to 0$.

(3) *Let $x \in R^n$ and $\xi \in S^{n-1}$. If $A \in \mathfrak{A}$ and $\psi_m = \phi_m(\cdot - x)e^{i\langle \cdot - x, \delta_m \xi\rangle}$, then*

$$\| A\psi_m - \sigma(A)(x, \xi)\psi_m \|_0 \to 0 \quad as \quad m \to \infty.$$

Consequently Range $\sigma(A) \subset$ spectrum A *for every* $A \in \mathfrak{A}$.

(We remark that if f is a function on R^n, support $f = [x \in R^n : f(x) \neq 0]^-$. If f_n is a sequence of functions on R^n and $x \in R^n$, then support $f_n \to x$ if and only if for every neighborhood N of x, there is an $M \in R$ such that $n > M$ implies support $f_n \subset N$.)

This theorem is a generalization of a lemma announced by Gohberg [4], and proved by Seeley in [7] (see Theorem 2.2 of [6] or Theorem 9 of [7]). Theorem 2 follows easily from the case of $T \in \mathcal{C}$, as shown in Section 7 of this paper. The case of $T \in \mathcal{C}$ is in Proposition 2 of the next section.

2. Prerequisites and further notation

We outline the definition of $\mathcal{C}$ and the σ-symbol. For this, let

$$F : L^2(R^n) \to L^2(R^n)$$

be the Fourier transform. F is an isometry of $L^2(R^n)$ onto $L^2(R^n)$.

We are interested first in operators in $B(L^2)$ of the form $F^{-1}\psi F$, where ψ is defined on $R^n - [0]$, $\psi(\lambda x) = \psi(x)$ for $\lambda > 0$ and $x \in R^n - [0]$ and

$$\psi \in B^\infty([y \in R^n : \| y \| > 1]).$$

In other words, consider those operators $T \in B(L^2)$ for which $(Tf)^\wedge = \psi f^\wedge$, where we use $f^\wedge$ for Ff.

$F^{-1}\psi F \leftrightarrow \psi$ gives a one to one correspondence of these operators with the above defined functions. We define $\sigma_0(F^{-1}\psi F)$ as the function in $B^\infty C$ given by $\sigma_0(F^{-1}\psi F)(x, y) = \psi(y)$ for $(x, y) \in R^n \times S^{n-1}$. The range of σ_0 are the functions in $B^\infty C$ which are independent of the first coordinate. Let us denote the set of such functions by $B^\infty(S^{n-1})$ and the corresponding operators by $\mathcal{K}_1$. Then

(2.1) $\sigma_0 : \mathcal{K}_1 \to B^\infty(S^{n-1})$. σ_0 is one to one and onto $B^\infty(S^{n-1})$.

Next we wish to extend σ_0^{-1} to a map of $B^\infty C$ into $B(L^2)$.

In [2], the authors use spherical harmonics as an example of a sequence $\{Y_i\}$, $i = 0, 1, 2, \cdots$, $Y_i \in B^\infty(S^{n-1})$, having the following properties:

(2.2) (a) $Y_0(x, y) = 1$ for $(x, y) \in R^n \times S^{n-1}$

(b) Each $k \in B^\infty C$ has a unique expansion of the form

$$k(x, y) = \sum_{i=0}^\infty a_i(x) Y_i(0, y), \qquad (x, y) \in R^n \times S^{n-1},$$

and convergence in the norm of BC (actually better convergence than that), where $a_i \in B^\infty(R^n)$.

(c) If $k = \sum_{i=0}^\infty a_i Y_i$ is such an expansion, then $T = \sum_{i=0}^\infty a_i \sigma_0^{-1}(Y_i)$ defines an operator in $B(L^2)$, the sum converging with respect to the operator norm, $\| \ \|$.

From (2.2), one extends σ_0^{-1} to a map of $B^\infty C$ into $B(L^2)$. Let $\mathcal{C}$ denote the range of this extension. The map $\sigma_0^{-1} : B^\infty C \to \mathcal{C}$ turns out to be one to one, which will be proved at the beginning of the next section (see Proposition 2 and the remarks after it). For the rest of this section we will assume this. Consequently, $\sigma_0 = (\sigma_0^{-1})^{-1}$ is a well defined map of $\mathcal{C}$, one to one and onto $B^\infty C$. The rest of this section is devoted to known properties of σ_0.

Notation. $\mathcal{C}' = [T \in \mathcal{C} : \sigma_0(T)|(R^n \times S^{n-1})$ has compact support.]

(2.3) (a) $\sigma_0 : \mathcal{C} \to B^\infty C$ is linear, one to one and onto. The identity $I \in \mathcal{C}$ and $\sigma_0(I) = 1$.

(b) If $S \in \mathcal{C}$ and $\psi \in B^\infty(R^n)$ then $\psi S \in \mathcal{C}$ and $\sigma_0(\psi S) = \psi \sigma_0(S)$.

(c) If $S \in \mathcal{C}$ and $\psi \in C_0^\infty$ then $\psi S \in \mathcal{C}'$.

(d) If $S \in \mathcal{C}'$, there is a $\psi \in C_0^\infty$ such that $\psi S = S$.

These are easy consequences of (2.2).

We shall use the H_m spaces, for m any integer, and recall their definition: H_m is the set of tempered distributions T on R^n whose Fourier transform $T^\wedge$ comes from a function for which

$$\| T \|_m^2 = \int (1 + \| \ \|^2)^{m/2} \, | T^\wedge |^2 < \infty.$$

We shall use the standard properties of these Hilbert spaces. For instance, if $m \geq 0$, then

$$H_m = [f \in L^2(R^n) : D_\alpha f \in L^2(R^n) \quad \text{for} \quad |\alpha| \leq m]$$

where the $D_\alpha f$'s are distribution derivatives. In particular, $H_0 = L^2(R^n)$. We will also need the following additional property of the H_m spaces (see (2.7)): if C is any compact subset of R^n and $H_m(C)$ is the set of elements of H_m with support contained in C, the natural injection of $H_m(C)$ into H_{m-1} is a compact operator.

We introduce the following further notation:

$$H_\infty = \bigcap H_m,$$

$\mathcal{R} = [R \in B(L^2) : R(H_\infty) \subset H_\infty ;\ R \mid H_\infty$ has a bounded extension $R_k \in B[H_k, H_{k+1}]$ for every integer k].

As usual $B(X, Y)$ denotes the bounded operators from X to Y where X and Y are normed linear spaces.

If m is a positive integer,

$\mathcal{R}_m = [R \in B(L^2) : R \mid H_{|k|}$ has a bounded extension $R_k \in B[H_k, H_{k+1}]$ for every integer k such that $-m \leq k \leq m - 1$].

Clearly $\mathcal{R} = \bigcap \mathcal{R}_m$.

We continue with known properties of σ_0.

(2.4) If $S_1, S_2 \in \mathcal{C}$ then $S_1 S_2 = S_3 + R$ where $S_3 \in \mathcal{C}$, $R \in \mathcal{R}$ and $\sigma_0(S_3) = \sigma_0(S_1)\sigma_0(S_2)$. Denote S_3 by $S_1 \circ S_2$.

(2.5) If $S \in \mathcal{C}$ then $S^* = S^\# + R$ where S^* is the adjoint of S, $S^\# \in \mathcal{C}$ and $\sigma_0(S^\#) = \sigma_0(S)^-$ (the complex conjugate of $\sigma_0(S)$). Clearly $(S^\#)^\# = S$.

For (2.4) and (2.5) see [1, Theorem 4, page 71]. For the C_β^∞ operators of [2] or [1] we have $R \in \mathcal{R}_{[\beta]}$ if $\beta > 1$ ($[\beta]$ is the largest integer n such that $n \leq \beta$). We will use only that $R \in \mathcal{R}_1$ which is proved in [2] as well as in [1].

(2.6) $\mathcal{R}^* = \mathcal{R}$, $\mathcal{R}_m^* = \mathcal{R}_m$. This follows easily from the duality between H_m and H_{-m}. See [6].

(2.7) If $\psi \in C_0^\infty$, $R \in \mathcal{R}_1$ then ψR and $R\psi$ are compact.

Proof. By (2.6) and by taking adjoints, it suffices to show $\psi R \in \mathcal{K}$. The proof is clear from the following diagram:

$$H_0 \xrightarrow{R} H_1 \xrightarrow{\psi} H_1(C) \xrightarrow{i} H_0 .$$

Here C is the support of ψ which is compact in R^n and i is the injection which is compact in $B(H_1(C), H_0)$.

3. The first extension

Our first task is to prove $\sigma_0^{-1} : B^\infty C \to B(L^2)$ is one to one, and then to show that $\sigma_0 : \mathcal{C} \to BC$ is continuous. For this we need a lemma concerning Fourier transforms. The statement is essentially Lemma 15 of [7]; the simpler proof is due to the author.

Recall that the Schwartz Space $\mathcal{S} = [f \in C^\infty(R^n) : \sup_{R^n} | u_\alpha D_\beta f | < \infty$ for every $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ where $\alpha_i, \beta_i \geq 0$ are integers]. If $f \in \mathcal{S}$, the Fourier transform,

$$f^\wedge(y) = (2\pi)^{-n/2} \int f e^{-i\langle \cdot , y \rangle},$$

maps $\mathcal{S}$ one-one onto $\mathcal{S}$.

If $f \in \mathcal{S}$ and $\delta > 0$, let $f_\delta(x) = \delta^{n/2} f(\delta x)$, $x \in R^n$. Also, if $x \in R^n$ and $r > 0$ then we write $B(x, r) = [y \in R^n : \| y - x \| < r]$.

LEMMA 1. *Let $\varepsilon > 0$. Then there is a $\psi \in \mathcal{S}$ such that* (a) *for every $\delta > 0$, $\int |\psi_\delta|^2 = 1$,* (b) *if $E_\varepsilon = [y \in R^n : \| y \| > \varepsilon]$ then there is a $\delta_0 > 0$ such that for every $\delta > \delta_0$, $\int_{E_\varepsilon} |\psi_\delta|^2 < \varepsilon$,* (c) *for any $\xi \in S^{n-1}$ and $\delta > 0$,*

$$\text{support } (\psi_\delta e^{i\langle \cdot, \delta\xi\rangle})^\wedge \subset [y \in R^n - \{0\} : \| y/\| y \| - \xi \| < \varepsilon].$$

Proof. Let $A(\xi) = [y \in R^n - \{0\} : \| y/\| y \| - \xi \| < \varepsilon]$. By uniform continuity of the map $y \to y/\| y \|$ on $[y \in R^n : \frac{1}{2} < \| y \| < 2]$, there is an $r > 0$ such that for every $\xi \in S^{n-1}$, $B(\xi, r) \subset A(\xi)$. Next note that there is a $\psi \in \mathcal{S}$ such that $\int |\psi|^2 = \int |\psi^\wedge|^2 = 1$ and support $\psi^\wedge \subset B(0, r)$ so that support $[(\psi^\wedge)(\cdot - \xi)] \subset B(\xi, r)$.

We show that ψ is the desired function. Let $T_\delta : x \to \delta x$, $x \in R^n$. For (a)

$$\int |\psi_\delta|^2 = \int \delta^n |\psi \circ T_\delta|^2 = \int |\psi|^2 = 1.$$

Similarly, for (b) $\int_{E_\varepsilon} |\psi_\delta|^2 = \int_{E(\delta\varepsilon)} |\psi|^2 < \varepsilon$ if δ is sufficiently large. For (c) first note $(f_\delta)^\wedge(y) = \delta^{-n/2} f^\wedge(y/\delta)$. Therefore support $(f_\delta)^\wedge = \delta$ (support $(f^\wedge)$). Next note that $(fe^{i\langle \cdot, \xi\rangle})^\wedge = (f^\wedge)(\cdot - \xi)$. From these two facts we have that

$$\begin{aligned} \text{support } [\psi_\delta e^{i\langle \cdot, \delta\xi\rangle}]^\wedge &= \delta \text{ support } [\psi e^{i\langle \cdot, \xi\rangle}]^\wedge \\ &= \delta \text{ support } (\psi^\wedge)(\cdot - \xi) \subset \delta B(\xi, r) \subset \delta A(\xi) = A(\xi). \end{aligned}$$

(Of course if $E \subset R^n$ we are using $\delta E = [\delta x : x \in E]$.), Q.E.D.

The proofs of the following two propositions use the argument of the corresponding Theorem 2.2 of [6].

PROPOSITION 1. *Theorem 2 is valid if in* (3), *$\mathcal{A}$ is replaced by $\mathcal{K}_1$.*

Proof. We wish to construct a sequence of C_0^∞ functions ϕ_m, $m = 1, 2, \cdots$, and a sequence of real numbers $\delta_m \uparrow \infty$, $m = 1, 2, \cdots$, satisfying the properties of Theorem 2 for $T \in \mathcal{K}_1$.

By Lemma 1 there is a sequence $f_m \in \mathcal{S}$ and $\delta_m \uparrow \infty$, $m = 1, 2, \cdots$, such that

(a) $\int |f_m|^2 = 1$,
(b) $\int_{F_m} |f_m|^2 < 1/m$, $F_m = [y \in R^n : \| y \| > 1/m]$ and
(c) support $[f_m e^{i\langle \cdot, \delta_m \xi\rangle}]^\wedge \subset [y \in R^n - [0] : \| y/\| y \| - \xi \| < 1/m]$.

Let $g_m = f_m(\cdot - x) e^{i\langle \cdot - x, \delta_m \xi\rangle}$. Since translation does not affect the support of the Fourier transform (if $f \in \mathcal{S}$ $(f(\cdot - x))^\wedge = e^{-i\langle \cdot, x\rangle} f^\wedge$) we have that

$$\text{support } \hat{g}_m \subset [y \in R^n - [0] : \| y/\| y \| - \xi \| < 1/m].$$

Let $T \in \mathcal{K}_1$, $x \in R^n$ and $\xi \in S^{n-1}$. Then

$$(Tf)^\wedge(y) = \sigma_0(T)(0, y/\| y \|) f^\wedge(y)$$

for every $f \in L^2(R^n)$. Notice that $\sigma_0(T)$ is continuous on $[0] \times S^{n-1}$ and that $\sigma_0(T)(x, \xi) = \sigma_0(T)(0, \xi)$. Now, using the fact that the Fourier transform is an isometry on $L^2(R^n)$ we get that

$$\| Tg_m - \sigma_0(T)(x, \xi)g_m \|_0 \to 0 \quad \text{as} \quad m \to \infty.$$

Now, there are $\theta_m \in C_0^\infty$, support $\theta_m \subset [y \in R^n : \| y \| \leq 2/m]$, $| \theta_m | \leq 1$ and $\theta_m = 1$ on $[y \in R^n : \| y \| \leq 1/m]$. Let $\phi_m = f_m \theta_m / \| f_m \theta_m \|_0$. Then ϕ_m and δ_m, $m = 1, 2, \cdots$, satisfy (1) and (2) of Theorem 2. For (3), note that

$$\psi_m = (g_m)\theta_m(\cdot - x)/\| f_m \theta_m \|_0 \quad \text{and} \quad \| f_m \theta_m \|_0 \geq M > 0$$

by (b). Then

$$\begin{aligned} \| T\psi_m - \sigma_0(T)(x, \xi)\psi_m \|_0 & \\ \leq \| Tg_m - \sigma_0(T)(x, \xi)g_m \|_0 / \| f_m \theta_m \|_0 & \\ + \| T - \sigma_0(T)(x, \xi) \| \, \| g_m - (g_m)\theta(\cdot - x) \|_0 / \| f_m \theta_m \|_0 . & \end{aligned}$$

By (b) $\| g_m - (g_m)\theta(\cdot - x) \|_0 \to 0$ from which (3) follows, Q.E.D.

PROPOSITION 2. *If ϕ_m and δ_m, $m = 1, 2, \cdots$, are the sequences constructed in Proposition* 1, *then for every $k \in B^\infty C$, $x \in R^n$ and $\xi \in S^{n-1}$,*

$$\| \sigma_0^{-1}(k)\psi_m - k(x, \xi)\psi_m \|_0 \to 0 \quad \textit{as} \quad m \to \infty,$$

where $\psi_m = \phi_m(\cdot - x)e^{i\langle \cdot - x, \delta_m \xi \rangle}$.

Proof. Properties (1) and (2) of Theorem 2 imply that $\| \psi_m \|_0 = 1$ and support $\psi_m \to x$. This means that

$$\| a\psi_m - a(x)\psi_m \|_0 \to 0 \quad \text{as} \quad m \to \infty \qquad \text{for every } a \in B^\infty(R^n). \tag{3.1}$$

Further, if $Y \in B^\infty(S^{n-1})$, $a \in B^\infty(R^n)$,

$$\begin{aligned} \| a\sigma_0^{-1}(Y)\psi_m - a(x)Y(x, \xi)\psi_m \|_0 &\leq \| a(\sigma_0^{-1}(Y) - Y(x, \xi)\psi_m \|_0 \\ &\quad + \| Y(x, \xi)(a\psi_m - a(x)\psi_m) \|_0 . \end{aligned}$$

The first of the terms on the right side tends to zero as $m \to \infty$ by Proposition 1, the second by (3.1). Therefore

$$\| a\sigma_0^{-1}(Y)\psi_m - a(x)Y(x, \xi)\psi_m \| \to 0 \quad \text{as} \quad m \to \infty. \tag{3.2}$$

The rest follows at once by additivity and continuity using (2.2), Q.E.D.

From Proposition 2, it follows that if $\sigma_0^{-1}(k) = 0$, then

$$| k(x, \xi) | = \| k(x, \xi)\psi_m \|_0 \to 0 \quad \text{as} \quad m \to \infty$$

so that $k = 0$. Therefore $\sigma_0^{-1} : B^\infty C \to \mathfrak{C}$ is one to one and its inverse defines $\sigma_0 : \mathfrak{C} \to B^\infty C$.

Theorem 2 for the case $A \in \mathfrak{C}$ is now just a restatement of Proposition 2.

LEMMA 2. *If* $A \in \mathcal{C}$ *and* $T \in \mathcal{K}^{\mathrm{loc}}$ *then*

$$|\sigma_0(A)|_0 \leq \| A + T \|.$$

Proof. Let $x \in R^n$ and $\xi \in S^{n-1}$. If $A \in \mathcal{C}$ we have by the part of Theorem 2 just proved, that there are ψ_m, $m = 1, 2, \cdots$, such that

$$\| \psi_m \|_0 = 1, \qquad \text{support } \psi_m \subset B(x, 1/m)$$

and

$$\| A\psi_m - \sigma_0(A)(x, \xi)\psi_m \|_0 \to 0 \quad \text{as} \quad m \to \infty.$$

If $f \in L^2(R^n)$ and X_m is the characteristic function of $B(x, 1/m)$ then

$$\left| \int \psi_m \bar{f} \right| = \left| \int \psi_m X_m \bar{f} \right| \leq \| \psi_m \|_0 \, \| X_m f \|_0 \to 0 \quad \text{as} \quad m \to \infty$$

by the dominated convergence theorem. Therefore $\psi_m \to 0$ weakly. Let $T \in \mathcal{K}^{\mathrm{loc}}$. Then $T\psi_m \to 0$ in L^2, because $T\psi_m = T\psi\psi_m$ if $\psi \in C_0^\infty$, $\psi = 1$ on on $B(x, 1)$ and $T\psi$ is compact. Consequently

$$\| A + T \| \geq \| (A + T)\psi_m \|_0 \geq | \sigma_0(A)(x, \xi) | \\ - \| (A + T)\psi_m - \sigma(A)(x, \xi)\psi_m \|_0 .$$

Since $\| (A + T)\psi_m - \sigma(A)(x, \xi)\psi_m \|_0 \to 0$ by the above, the result follows, Q.E.D.

We now prove the necessary algebraic facts for the first extension. It is convenient to introduce the following C^* algebra.

DEFINITION. $\mathfrak{M} = [T \in B(L^2) : \psi T - T\psi \in \mathcal{K} \text{ for all } \psi \in C_0^\infty]$.

LEMMA 3. (a) $\mathfrak{M}$ *is a* C^* *algebra and* $\mathcal{K}^{\mathrm{loc}}$ *is a closed (self adjoint) two sided ideal in* $\mathfrak{M}$.

(b) $\mathfrak{M} = [T \in B(L^2) : \psi T - T\psi \in \mathcal{K}^{\mathrm{loc}}$ *for every* $\psi \in C_0^\infty]$.

(c) $\mathcal{R}_1 \subset \mathcal{K}^{\mathrm{loc}}$.

(d) $\mathcal{C} \subset \mathfrak{M}$.

Proof. (a) Let M_1, $M_2 \in \mathfrak{M}$ and $\psi \in C_0^\infty$.

$$\psi M_1 M_2 - M_1 M_2 \psi = (\psi M_1 - M_1 \psi)M_2 + M_1(\psi M_2 - M_2 \psi);$$
$$(\psi M_1 - M_1 \psi)^* = M_1^* \bar{\psi} - \bar{\psi} M_1^*.$$

Using these formulas, it is easy to see that $\mathfrak{M}$ is a C^* algebra. $\mathcal{K}^{\mathrm{loc}}$ is closed in $B(L^2)$ since $\mathcal{K}$ is closed. $\mathcal{K}^{\mathrm{loc}}$ is a vector space and if $T \in \mathcal{K}^{\mathrm{loc}} \subset \mathfrak{M}$ and $M \in \mathfrak{M}$, then $TM \in \mathfrak{M}$ so that $\psi(TM) - (TM)\psi \in \mathcal{K}$. $T \in \mathcal{K}^{\mathrm{loc}}$ implies $\psi TM \in \mathcal{K}$ so that $TM\psi \in \mathcal{K}$. Therefore $TM \in \mathcal{K}^{\mathrm{loc}}$ and this means $\mathcal{K}^{\mathrm{loc}}$ is a right ideal in $\mathfrak{M}$. $T^*\psi = (\bar{\psi}T)^*$ and $\psi T^* = (T\bar{\psi})^*$; it follows, since $\mathcal{K}$ is a self adjoint, that $\mathcal{K}^{\mathrm{loc}}$ is self adjoint. Then $MT = (T^*M^*)^* \in \mathcal{K}^{\mathrm{loc}}$; hence $\mathcal{K}^{\mathrm{loc}}$ is a two-sided ideal.

(b) Let $\mathfrak{M}_1$ be the right side of the equality in the statement of (b).

Clearly $\mathfrak{M} \subset \mathfrak{M}_1$. Let $T \in \mathfrak{M}_1$ and $\psi \in C_0^\infty$. There is a $\phi \in C_0^\infty$ such that $\psi\phi = \psi$. Then $\psi T - T\psi \in \mathcal{K}^{\mathrm{loc}}$ which implies that $\psi T - \phi T\psi \in \mathcal{K}$; $\phi T - T\phi \in \mathcal{K}^{\mathrm{loc}}$ implies that $\phi T\psi - T\psi \in \mathcal{K}$. By addition, $\psi T - T\psi \in \mathcal{K}$ which means $T \in \mathfrak{M}$.

(c) This is (2.7).

(d) Let $S \in \mathcal{C}$ and $\psi \in C_0^\infty$. Note that $\psi \in \mathcal{C}$; therefore

$$\psi S - \psi \circ S \in \mathcal{R}_1 \quad \text{and} \quad S\psi - S \circ \psi \in \mathcal{R}_1$$

by (2.4). Since σ_0 is one-one, $\psi \circ S = S \circ \psi$ which implies that

$$\psi S - S\psi \in \mathcal{R}_1 \subset \mathcal{K}^{\mathrm{loc}}$$

by addition and (c); $S \in \mathfrak{M}$ then follows from (b), Q.E.D.

LEMMA 4 (the first extension).

(a) $\mathcal{C} \cap \mathcal{K}^{\mathrm{loc}} = \{0\}$.

(b) *Let* $\mathcal{B} = \mathcal{C} + \mathcal{K}^{\mathrm{loc}}$ *(vector space direct sum). Then* $\mathcal{B}$ *is a self-adjoint algebra and* $\mathcal{K}^{\mathrm{loc}}$ *is a closed (self-adjoint) two-sided ideal of* $\mathcal{B}$.

(c) *Let* $\sigma_1 : \mathcal{B} \to B^\infty C$ *be defined by* $\sigma_1(S + K) = \sigma(S)$ *where* $S \in \mathcal{C}$, $K \in \mathcal{K}^{\mathrm{loc}}$. *Then* σ_1 *is a continuous* $*$ *algebra homomorphism of* $\mathcal{B}$ *onto* $B^\infty C$ *with kernel* $\mathcal{K}^{\mathrm{loc}}$.

(d) $| \sigma_1(A) |_0 \leq \| A \|$ *for every* $A \in \mathcal{B}$.

Proof. (a) If $A \in \mathcal{C} \cap \mathcal{K}^{\mathrm{loc}}$, then $\sigma_0(A) = 0$ by Lemma 2; $A = 0$ by (2.3).

(b) The assertion concerning $\mathcal{K}^{\mathrm{loc}}$ follows from Lemma 3, since also by Lemma 3, $\mathcal{B} \subset \mathfrak{M}$. That $\mathcal{B}$ is an algebra now follows from (2.4), that it is self adjoint follows from (2.5).

(c) We now show that σ_1 is a $*$ homomorphism. Let $B_i \in \mathcal{B}$; then

$$B_i = S_i + T_i$$

where $S_i \in \mathcal{C}$, $T_i \in \mathcal{K}^{\mathrm{loc}}$, $i = 1, 2$. Then $B_1 B_2 = S_1 \circ S_2 + T_3$ by (2.4) and the fact that $\mathcal{K}^{\mathrm{loc}}$ is a two-sided ideal containing $\mathcal{R}_1$. Similarly

$$B_1^* = S_1^\# + T_4$$

by (2.5) where $T_i \in \mathcal{K}^{\mathrm{loc}}$, $i = 3, 4$. Therefore

$$\sigma_1(B_1 B_2) = \sigma_0(S_1 \circ S_2) = \sigma_0(S_1)\sigma_0(S_2) = \sigma_1(B_1)\sigma_1(B_2)$$

and

$$\sigma_1(B_1^*) = \sigma_0(S_1^\#) = \sigma_0(S_1)^- = \sigma_1(B_1)^-.$$

σ_1 is onto because σ_0 is onto. Kernel $\sigma_1 = \mathcal{K}^{\mathrm{loc}}$ because σ_0 is one-one.

(d) $| \sigma_1(S_1 + M_1) |_0 = | \sigma_0(S_1) |_0 \leq \| S_1 + M_1 \|$ by Lemma 2. This proves that σ_1 is continuous, Q.E.D.

4. The second extension

LEMMA 5 (the second extension). *Let* $\sigma_2 : \mathcal{B}^- \to (B^\infty C)^-$ *be the unique continuous extension of* σ_1 *to the* C^* *algebra* $\mathcal{B}^-$ *where* $(B^\infty C)^-$ *is the closure of*

$B^\infty C$ in BC. Then σ_2 is a continuous $*$ homomorphism of $\mathcal{B}^-$ onto $(B^\infty C)^-$ with kernel $\mathcal{K}^{\mathrm{loc}}$. $|\sigma(B)|_0 \leq \| B \|$ for every $B \in \mathcal{B}^-$.

Let $\kappa : \mathcal{B}^- \to \mathcal{B}^-/K^{\mathrm{loc}}$ be defined by

$$\kappa(A) = A + K^{\mathrm{loc}}$$

and $\gamma : B^-/K^{\mathrm{loc}} \to (B^\infty C)^-$ defined by

$$\gamma \circ \kappa = \sigma_2 .$$

Then γ is an isometric $*$ isomorphism of $\mathcal{B}^-/\mathcal{K}^{\mathrm{loc}}$ onto $(B^\infty C)^-$.

To prove Lemma 5, we shall use some facts about $B*$ algebras [5 p. 241 and pp. 311–314].

1. A $*$ homomorphism of a B^*-algebra into a B^*-algebra is continuous and has closed range.
2. A closed two-sided ideal of a B^* algebra is self adjoint and the quotient space is a B^* algebra.
3. A $*$ isomorphism of a B^* algebra into a B^* algebra is isometric.

Proof of Lemma 5. The lemma follows easily from 1, 2, and 3 above once we prove kernel $\sigma_2 = \mathcal{K}^{\mathrm{loc}}$. Clearly $\mathcal{K}^{\mathrm{loc}} \subset$ kernel σ_2. Let $B \in$ kernel σ_2 and $\psi \in C_0^\infty$. There is a sequence $B_n \in \mathcal{B}$ such that $B_n \to B$. From (2.3) we see that $\psi\mathcal{C} \subset \mathcal{C}'$; therefore

$$\psi B \in (\psi(\mathcal{C} + \mathcal{K}^{\mathrm{loc}}))^- \subset (\mathcal{C}' + \mathcal{K})^-$$

is contained in the algebra considered in [3] or [7]. Since $\sigma_2(\psi B) = 0$, it follows from Theorem 2 of [3] or Corollary 29 of [7] that $\psi B \in \mathcal{K}$. Since $B \in \mathfrak{M}$, we have also that $B\psi \in \mathcal{K}$ so that $B \in \mathcal{K}^{\mathrm{loc}}$. For completeness, we shall sketch a separate argument to show that $\psi B \in \mathcal{K}$. It is the argument of the proof of Theorem 4 of [3] adapted to the algebras of this paper.

We wish to show that $\psi B \in \mathcal{K}$. Since there is a $\phi \in C_0^\infty$ such that $\phi\psi = \psi$, it suffices to show that $\psi B \in \mathcal{K}^{\mathrm{loc}}$. Thus it is enough to show that the kernel of $\sigma_2|(\mathcal{C}' + \mathcal{K}^{\mathrm{loc}})^-$ is $\mathcal{K}^{\mathrm{loc}}$.

We will need the one point compactification of R^n which is identified with S^n. Then $C(S^n \times S^{n-1})$ ($=$ the continuous functions on $S^n \times S^{n-1}$) is imbedded isometrically into $BC(R^n \times S^{n-1})$ by restriction.

Recall that $\mathcal{H}_1 = [T \in B(L^2) : (Tf)^\wedge(y) = g(y/\| y \|)f^\wedge(y)$ and if $k(x, \xi) = g(\xi)$, $(x, \xi) \in R^n \times S^{n-1}$, then $k \in B^\infty C]$.

Let $\mathcal{H}_2 = [T \in B(L^2) : T$ is a multiplication by $\psi + c$, where $\psi \in C_0^\infty$ and c is a constant$]$.

Note that if $T \in \mathcal{H}_1$ then $T \in \mathcal{C}$ and $\sigma_0(T) = k$; if $T \in \mathcal{H}_2$, then $T \in \mathcal{C}$ and $\sigma_0(T)(x, \xi) = \psi(x) + c$. It is easily seen from the theory of multiplication operators on $B(L^2)$ and the Stone-Weierstrass theorem that $\sigma_2|\mathcal{H}_1^-$ and $\sigma_2|\mathcal{H}_2^-$ are isometries with $C(S^{n-1})$ and $C(S^n)$ respectively, where $C(S^{n-1})$ and $C(S^n)$ are imbedded in $C(S^n \times S^{n-1})$ in the obvious way. It follows easily that $\gamma|\kappa(\mathcal{H}_1^-)$ and $\gamma|\kappa(\mathcal{H}_2^-)$ are isometries with $C(S^{n-1})$ and $C(S^n)$ also.

Let $\mathcal{H}_3$ be the algebra generated by $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$. Then by using (2.2) and (2.3)(d) it is easy to see that

$$(\mathcal{C}' + \mathcal{K}^{\text{loc}})^- \subset (\mathcal{H}_3 + \mathcal{K}^{\text{loc}})^-.$$

Therefore it is sufficient to show that kernel $\sigma_2 \mid (\mathcal{H}_3 + \mathcal{K}^{\text{loc}})^-$ is $\mathcal{K}^{\text{loc}}$. By continuity we have

$$\kappa((\mathcal{H}_3 + \mathcal{K}^{\text{loc}})^-) \subset (\kappa(\mathcal{H}_3 + \mathcal{K}^{\text{loc}}))^- = \kappa(\mathcal{H}_3)^-,$$

so it is sufficient to show that $\gamma|(\kappa(\mathcal{H}_3))^-$ is one-one.

We note that $\mathcal{B}^-/\mathcal{K}^{\text{loc}}$ is a commutative B^* algebra with identity. For if $B_1, B_2 \in \mathcal{B}$, then

$$B_1 B_2 - B_2 B_1 \in \text{kernel } \sigma_1 = \mathcal{K}^{\text{loc}};$$

and by continuity the same holds for $\mathcal{B}^-$.

Let $\mathcal{L} = \kappa(\mathcal{H}_3)^-$; then $\mathcal{L}$ is a commutative B^* algebra with identity. The finite sums $\sum \kappa(a_i)\kappa(T_i)$, $a_i \in \mathcal{H}_1^-$, $T_i \in \mathcal{H}_2^-$ are dense in $\mathcal{L}$ and so

$$\gamma(\mathcal{L}) \subset C(S^n \times S^{n-1}).$$

Let μ be a multiplicative linear functional on L. Then $\mu|\kappa(\mathcal{H}_1^-)$ and $\mu|\kappa(\mathcal{H}_2^-)$ are multiplicative linear functionals, so by the above stated isometries there is a $\xi \in S^{n-1}$ and an $x \in S^n$ such that $\mu(H_1) = \gamma(H_1)(\xi)$ and

$$\mu(H_2) = \gamma(H_2)(x)$$

whenever $H_1 \in \kappa(\mathcal{H}_1^-)$ and $H_2 \in \kappa(\mathcal{H}_2^-)$. Since μ is continuous and the finite sums are dense, $\mu(L) = \gamma(L)(x, \xi)$ for all $L \in \mathcal{L}$. Now if $\gamma(L) = 0$ then $\mu(L) = 0$ for every multiplicative linear functional μ; since $\mathcal{L}$ is semisimple, $L = 0$, Q.E.D.

We postpone the characterization of $(B^\infty C)^-$ to Section 6.

5. The third extension

In this section we define $\mathcal{A}$ and extend σ_2 to $\mathcal{A}$. We then prove (d) of Theorem 1.

Definition. $\mathcal{A} = [A \in B(L^2) : \psi A$ and $A\psi \in \mathcal{B}^-$ for all $\psi \in C_0^\infty]$.

We will need to know that $\mathcal{A} \subset \mathfrak{M}$; this, together with several other descriptions of $\mathcal{A}$ is the subject of the next lemma. The definition of $\mathfrak{M}$ is given in Section 3.

Lemma 6. *Let*

$$\mathcal{A}_1 = [A \in \mathfrak{M} : \psi A \in \mathcal{B}^- \quad \textit{for all} \quad \psi \in C_0^\infty]$$

$$\mathcal{A}_2 = [A \in \mathfrak{M} : A\psi \in \mathcal{B}^- \quad \textit{for all} \quad \psi \in C_0^\infty]$$

$$\mathcal{A}_3 = [A \in B(L^2) : \psi A \textit{ and } A\psi \in (\mathcal{C}' + \mathcal{K})^- \quad \textit{for all} \quad \psi \in C_0^\infty].$$

Then $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}$.

Proof. Let $\psi \in C_0^\infty$. $\mathfrak{A}_1 = \mathfrak{A}_2$ follows from the fact that $\psi A - A\psi \in \mathcal{K}$ for $A \in \mathfrak{M}$ and $\mathcal{K} \subset \mathfrak{B}^-$. Next we prove that $\mathfrak{A} \subset \mathfrak{M}$ from which $\mathfrak{A} = \mathfrak{A}_i$, $i = 1, 2$ is immediate. For this let $A \in \mathfrak{A}$. There is a $\phi \in C_0^\infty$ such that $\phi\psi = \psi$. Then $\psi A \in \mathfrak{B}^-$,

$$\phi(\psi A) - (\psi A)\phi \in \text{kernel } \sigma_2 = \mathcal{K}^{\text{loc}}$$

and

$$\psi(A\phi) - (A\phi)\psi \in \text{kernel } \sigma_2 = \mathcal{K}^{\text{loc}},$$

so that $\psi A - A\psi \in \mathcal{K}^{\text{loc}}$. Thus $A \in \mathfrak{M}$ by Lemma 3.

Clearly $\mathfrak{A}_3 \subset \mathfrak{A}$. If $A \in \mathfrak{A}$, then

$$\psi A = \phi\psi A \in \phi\mathfrak{B}^- \subset (\phi(\mathcal{C} + \mathcal{K}^{\text{loc}}))^- \subset (\mathcal{C}' + \mathcal{K})^-.$$

Since $\psi A - A\psi \in \mathcal{K}$, we have $A\psi \in (\mathcal{C}' + \mathcal{K})^-$ also, Q.E.D.

LEMMA 7 (the third extension). *$\mathfrak{A}$ is a C^* algebra containing $\mathfrak{B}^-$. There is a unique continuous $*$ homomorphism σ from $\mathfrak{A}$ into $BC(R^n \times S^{n-1})$ such that $\sigma|\mathfrak{B}^- = \sigma_2$.*

Proof. (i) *$\mathfrak{A}$ is a C^* algebra containing $\mathfrak{B}^-$.* Clearly, $\mathfrak{A}$ is a linear space. Let $A_1, A_2 \in \mathfrak{A}$ and $\psi \in C_0^\infty$; there is a $\phi \in C_0^\infty$ such that $\phi\psi = \psi$. Then

$$(5.1)\quad \psi(A_1 A_2) = (\psi A_1 - A_1\psi)A_2 + (A_1\phi)(\psi A_2) \in \mathcal{K} + \mathfrak{B}^- \subset \mathfrak{B}^-$$

by Lemma 6; therefore $\mathfrak{A}$ is an algebra. If $A \in \mathfrak{A}$

$$(5.2)\quad \psi A^* = (A\bar{\psi})^* \in \mathfrak{B}^-$$

so that $\mathfrak{A}$ is self adjoint; $\mathfrak{A}$ is closed since $\mathfrak{B}^-$ is closed. Clearly $\mathfrak{A}$ contains $\mathfrak{B}^-$.

(ii) *Definition of σ.* Let $x \in R^n$, $\xi \in S^{n-1}$. Define

$$\sigma(A)(x, \xi) = \sigma_2(\psi A)(x, \xi)$$

where $\psi \in C_0^\infty$, $\psi(x) = 1$ and $A \in \mathfrak{A}$.

Suppose $\psi_1, \psi_2 \in C_0^\infty$ and $\psi_1(x) = \psi_2(x) = 1$. Then

$$\sigma_2(\psi_1 A)(x, \xi) = \psi_2(x)\sigma_2(\psi_1 A)(x, \xi)$$
$$= \sigma_2(\psi_2\psi_1 A)(x, \xi) = \sigma_2(\psi_1\psi_2 A)(x, \xi) = \sigma_2(\psi_2 A)(x, \xi).$$

Therefore σ is well defined. σ extends σ_2, as is easily seen from the definition.

(iii) *σ is a $*$ homomorphism from $\mathfrak{A}$ into BC.* Let $A \in \mathfrak{A}$. If $x \in R^n$ and $\psi \in C_0^\infty$, $\psi = 1$ on a neighborhood N of x in R^n, then $\sigma(A)(y, \xi) = \sigma_2(\psi A)(y, \xi)$ for every $(y, \xi) \in N \times S^{n-1}$. This shows that $\sigma(A)$ is a continuous function on $R^n \times S^{n-1}$. Also,

$$|\sigma(A)(x, \xi)| = |\sigma_2(\phi A)(x, \xi)| \le \|\phi A\| \le \|A\|,$$

where $\phi \in C_0^\infty$, $0 \le \phi \le 1$ and $\phi(x) = 1$. Therefore $\sigma(A) \in BC$ and $|\sigma(A)|_0 \le \|A\|$.

Let $x \in R^n$, $\xi \in S^{n-1}$. Let $\psi(x) = 1$ in (5.1); then it is easy to see that

$\sigma(A_1 A_2)(x, \xi) = \sigma(A_1)(x, \xi)\sigma(A_2)(x, \xi)$. Similarly, using (5.2) one sees that $\sigma(A^*)(x, \xi) = \sigma(A)(x, \xi)^-$. The simple proof is left to the reader.

If σ' is a homomorphism of $\mathcal{A}$ into BC which extends σ_2, then $\sigma'(A)(x, \xi) = \sigma_2(\psi A)(x, \xi)$ if $\psi(x) = 1$, so that σ is unique, Q.E.D.

LEMMA 8. *The kernel of σ is $\mathcal{K}^{\text{loc}}$. Consequently, as in Lemma 5, σ induces an isometric $*$ isomorphism of $\mathcal{A}/\mathcal{K}^{\text{loc}}$ into BC which extends γ.*

Proof. Since σ extends σ_2, kernel $\sigma \supset \mathcal{K}^{\text{loc}}$. Suppose $A \in$ kernel σ. Then $\psi A \in$ kernel $\sigma|\mathcal{B}^- = \mathcal{K}^{\text{loc}}$. Then $\psi A = \phi\psi A \in \mathcal{K}$ where $\phi\psi = \psi$, so that $A \in \mathcal{K}^{\text{loc}}$, Q.E.D.

We now prove Theorem 1(d). This will follow immediately from the following slightly more general result.

LEMMA 9. *Let $\mathcal{G}$ be a subalgebra of $B(L^2)$ and let $\mathcal{B}^- \subset \mathcal{G}$. Also suppose there is an algebra homomorphism σ' of $\mathcal{G}$ into the continuous (not necessarily bounded) functions on $R^n \times S^{n-1}$ with kernel $\mathcal{K}^{\text{loc}}$, which is an extension of $\sigma|\mathcal{B}^-$. Then $\mathcal{G} \subset \mathcal{A}$ and $\sigma' = \sigma|\mathcal{G}$.*

Proof. We will need the fact that $B^\infty C^- = UC$ (= the bounded uniformly continuous functions on $R^n \times S^{n-1}$); this will be proved in the next section.

Let $G \in \mathcal{G}$. If $\psi \in C_0^\infty$, then $\sigma'(\psi G) = \psi\sigma'(G) \in UC$, so that there is an $A \in \mathcal{B}^-$ such that $\psi G - A \in$ kernel $\sigma' = \mathcal{K}^{\text{loc}}$. But $\mathcal{K}^{\text{loc}} \subset \mathcal{B}^-$ so that $\psi G \in \mathcal{B}^-$. Since $\psi G - G\psi \in$ kernel $\sigma' = \mathcal{K}^{\text{loc}}$, $G\psi \in \mathcal{B}^-$ also. Therefore $G \in \mathcal{A}$.

$$\sigma'(G)(x, \xi) = \sigma'(\psi G)(x, \xi) = \sigma(G)(x, \xi) \quad \text{if} \quad \psi(x) = 1,$$

because σ' extends $\sigma|\mathcal{B}^-$, Q.E.D.

Proof of Theorem 1(d). Suppose $(\mathcal{A}', \sigma')$ satisfies (a)–(c) of Theorem 1. Then $\mathcal{A}' \supset \mathcal{B}^-$. $\sigma'|\mathcal{C} = \sigma_0$ and $\sigma'|\mathcal{K}^{\text{loc}} = 0$ so that $\sigma'|\mathcal{B} = \sigma_1$. Since a $*$ homomorphism on a C^* algebra is continuous $\sigma'|\mathcal{B}^- = \sigma_2$. Now, an application of Lemma 9 completes the proof, Q.E.D.

The proof of Theorem 1 is now complete, except for the statement about the range of σ.

6. The range of σ

In this section we show that σ maps $\mathcal{A}$ onto $BC(R^n \times S^{n-1})$ and $\mathcal{B}^-$ onto the bounded uniformly continuous functions on $R^n \times S^{n-1}$.

DEFINITION. Let $UC(R^n)$ be the set of bounded, uniformly continuous functions on R^n.

We will consider $UC(R^n)$ as a closed subspace of the sup-normed Banach space of bounded continuous functions on R^n.

LEMMA 10. *$B^\infty(R^n)$ is a dense subset of $UC(R^n)$.*

Proof. It follows from the mean value theorem on R^n that if $x, y \in R^n$ and

$f \in B^\infty(R^n)$, then

$$|f(x) - f(y)| \leq n^{1/2}(\max_{|\alpha|=1} \| D_\alpha f \|_\infty) \| x - y \|$$

where $\| g \|_\infty = \sup_{R^n} | g |$ for $g \in UC(R^n)$. Therefore $B^\infty(R^n) \subset UC(R^n)$.

Suppose $f \in UC(R^n)$. Let ϕ_n, $n = 1, 2, \cdots$, be an approximate identity in R^n. That is, $\phi_n \in C_0^\infty$, support $\phi_n \to 0$ as $n \to \infty$, $\phi_n \geq 0$, and $\int \phi_n = 1$. Then $f * \phi_n$ (the convolution of f with ϕ_n) converges to f, uniformly on R^n. Also, $f * \phi_n \in C^\infty(R^n)$ and $D_\alpha(f * \phi_n) = f * D_\alpha \phi_n$. Since $f \in L^\infty$ and $D_\alpha \phi_n \in C_0^\infty \subset L^1$, we have that

$$\| D_\alpha(f * \phi_n) \|_\infty = \| f * (D_\alpha \phi_n) \|_\infty \leq \| f \|_\infty \int | D_\alpha \phi_n |.$$

This means that $f * \phi_n \in B^\infty(R^n)$, Q.E.D.

Definition. $UC = UC(R^n \times S^{n-1}) = [k \in BC(R^n \times S^{n-1}) : k$ is uniformly continuous on $R^n \times S^{n-1}]$.

UC will be considered as a closed subspace of BC; and it follows from the mean value theorem that $B^\infty C \subset UC$.

Lemma 11. *$(B^\infty C)^- = UC$. Consequently $\sigma(\mathfrak{B}^-) = UC$.*

Proof. Let $\eta_1 : (x, \xi) \to x$ and $\eta_2 : (x, \xi) \to \xi$ be the coordinate functions for $R^n \times S^{n-1}$. Then $f \to f(\eta_1)$ for $f \in UC(R^n)$ is an isometry of $UC(R^n)$ into $BC(R^n \times S^{n-1})$. Similarly $f \to f(\eta_2)$ for $f \in C(S^{n-1})$ (the continuous functions on S^{n-1}) is an isometry of $C(S^{n-1})$ into $BC(R^n \times S^{n-1})$.

We observe that $C(S^{n-1}) \subset (B^\infty C)^-$ by the Stone-Weierstrass theorem, and $UC(R^n) \subset (B^\infty C)^-$ by Lemma 10. This $\subset$ symbol is meant in the sense of the above mentioned isometries.

Let $k \in UC$ and $\varepsilon > 0$. Then for each $\xi_0 \in S^{n-1}$, there exists $\delta > 0$ such that if $\| \xi - \xi_0 \| < \delta$ for $\xi \in S^{n-1}$, then $| k(x, \xi) - k(x, \xi_0) | < \varepsilon$ for every $x \in R^n$.

By compactness of S^{n-1}, there exist $\xi_i \in S^{n-1}$ and $\delta_i > 0$, for $i = 1, \cdots, m$ such that the above holds for ξ_i with $\delta = \delta_i$, and the sets

$$U_i = [\xi \in S^{n-1} : \| \xi - \xi_i \| < \delta_i] \qquad i = 1, \cdots, m$$

are a covering for S^{n-1}. Next we observe that there exist $\psi_i \in C(S^{n-1})$, $i = 1, \cdots, m$ such that $0 \leq \psi_i \leq 1$, support $\psi_i \subset U_i$ and $1 = \sum_{i=1}^m \psi_i$. Let $k_1 = \sum_{i=1}^m k(\eta_1, \xi_i)\psi_i(\eta_2)$. Then $k_1 \in (B^\infty C)^-$ and $| k_1 - k |_0 < \varepsilon$, Q.E.D.

Lemma 12. *σ maps $\mathfrak{A}$ onto $BC(R^n \times S^{n-1})$.*

Proof. Let f be a C_0^∞ function on R such that $0 < f(t) \leq 1$ for $| t | < \frac{3}{4}$ and $f(t) = 0$ if $| t | \geq \frac{3}{4}$. Let g be a C_0^∞ function on R such that $g(t) = 1$ for $| t | \leq 1$, $0 \leq g(t) \leq 1$ for $1 \leq | t | \leq 2$, and $g(t) = 0$ for $| t | > 1$.

Let $f_n(t) = f(t - n)$, $g_n(t) = g(t - n)$, $n = 0, 1, \cdots$, and

$$\theta = \sum_{n=0}^\infty f_n > 0.$$

Let ψ_n and ϕ_n, $n = 0, 1, \cdots$, be the functions on R^n defined by

$$\psi_n(x) = f_n(\| x \|^2)/\theta(\| x \|^2), \qquad \phi_n(x) = g_n(\| x \|^2).$$

Then (1) $\psi_n, \phi_n \in C_0^\infty$, (2) $\psi_n \phi_n = \psi_n$, (3) $\sum_{n=0}^\infty \psi_n = 1$, (4)

$$\text{support } \psi_n \cap \text{support } \psi_{n+2} = \phi,$$

(5) $\sum_{n=0}^\infty \phi_n \leq 4$, (6) $0 \leq \phi_n, \psi_n \leq 1$, (7) $\phi_n(x) = 0$ if $\| x \|^2 \geq n + 2$.

Now let $k \in BC$. Then $\phi_n k \in UC$ so by Lemmas 5 and 11 there is an $A_n \in \mathcal{B}^-$ such that $\sigma(A_n) = \phi_n k$ and

$$\| A_n \| \leq | \phi_n k |_0 + 1 \leq | k |_0 + 1.$$

If $h \in C_0^\infty$ define

$$B_i h = \sum_{m=0}^\infty \psi_{2m+i} A_{2m+i} \phi_{2m+i} h, \qquad i = 0, 1;$$

by (7) this is a finite sum. It is easy to see that B_i is linear; we now show that B_i is a bounded operator on C_0^∞ in the L^2 norm and therefore extends by continuity to an element $B_i \in B(L^2)$. By (4) we have that the terms of the sum are orthogonal so that

$$\begin{aligned}
&\| B_i h \|^2 \\
&= \sum_{m=0}^\infty \| \psi_{2m+i} A_{2m+i} \phi_{2m+i} h \|^2 \leq (| k |_0 + 1)^2 \sum_{m=0}^\infty \| \phi_{2m+i} h \|^2 \\
&= (| k |_0 + 1)^2 \sum_{m=0}^\infty \int | \phi_{2m+i} |^2 | h |^2 \\
&= (| k |_0 + 1)^2 \int \sum_{m=0}^\infty | \phi_{2m+i} |^2 | h |^2 \text{ by the monotone convergence theorem} \\
&\leq (| k |_0 + 1)^2 \int \sum_{m=0}^\infty (| \phi_{2m+i} |) | h |^2 \qquad \text{by (6)} \\
&\leq 4(| k |_0 + 1)^2 \| h \|^2 \qquad \text{by (5).}
\end{aligned}$$

If $\psi \in C_0^\infty$, there is an integer $M \geq 0$ such that $\psi\psi_{2m+i} = \psi\phi_{2m+i} = 0$ for $m > M$, $i = 0, 1$. Then

$$\psi B_i = \sum_{m=0}^M \psi\psi_{2m+i} A_{2m+i} \phi_{2m+i} \quad \text{and} \quad B_i \psi = \sum_{m=0}^M \psi_{2m+i} A_{2m+i} \phi_{2m+i} \psi$$

so that ψB_i and $B_i \psi \in \mathcal{B}^-$. Therefore $B_i \in \mathcal{A}$, $i = 0, 1$.

Let $A = B_0 + B_1 \in \mathcal{A}$. Let $x \in R^n$, $\xi \in S^{n-1}$ and $\psi(x) = 1$.

Then

$$\begin{aligned}
\sigma(A)(x, \xi) &= \sigma(\psi B_0)(x, \xi) + \sigma(\psi B_1)(x, \xi) \\
&= \sum_{m=0}^M \psi(x)\psi_{2m}(x)\psi_{2m}(x)k(x, \xi)\phi_{2m}(x) \\
&\qquad + \sum_{m=0}^M \psi(x)\psi_{2m+1}(x)\psi_{2m+1}(x)k(x, \xi)\phi_{2m+1}(x) \\
&= k(x, \xi) \sum_{m=0}^\infty \psi_m(x) = k(x, \xi)
\end{aligned}$$

by (3), (2) and the fact that $\phi_j(x) = 0$ if $j > 2M + 1$, Q.E.D.

The proof of Theorem 1 is now complete.

7. Concluding remarks

In this section we discuss the relation of the C_β^∞ operators of [2], $\beta \geq 0$ to the algebra $\mathfrak{A}$. We will also complete the proof of Theorem 2. Notice that $\mathfrak{C} = \bigcap_\beta(C_\beta^\infty \text{ operators})$.

Let C_β denote the C_β^∞ operators. Then we wish to remark that

$$(C_\beta + \mathfrak{K}^{\text{loc}})^- = \mathfrak{B}^- \quad \text{if } \beta > 0$$

and in any case, $C_\beta + \mathfrak{K}^{\text{loc}} \subset \mathfrak{A}$ and $\sigma|C_\beta$ is the σ-symbol of [2]. This follows from the fact (see [2]) that if $H \in C_\beta$ and σ_0 denotes the σ-symbol of [2] on C_β (an extension of that on $\mathfrak{C}$) then $H = \sum_{i=0}^\infty a_i T_i$, $a_i \in C_\beta(R^n)$, $T_i \in \mathfrak{C}$ and $\sigma_0(H) = \sum_{i=0}^\infty a_i \sigma_0(T_i)$, the first sum converging in $B(L^2)$, the second in BC. It is easily seen that if ψ is a bounded, continuous function on R^n, then the corresponding multiplication $\psi \in \mathfrak{A}$, and $\sigma(\psi)(x, \xi) = \psi(x)$. Therefore $H \in \mathfrak{A}$ and

$$\sigma(H) = \sum_{i=0}^\infty \sigma(a_i)\sigma(T_i) = \sum_{i=0}^\infty a_i \sigma_0(T_i) = \sigma_0(H).$$

If $\beta > 0$ then $C_\beta(R^n) \subset UC(R^n)$ so that $H \in \mathfrak{B}^-$. Thus

$$\mathfrak{B} = \mathfrak{C} + \mathfrak{K}^{\text{loc}} \subset C_\beta + \mathfrak{K}^{\text{loc}} \subset \mathfrak{B}^-;$$

this means that $(C_\beta + \mathfrak{K}^{\text{loc}})^- = \mathfrak{B}^-$.

Finally we finish the proof of Theorem 2. In Section 3, Proposition 2, we proved it for the case $A \in \mathfrak{C}$. We will show that the same functions ϕ_m and numbers δ_m work for $\mathfrak{A}$. In the proof of Lemma 2 it is shown that $T\psi_m \to 0$ if $T \in \mathfrak{K}^{\text{loc}}$; this establishes the theorem for $A \in \mathfrak{B} = \mathfrak{C} + \mathfrak{K}^{\text{loc}}$. Then an easy continuity argument proves the assertion for $A \in \mathfrak{B}^-$. Finally, if $x \in R^n$ and $\xi \in S^{n-1}$, there is a compact neighborhood N of x such that support $\psi_m \subset N$. Also there is a $\psi \in C_0^\infty$ such that $\psi = 1$ on N so that $\psi\psi_m = \psi_m$. Therefore, if $A \in \mathfrak{A}$, we have $A\psi_m = A\psi\psi_m$ and $\sigma(A\psi)(x, \xi) = \sigma(A)(x, \xi)$. The theorem follows from these facts and its truth for $\mathfrak{B}^-$.

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