EXTREMAL PROCESSES, II

BY

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1. Introduction. An extremal process is defined to be a stochastic process, $\{Y(t), t \geq 0\}$, having the following property:

$$Y(t_1); Y(t_2); \cdots; Y(t_n) \text{ has the same joint distribution law as does}$$
$$U_1; \max(U_1, U_2); \cdots; \max(U_1, U_2, \cdots, U_n),$$

(1.1) where $0 \leq t_1 < \cdots < t_n$, n is arbitrary, $U_1, \cdots, U_n$ are independent random variables, and

$$P(U_i < u) = P(Y(t_i - t_{i-1}) < u) \quad (t_0 = 0),\ i = 1, \cdots, n.$$

Notice the formal similarity between this definition of an extremal process and the definition of an i.d., stationary increment process. If "$\max(U_1, \cdots, U_i)$" is replaced by "$U_1 + \cdots + U_i$" the definition of an extremal process becomes the definition of an i.d. process.

The motivating examples of extremal processes are studied in [2]. These arise in connection with limiting distributions of maxima of independent and identically distributed random variables.

Another example of an extremal process is the following. Let $\{Z(t), t \leq 0\}$ be an i.d. stationary increment process. Define $Y(t)$ = maximum positive discontinuity of $Z(u)$, $0 \leq u \leq t$, with the understanding that this has value 0 if there are no positive discontinuities. It is easy to see that $Y(t)$ is an extremal process.

The purpose of this paper is to describe the general structure of extremal processes, and to generalize and unify the results of [2]. In Section 7 and thereafter extensions are made to multivariate processes.

We also refer the reader to a paper by Lamperti [5], where the same class of processes is studied from a somewhat different point of view.

2. The function Q. There is no loss in generality in supposing that $Y(t)$ is a separable process and is non-decreasing in t with probability 1, [1]. (In fact, Theorem 4.1 describes an explicit representation of the process having these properties.) Hence, for any fixed u, $P(Y(t) < u)$ is monotone in t. The definition of an extremal process implies that

$$P(Y(t_1 + t_2) < u) = P(Y(t_1) < u)P(Y(t_2) < u)$$

for all t_1, t_2. Hence, there exists a constant $Q = Q(u)$ such that

$$P(Y(t) < u) = \exp -tQ(u). \tag{2.1}$$

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The function Q plays an important role in the analysis that follows. The main properties of Q are the following:

(a) Q is monotone non-increasing in u.

(b) Q is continuous from the left (by convention). Suppose $[a, b]$ is the smallest interval on which the total variation of Q is concentrated. (a may be $-\infty$, and b may be $+\infty$.) Then $Q(a - 0) = +\infty$, $Q(b + 0) = 0$.

Conversely, given any function Q satisying (a), (b), a separable non-decreasing extremal process can be determined by defining $P(Y(t) < u)$ by (2.1) and requiring the joint distributions to be given by (1.1). It need only be noted that such a definition of $Y(t)$ satisfies the consistency requirements of Kolmogorov.

We want to point out that the three examples studied in [2] have Q's given by (I) $Q(u) = u^{-\alpha}$, $u > 0$, $\alpha > 0$; (II) $Q(u) = (-u)^{\alpha}$, $u < 0$, $\alpha > 0$; (III) $Q(u) = e^{-u}$.

3. Maximum discontinuity of an i.d. process. Here, we take a closer look at the example mentioned in Section 1. Let $\{Z(t), t \geq 0\}$ be an i.d. stationary increment process with Lévy representation

$$E \exp i\theta Z(t) = \exp t \int_{-\infty}^{\infty} \left(\exp i\theta u - 1 - \frac{i\theta u}{1 + u^2}\right)\left(\frac{1 + u^2}{u^2}\right) dG(u), \tag{3.1}$$

where G is a mass function with finite total mass [1].

Define

$$\begin{aligned} Y(t) &= \text{maximum positive discontinuity of } Z(u), 0 \leq u \leq t, \\ &\qquad\qquad \text{if there are positive discontinuities,} \\ &= 0, \qquad\qquad \text{if there are no positive discontinuities.} \end{aligned} \tag{3.2}$$

As noted in Section 1, $\{Y(t)\}$ is an extremal process. We want to show that its parameter Q is given by

$$\begin{aligned} Q(u) &= -\infty, & u < 0, \\ &= \int_u^{\infty} \left(\frac{1 + x^2}{x^2}\right) dG(x), & u > 0. \end{aligned} \tag{3.3}$$

Let $u > 0$ be fixed. We will evaluate $P(Y(t) \geq u)$ in the following way. Given any $\varepsilon > 0$, $Z(t)$ can be represented as a sum of three independent processes, $Z(t) = Z_1(t) + Z_2(t) + Z_3(t)$, with the characteristic functions given by (3.1) except that the integral for $Z_2(t)$ is $\int_{\varepsilon}^{\infty}$, the integral for $Z_1(t)$ is $\int_{-\infty}^{-\varepsilon}$ and the integral for $Z_3(t)$ is $\int_{-\varepsilon}^{\varepsilon}$. It is possible to choose $\varepsilon = \varepsilon(u) > 0$ so that the discontinuities of $Z_1(t)$ and $Z_3(t)$ are less than u. (These are algebraic values of discontinuities under discussion, and not absolute values.) Hence to evaluate $P(Y(t) \geq u)$ there is no loss of generality in supposing that the original $Z(t)$ in terms of which $Y(t)$ is defined is $Z_2(t)$. Now, $Z_2(t)$ is distributed like $Z_4(t) + t \int_\varepsilon^\infty (1/u)\, dG(u)$, where $Z_4(t)$ is a *compound Poisson* process with

$$E \exp i\theta Z_4(t) = \exp t \int_\varepsilon^\infty (\exp i\theta u - 1)(1 + u^2)/(u^2)\, dG(u).$$

Hence, we may as well suppose that the $Z(t)$ in (3.2) is just $Z_4(t)$. For a compound Poisson process with parameter λ, and positive jumps with c.d.f. H, the maximum discontinuity in $[0, t)$, $Y(t)$, satisfies

$$\begin{aligned} P(Y(t) \geq u) &= \textstyle\sum_{k=0}^\infty (\lambda t)^k (\exp -\lambda t)(1 - H^k(u))/k! \\ &= 1 - \exp -\lambda t(1 - H(u)). \end{aligned}$$

In particular, for $Z_4(t)$,

$$\lambda = \int_\varepsilon^\infty (1 + x^2)/(x^2)\, dG(x), \qquad H(u) = \frac{1}{\lambda} \int_\varepsilon^u (1 + x^2)/(x^2)\, dG(x).$$

Hence, $\lambda(1 - H(u)) = \int_u^\infty (1 + x^2)/(x^2)\, dG(x)$, and Q is given by (3.3).

4. A representation for $Y(t)$. We want to give an explicit representation for $Y(t)$, $0 < t_0 \leq t$. Suppose $Y(t)$ is determined by Q. Define a process $Y^*(t)$, $t_0 \leq t$, as follows. Let $\{W_n, n = 1, 2, \cdots\}$ be a sequence of independent, exponential (parameter 1) random variables. These random variables are independent of all other random variables to be considered. Let R_y denote the c.d.f. defined by

$$\begin{aligned} R_y(u) &= 0, & u \leq y, \\ &= 1 - Q(u)/Q(y), & u > y. \end{aligned}$$

(a) $Y^*(t_0)$ has the same distribution as does $Y(t_0)$.

(b) Under the condition that $Y^*(t_0) = x$, $Y^*(t)$ remains equal to x for a random amount of time, $W_1/Q(x)$. The process then jumps to a height Z_1 ($>x$), where Z_1 has the c.d.f. R_x. The process then remains at height Z_1 a random amount of time $W_2/Q(Z_1)$. The process then jumps to a height Z_2 ($>Z_1$), where the c.d.f. of Z_2 (conditional on the given values of all preceding random variables) is R_{Z_1}, etc.

It should be evident that $Y(t)$ and $Y^*(t)$ are Markov processes. We want to show that they are essentially the same.

THEOREM 4.1. *For any $t_0 < t_1 < \cdots < t_n$ the joint distribution laws of*

$$(Y^*(t_0), Y^*(t_1), \cdots, Y^*(t_n)) \quad \textit{and of} \quad (Y(t_0), Y(t_1), \cdots, Y(t_n))$$

are the same.

Proof. It is sufficient to show that the infinitesimal generators of both processes are the same [3]. The details are the same as those of the proof of Theorem 4.2 of [2] and will not be repeated here.

5. The case of continuous Q. Some interesting properties hold for extremal processes whose parameter Q is continuous. These properties were already shown in [2] to hold for the three examples (I), (II), (III), whose Q's are continuous. These properties are the following.

THEOREM 5.1. *Suppose Q is continuous.* (*This implies, incidentally, that $Q(u)$ approaches ∞ as u approaches a from the right.*) *Define*

$$C(t_1, t_2) = 1 \quad \text{if} \quad Y(t_2) > Y(t_1),$$
$$= 0 \quad \text{otherwise.} \qquad (0 < t_1 < t_2)$$

(a) *If $0 < t_1 < t_2 < \cdots < t_k$, then $C(t_1, t_2), \cdots, C(t_{k-1}, t_k)$ are mutually independent.*

(b) *Suppose $0 < s < u$. Then $Y(t)$ has a finite number of discontinuities in (s, u) and this number is Poisson distributed with parameter $-\log s/u$.*

(c) *Suppose $0 < s < u, x < v$. Let $J(x, v)$ be the number of discontinuities in $Y(t)$ from time s up to the time for which $Y(t)$ first exceeds v, subject to the condition that $Y(s) = x$. Then, under the condition that $Y(s) = x$, $J(x, v)$ is Poisson distributed with parameter $-\log (Q(v)/Q(x))$.*

Proof. Again the proof is closely related to details in [2] and will not be repeated. We point out the proof follows from the proofs of Theorems 4.1 and 4.5 of [2] by an appropriate change of variables on Q.

Remark. The assertions of Theorem 5.1 are in general not true if Q is not continuous.

6. The multivariate case. The remainder of this paper is concerned with multivariate versions of the processes studied above.

For motivation, we first look at an example involving the limiting distribution of the k largest of n independent and identically distributed random variables. This example is a multivariate version of the examples which constitute the paper [2]. Let $\{X_i, n = 1, 2, \cdots\}$ be a sequence of independent and identically distributed random variables and define $Y_n^{(i)} = i$-th largest among $X_1, \cdots, X_n$, $i = 1, 2, \cdots$. Suppose that, for fixed k, the random vector

$$(Y_n^{(1)} - a_n)/b_n, \cdots, (Y_n^{(k)} - a_n)/b_n$$

converges in law to a non-degenerate k-dimensional distribution. The paper by Lamperti [5] studies the "invariance principle" in its relevance to this convergence. Since, in what follows, we are interested in phenomena which "occur in the limit" rather than in the nature of the convergence, we assume at the outset that the X_n's already have one of the three limit type distributions,

$$P(X_n < t) = 0, \qquad t \le 0, \tag{6.1}$$
$$= \exp -\lambda t^{-\alpha}, \quad t > 0 \qquad (\alpha > 0),$$

or

$$(6.2)\qquad \begin{aligned} P(X_n < t) &= \exp -\lambda(-t)^\alpha, \quad t \le 0 \qquad (\alpha > 0) \\ &= 1, \qquad\qquad t > 0, \end{aligned}$$

or

$$(6.3)\qquad P(X_n < t) = e^{-\lambda(\exp -t)}, \quad -\infty < t < \infty,$$

λ being a positive parameter. These three distributions can be combined in one as

$$(6.4)\qquad P(X_n < t) = \exp -\lambda Q(t),$$

with an appropriate identification of the function Q. Appropriate centering and norming constants for (6.1), (6.2), (6.3) are

$$(a_n = 0, b_n = 1/n^{1/\alpha}), \quad (a_n = 0, b_n = n^{1/\alpha}), \quad (a_n = \log n, b_n = 1),$$

respectively. It is easy to verify that the joint limiting distributions do exist. The limiting probabilities are described as follows. If

$$v_k < u_k < v_{k-1} < u_{k-1} < \cdots < v_1 < u_1,$$

then

$$(6.5)\qquad \begin{aligned} &\lim_{n\to\infty} P(\textstyle\bigcap_{i=1}^k [v_i \le (Y_n^{(i)} - a_n)/b_n < u_i]) \\ &\quad = \lambda^{k-1}[\exp -\lambda Q(u_k) - \exp -\lambda Q(v_k)] \textstyle\prod_{i=1}^{k-1} [Q(v_i) - Q(u_i)], \end{aligned}$$

with the Q as described in (6.4). This can be verified by an elementary calculation which we leave to the reader. Since $Y_n^{(k)} \le \cdots \le Y_n^{(1)}$, the probabilities (6.5) determine the probability of any Borel set in k-space. We will refer to the k-dimensional c.d.f. determined by (6.5) as

$$(6.6)\qquad F(\lambda, Q; y_1, \cdots, y_k).$$

Consider now a stochastic vector process $Y_n(t)$ defined as

$$Y_n(t) = (Y_{[nt]+1}^{(1)} - a_n)/b_n, \cdots, (Y_{[nt]+1}^{(k)} - a_n)/b_n, \qquad 0 \le t < \infty.$$

The "limiting process" $Y(t)$ "approached by $Y_n(t)$" provides one example of a multivariate extremal process. The limiting c.d.f. of $Y_n(t)$ is $F(\lambda t, Q, \cdot)$. The limiting joint distribution of m vectors $Y_n(t_1), \cdots, Y_n(t_m)$ will be described in Section 10. Just as was done in the 1-dimensional case, we will avoid these heuristics and will directly define a k-dimensional extremal process $Y(t) = Y_1(t), \cdots, Y_k(t)$ to be one in which the joint distribution law of the m k-vectors $Y(t_1), \cdots, Y(t_m)$ coincides with the limiting distribution of $Y_n(t_1), \cdots, Y_n(t_m)$.

7. Definition of multivariate extremal distribution. Let Q, a and b be as defined in Section 2.

DEFINITION 7.1. A random vector $Y_1, \cdots, Y_k$ will be said to have a k-dimensional extremal distribution with parameters λ, Q if it has the following properties:

(a) $a \leq Y_k \leq \cdots \leq Y_1 \leq b$, with probability 1.

(b) If $a < v_k < u_k < v_{k-1} < u_{k-1} < \cdots < v_1 < u_1 < b$, then

$$\begin{aligned} &P(\textstyle\bigcap_{i=1}^{k} [v_i \leq Y_i < u_i]) \\ &\quad = \lambda^{k-1}(\exp -\lambda Q(u_k) - \exp -\lambda Q(v_k)) \textstyle\prod_{i=1}^{k-1} [Q(v_i) - Q(u_i)], \end{aligned} \tag{7.1}$$

with the understanding that $\prod_{i=1}^{0} = 1$.

The probability of any Borel set in k-space is determined by (7.1). Just as in (6.6) we denote the c.d.f. of the k variables by $F(\lambda, Q; y_1, \cdots, y_k)$. Some of the properties of this distribution are the following:

(7.2) (a) If $Y_1, \cdots, Y_k$ is extremal with parameters λ, Q, then $Y_1, \cdots, Y_r$, $(r \leq k)$, is extremal with parameters λ, Q.

(b) The marginal distribution of Y_j is given by

$$P(Y_j < u) = \textstyle\sum_{i=0}^{j-1} \lambda^i e^{-\lambda Q(u)} Q^i(u)/i!.$$

In case Q is absolutely continuous with respect to Lebesgue measure,

$$P(Y_j < u) = \frac{\lambda^j}{(j-1)!} \int_a^u e^{-\lambda Q(x)} Q^{j-1}(x)(-Q'(x))\, dx. \tag{7.3}$$

When Q is absolutely continuous, properties (a) and (b) can be deduced from (7.1) in a routine way. Otherwise, the verification is somewhat tedious. However, Theorem 9.2, below, provides a fairly easy proof. (See remarks following Theorem 9.2.)

8. The operation$*$. Now we want to define an operation on vectors which generalizes the *maximum* for scalars, and which arises in a natural way with extremal processes.

DEFINITION 8.1. Suppose

$$a = (a_1, \cdots, a_k), \quad b = (b_1, \cdots, b_k), a_1 \geq \cdots \geq a_k, b_1 \geq \cdots \geq b_k.$$

By $a * b$ is meant a new k-vector whose elements are the ordered values, in decreasing order, of the k largest elements of the combined set of a_i's and b_i's. For example, if $a = (7, 3, 0)$, $b = (5, 5, -3)$, then $a * b = (7, 5, 5)$. Notice that $a * b = b * a$ and $a * (b * c) = (a * b) * c$. Notice also that if $k = 1$, then $a * b = \max(a, b)$.

The connection between the operation $*$ and extremal distributions is the following.

THEOREM 8.1. *Suppose X, Y are independent extremal k-vectors with parameters λ_1, Q; λ_2, Q respectively. Then $X * Y$ is an extremal k-vector with parameters $\lambda_1 + \lambda_2$, Q.*

A direct proof is quite tedious. However, this also will follow easily from Theorem 9.2. (See remarks following that theorem.)

9. A realization of the extremal distribution. Suppose $(Z(t),\, t \geq 0)$ is an i.d. stationary increment process with Lévy representation

$$E \exp i\theta Z(t) = \exp t \int_{-\infty}^{\infty} \left(\exp i\theta u - 1 - \frac{i\theta u}{1+u^2}\right)\left(\frac{1+u^2}{u^2}\right) dG(u) \tag{9.1}$$

where G is a mass function with finite total mass. We define $Y_1(t), \cdots, Y_k(t)$ as follows:

$$\begin{aligned} Y_i(t) &= i\text{-th largest discontinuity of } Z(u),\, 0 \leq u < t \\ &\qquad\qquad \text{if this discontinuity is not } \leq 0, \\ &= 0 \qquad\qquad \text{otherwise,} \end{aligned}$$

$i = 1, \cdots, k$. If there are fewer than i positive discontinuities, then $Y_i(t)$ is understood to equal 0.

THEOREM 9.1. *Suppose $t > 0$. Then $Y_1(t), \cdots, Y_k(t)$ has the extremal distribution with parameters t, Q,*

$$\begin{aligned} Q(u) &= \infty, & u &< 0, \\ &= \int_u^{\infty} \frac{1+x^2}{x^2}\, dG(x), & u &\geq 0. \end{aligned} \tag{9.2}$$

Proof. The proof is practically the same as the proof of (3.3), so we omit the details.

Not every extremal distribution can be realized in this way, but the distribution described in Theorem 9.1 is actually quite close to the most general one. The facts are as follows.

THEOREM 9.2. *Given any positive λ and Q there exists an i.d. stationary increment process $Z(u)$, and a monotone, non-decreasing function h, such that $h(Y_1(\lambda)), \cdots, h(Y_k(\lambda))$ is extremal with parameters λ, Q.*

Proof. First choose G in (9.1) so that $R(u) = \int_u^{\infty} [(1 + x^2)/x^2]\, dG(x)$ is continuous and strictly decreasing in $(0, \infty)$, and $R(0) = \infty$. This can be done in many ways and the particular choice is irrelevant. Define g by $g(u) = R^{-1}(Q(u))$. Suppose $[a, b]$ is the carrier of Q. The function g does not have an inverse in the conventional sense, but define $g^{-1}(y) = \sup (u : g(u) < y)$. Since g is left continuous it follows that $(g(u) \leq y)$ if and only if $(u \leq g^{-1}(y))$. Now define $h = g^{-1}$. Suppose

$$a < v_k < u_k < \cdots < v_1 < u_1 < b.$$

Then

$$P\left(\prod_{i=1}^{k} (v_i \leq h(Y_i) < u_i)\right) = P\left(\prod_{i=1}^{k} [g(v_i) \leq Y_i < g(u_i)]\right).$$

So by (7.1) and Theorem 9.1, $h(Y_1), \cdots, h(Y_k)$ is extremal with parameters $t, R(R^{-1}(Q)) = Q$. Choose $t = \lambda$ and the proof is complete.

Remarks. Theorem 9.2 can be used to provide easy proofs of the properties (7.2) and of Theorem 8.1. Since any extremal distribution can be represented in terms of an i.d. process, the properties of i.d. processes make such verifications easy. For example, consider Theorem 8.1. Suppose $X = X_1, \cdots, X_k$ and $Y = Y_1, \cdots, Y_k$ are the k largest discontinuities of $Z(t)$ over the intervals $[0, \lambda_1)$, $[\lambda_1, \lambda_1 + \lambda_2)$. Then by the independence of increments for $Z(t)$, it is clear that $X * Y$ has the distribution of the k largest discontinuities of $Z(t)$ over $[0, \lambda_1 + \lambda_2)$, which proves the theorem. Property (7.2) (a) is obvious and (7.2) (b) is verified by a direct computation in the same way.

10. Definition of extremal process.

Definition 10.1. A vector-valued process $Y(t) = Y_1(t), \cdots, Y_k(t)$, $t \geq 0$, is said to be an extremal process, with parameter Q if it has the following properties.

(a) For any positive t, $Y(t)$ has the k-dimensional c.d.f. $F(t, Q; \cdot)$.

(b) If $0 < t_1 < \cdots < t_m$, then the joint distribution of the m k-vectors $Y(t_1), \cdots, Y(t_m)$ is the same as the joint distribution of

$$U_1, (U_1 * U_2), \cdots, (U_1 * \cdots * U_m),$$

where $U_1, \cdots, U_m$ are independent with distributions

$$F(t_1, Q; \cdot), F(t_2 - t_1, Q; \cdot), \cdots, F(t_m - t_{m-1}, Q; \cdot)$$

respectively.

Remarks. (a) Given any Q, an extremal process can be defined on $(0, \infty)$ by specifying the joint distributions as in Definition 10.1. It only needs to be noted that Theorem 8.1 guarantees that the consistency requirements of Kolmogorov are satisfied.

(b) From Theorem 9.2, it follows that any extremal process can be realized as

$$h(Y_1(t)), \cdots, h(Y_k(t))$$

where $Y_1(t), \cdots, Y_k(t)$ are the k largest discontinuities of $Z(u)$, $0 \leq u < t$.

(c) From property (b) of Definition 10.1 it follows that extremal processes are Markovian.

11. A representation for $Y(t)$. We want to give an explicit representation for $Y(t)$, $0 < t_0 \leq t$, which is analogous to the representation given in the 1-dimensional case (Section 4) and which describes in a simple way the evolution of the process. We will do this by describing a process $Y^*(t)$ which is equivalent to $Y(t)$ in the sense that the finite-dimensional distributions agree. What follows makes sense only for $k \geq 2$. Let $(W_n, n = 1, 2, \cdots)$ be a sequence of independent, exponential (parameter 1) random variables. These are independent of all other random variables to be considered. Let R_y denote the c.d.f. defined by

$$\begin{aligned} R_y(u) &= 0, & u &\le y, \\ &= 1 - Q(u)/Q(y), & u &> y. \end{aligned}$$

$Y^*(t)$ is described as follows.

(a) $Y^*(t_0)$ has the k-dimensional c.d.f. $F(t_0, Q; \cdot)$.

(b) Under the condition that $Y^*(t_0) = (x_1, \cdots, x_k) = x$, the vector $Y^*(t)$ remains equal to x for a random amount of time $W_1/Q(x_k)$. The process then jumps to a new set of heights as follows. Z_1 $(>x_k)$ is a random variable whose c.d.f. (conditional on $Y^*(t_0)$ and W_1) is R_{x_k}.

$$\begin{aligned} &\text{If } x_k < Z_1 \le x_{k-1}, \text{ the process moves to } (x_1, \cdots, x_{k-2}, x_{k-1}, Z_1), \\ &\text{if } x_{k-1} < Z_1 \le x_{k-2}, \text{ the process moves to } (x_1, \cdots, x_{k-2}, Z_1, x_{k-1}), \\ &\qquad \vdots \\ &\text{if } x_2 < Z_1 \le x_1, \text{ the process moves to } (x_1, Z_1, \cdots, x_{k-2}, x_{k-1}), \\ &\text{if } x_1 < Z_1, \text{ the process moves to } (Z_1, x_1, \cdots, x_{k-2}, x_{k-1}). \end{aligned} \tag{11.1}$$

From this new set of heights, this procedure then repeats itself again, and so on.

THEOREM 11.1. *For any $0 < t_0 < t_1 < \cdots < t_n$, the joint distribution laws of $Y^*(t_0), \cdots, Y^*(t_n)$ and of $Y(t_0), \cdots, Y(t_n)$ are the same.*

Proof. Again the proof is almost the same as the proof of Theorem 4.2 of [2], so we refer the reader to it for details.

12. The marginal process $Y_i(t)$. Here we want to describe the marginal process $Y_i(t)$ which is Markovian for any i. A simple preliminary lemma is the following:

LEMMA 12.1. *Suppose $s > 0$. The conditional distribution of $Y_{i-1}(s)$ given $Y_i(s)$ does not depend on s. More generally, the conditional distribution of $Y_1(s), \cdots, Y_{i-1}(s)$ given $Y_i(s)$ does not depend on s.*

Proof. The second assertion is the easier one to verify. The verification is immediate via (7.1) and (7.3) in case Q is absolutely continuous. Otherwise the direct details are a little messy and we omit them. An indirect verification using the representation given in Theorem 9.2 is not too difficult.

Using Lemma 12.1 we can now define

$$P(Y_{i-1}(s) < u \mid Y_i(s) = v) = F_{v,i}(u),$$

$F_{v,i}$ being a c.d.f. which does not depend on s. Now using the properties of $Y^*(t)$ given in Section 11, it is easy to see that $Y_i(t)$ evolves as follows.

THEOREM 12.1. *Suppose $t_0 > 0$. Given that $Y_i(t_0) = x$, $Y_i(t)$ remains equal to x an exponential amount of time W_1 (parameter $Q(x)$), then jumps to a height Z_1 ($>x$), where*

$$P(Z_1 < z) = 0, \quad z < x,$$
$$= 1 - [1 - F_{x,i}(z)]Q(z)/Q(x).$$

From this new height the procedure repeats itself again, and so on.

Proof. That W_1 is exponential with parameter $Q(x)$ is clear from (11.1). The assertion about Z_1 is verified as follows.

$$P(Z_1 < z \mid Y_i(t_0) = x) = \int_x^\infty P(Z_1 < z \mid Y_i(t_0) = x, Y_{i-1}(t_0) = y)\, dF_{x,i}(y)$$
$$= \int_x^z 1\, dF_{x,i}(y) + \int_z^\infty [1 - Q(z)/Q(x)]\, dF_{x,i}(y)$$
$$= 1 - (1 - F_{x,i}(z))Q(z)/Q(x).$$

13. The discontinuity points are Poisson distributed. Here we want to describe the discontinuities of the vector process $Y(t)$ in (s, u), $0 < s < u$. The proofs are similar to those in [2] so we leave them to the reader. Suppose that Q is continuous and define

$$C_i(s, u) = 1 \quad \text{if } Y_i(u) > Y_i(s),$$
$$= 0 \quad \text{otherwise},$$

$i = 1, \cdots, k$, and also define $C(s, u) = (C_1(s, u), \cdots, C_k(s, u))$. (It is evident from the definition of an extremal process that $C_1 \leq C_2 \leq \cdots \leq C_k$, so if one of the entries in C is 1 then all the following ones are also.) Define also

$$M_i(s, u) = \text{number of discontinuities of } Y_i(t) \text{ in } (s, u), \quad i = 1, \cdots, k.$$

THEOREM 13.1. *Suppose that Q is continuous.*

(a) *If $0 < t_1 < \cdots < t_n$ then $C(t_1, t_2), C(t_2, t_3), \cdots, C(t_{n-1}, t_n)$ are independent random vectors.*

(b) *$M_1(s, u), M_2(s, u), \cdots, M_k(s, u)$ has a joint multivariate Poisson distribution. Specifically, it has the distribution of*

$$X_1, X_1 + X_2, \cdots, X_1 + \cdots + X_k,$$

where the X_i's are independent random variables, each Poisson distributed with parameter $\log (u/s)$.

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