A CHARACTERIZATION OF GROUPS HAVING PROPERTY P

BY

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R. W. Carter [2] has studied in great detail the groups that have property P. A finite solvable group G is said to have property P if the system normalizers of G are Carter subgroups of G. In this paper we give a characterization of groups having property P. We prove that a finite solvable group G has property P if and only if it has a subgroup U such that

(a) U is a maximal nilpotent subgroup

(b) there is a normal subgroup $N \neq \{1\}$ of G with $U \cap N \subseteq z(G)$ or $U = G$

(c) property (b) is satisfied by the image of U in any homomorphic image.

It is well known that the covering and avoidance property does not characterize the system normalizers of G. We deduce from the above result that maximal nilpotent groups of finite solvable groups having the covering and avoidance property are the system normalizers (it turns out that they are even Carter subgroups of the given group). At the end of this paper we give a property which characterizes Carter subgroups of a finite solvable group.

These results are extracted from my thesis for the degree of Ph.D. at the University of Notre Dame (1965). I take this opportunity to express my gratitude to Professor H. Zassenhaus for his help and encouragement in supervising this research.

All groups considered in this paper are finite solvable. The notations are that of [6].

LEMMA I. *If U is a maximal nilpotent subgroup of a finite solvable group and N is a normal subgroup of G satisfying $U \cap N \subseteq z(G)$, then UN/N is a maximal nilpotent subgroup of G/N.*

Proof. Let G be a group of least order for which the result is false. If $U \cap N = N_1 \neq 1$, then $UN_1/N_1 = U/N_1$ is a maximal nilpotent subgroup of G/N_1 and $U/N_1 \cap N/N_1 \subseteq z(G/N_1)$, which implies by least criminality of G that $UN/N \cong (U/N_1 \cdot N/N_1)/(N/N_1)$ is a maximal nilpotent subgroup of G/N, which gives us a contradiction. So we can assume $U \cap N = \{1\}$. Let M be a minimal normal subgroup of G, which is contained in N. Then UM/M is not a maximal nilpotent subgroup of G/M because otherwise $(UM/M \cdot N/M)/(N/M)$ is a maximal nilpotent subgroup of $(G/M)/(N/M)$, i.e. UN/N is a maximal nilpotent subgroup of G/N, which contradicts the least criminality of G.

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Let UM/M be a proper subgroup of a nilpotent group V/M contained in G/M. Because G is a least criminal, $V = G$. Let K be a minimal covering of G/M. So we get by [8] that $G = K \cdot M$ where K is nilpotent. Now $K \cap M = \{1\}$ because it is normal in G.

Since M is a minimal normal subgroup of G, M is an elementary abelian p-group for some prime p dividing the order of the group G. Let U_p, K_p be the sylow p-subgroups of U and K respectively. Let U^p, K^p be the p-compliments of U and K. By Hall's theorem [7], $U^p \subseteq g^{-1}K^p g$ for some g in G. Let $K_1 = g^{-1} \cdot K \cdot g$. Again we have $G = K_1 \cdot M$, where K_1 is nilpotent, $K_1 \cap M = \{1\}$ and $U^p \subseteq K_1^p$. Since U is maximal nilpotent we have $(C_G(U^p))_p = U_p$. Now

$$(K_1)_p \subseteq (C_G(K_1^p))_p \subseteq (C_G(U^p))_p = U_p .$$

Therefore $(K_1)_p = U_p$. Thus we have

$$U = U_p \cdot U^p \subseteq (K_1)_p \cdot K_1^p = K_1$$

and therefore $U = K_1$. Thus $UM/M = G/M$ which is a contradiction proving the lemma.

THEOREM 1. *A finite solvable group G has property P if and only if it has a subgroup U such that*

(a) *U is a maximal nilpotent subgroup*

(b) *there is a normal subgroup $N \neq \{1\}$ of G with $U \cap N \subseteq z(G)$ or $U = G$*

(c) *property* (b) *is satisfied by the image of U in any homomorphic image.*

Proof. If G is a group having property P, then by [3] and [7] any system normalizer U of G satisfies properties (a)–(c).

Conversely let G be a group of least order for which the result is not true. Without loss of generality, we can assume $U \neq G$. Let $N \neq \{1\}$ be any normal subgroup of G satisfying $U \cap N \subseteq z(G)$. Then UN/N is a maximal nilpotent subgroup of G/N by Lemma I. Therefore UN/N satisfies properties (a)–(c) and so is a Carter subgroup of G/N. Now

$$N_{U \cdot N}(U) = U \cdot (N_{U \cdot N}(U) \cap N) = U$$

since U is maximal nilpotent. This together with $N_{G/N}(UN/N) = UN/N$ implies that U is a Carter subgroup of G. By [8], U contains a system normalizer of G properly. Therefore U does not avoid at least one of the eccentric chief factors say G_{i-1}/G_i where

$$G = G_0 \supset G_1 \supset \cdots \supset G_m = \{1\}$$

is a chief series of G such that $G_{m-1} \cap U \subseteq z(G)$. Now UG_i/G_i satisfies the properties of our theorem in G/G_i and order of G/G_i is less than the order of G. Therefore UG_i/G_i avoids all the eccentric chief factors and in particular the eccentric chief factor G_{i-1}/G_i which is a contradiction. This proves our result.

COROLLARY 1. *If the system normalizers of G are maximal nilpotent, then they are Carter subgroups.*

THEOREM 2. *If U is a subgroup of a finite solvable group G that satisfies the following properties*

(a) *U is a maximal nilpotent subgroup of G*

(b) *if* $\{1\} = N_0 \subset N_1 \subset \cdots \subset N_s = G$ *is a chief series of G, then* UN_i/N_i *is a maximal nilpotent subgroup of* G/N_i. *Then U is a Carter subgroup of G.*

Proof. Let G be a group of least order for which the result is not true. Let $H = U \cdot N_1$. Then $N_H(U) = U \cdot (N_H(U) \cap N_1)$. Now U and $N_H(U) \cap N_1$ are nilpotent normal subgroups of $N_H(U)$. Therefore $N_H(U)$ is nilpotent and hence is equal to U. Now consider G/N_1. Since $| G/N_1 | < | G |$, G/N_1 is not a criminal. Now UN_1/N_1 satisfies all the conditions of the theorem. Therefore UN_1/N_1 is a Carter subgroup of G/N_1 : i.e.

$$N_{G/N_1}(UN_1/N_1) = UN_1/N_1 .$$

Now the same argument as in the previous Theorem gives us that $N_G(U) = U$ which gives us a contradiction and proves our result

COROLLARY 2. *A subgroup U of G is a Carter subgroup of G if and only if it satisfies the following conditions.*

(a) *U is a maximal nilpotent subgroup of G.*

(b) *if* σ *is any homomorphism of G, then* U^σ *is a maximal nilpotent subgroup of* G^σ.

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