EXTENSIONS OF ELEMENTARY ABELIAN GROUPS OF ORDER 2^{2n} BY $S_{2n}(2)$ AND THE DEGREE 2-COHOMOLOGY OF $S_{2n}(2)$[1]

BY

ULRICH DEMPWOLFF

Following Artin's notation (see [1]) we denote by $S_{2n}(2)$ the symplectic group of dimension $2n$ over the field F_2 of 2 elements which is defined as a subgroup of $GL(V)$, V a $2n$-dimensional vector space over F_2 leaving a non-degenerate, skew-symmetric scalar product invariant. We want to show:

THEOREM. *Let G be a finite group which satisfies the following conditions*:

(i) $V \lhd G$, *V is elementary abelian of order* 2^{2^n},

(ii) $G/V \simeq S_{2n}(2)$,

(iii) $C_G(V) \subseteq V$.

Then G/V acts on V faithfully and one can define a skew symmetric, non-degenerate scalar product which is G/V-invariant. If $n \geq 2$ then either G splits over V or G is a uniquely determined (up to equivalence of extensions) nonsplit extension of V by $S_{2n}(2)$ and such nonsplit extensions do exist.

We have a corollary.

COROLLARY. *If V denotes the standard F_2-module for $S_{2n}(2)$, then*

$$\dim_{F_2} H^2(S_{2n}(2), V) = 1 \quad \text{if} \quad n \geq 2.$$

Remark. By a result of Pollatsek [7], it is known that

$$\dim_{F_2} H^1(S_{2n}(2), V) = 1.$$

A recent result of R. Griess [5] shows that $\dim_{F_2} H^2(S_{2n}(2), V) \geq 1$. We will show that $\dim_{F_2} H^2(S_{2n}(2), V) \leq 1$ which will imply the theorem. The proof follows the same line of arguments as in [2], [3]. Thus we consider for $v \in V^{\#}$ the stabilizer H of v in G and determine the structure of $O_2(H)$. Then we study the action of $H/O_2(H)$ on $O_2(H)$. The information obtained in this way will enable us to determine the structure of G in terms of generators and relations. As the arguments used in the proof are very computational, a more group theoretic proof is certainly more desirable. We prove the theorem by a series of lemmas.

By our assumptions we may always V consider as an F_2 vector space of dimension $2n$ acted upon by G/V as a subgroup of $GL(2n, 2)$ faithfully. We will always denote by E_{ij} a square-matrix whose entries are all 0 with the sole exception of the entry 1 for the index pair (i, j). If the matrices which

Received January 22, 1973.

[1] This work was supported by the FAZIT-Stiftung.

are in question have dimension $2m$ we set for $i + j \neq 2m + 1$,

$$t_{ij} = I_{2m} + E_{ij} + E_{2m+1-j,2m+1-i}$$

and for $1 \leq i \leq 2m$,

$$t_{i,2m+1-i} = I_{2m} + E_{i,2m+1-i}$$

where I_{2m} is the $2m$-dimensional identity matrix.

It is easy to check that the subgroup X generated by t_{ij} for $i + j \leq 2m + 1$ of $GL(2m, 2)$ is isomorphic to $S_{2m}(2)$. Moreover the subgroups

$$B_0 = \langle t_{ij} \mid t_{ij} \in X;\ i < j\rangle \quad \text{and} \quad N_0 = \langle t_{i,i+1}\, t_{i+1,i}\, t_{i,i+1} \mid 1 \leq i \leq m\rangle$$

form a (B, N)-pair for X. Note that $t_{ij} = t_{2m+1-j,2m+1-i}$.

Throughout the proof we will always denote by t_{ij} either matrices of the type described above or elements of the automorphism group of a F_2-vector-space which act in respect to a fixed basis $v_1, \cdots, v_{2m}$ as described by the matrices.

(1) *Let $\mathfrak{V}$ be a $2m$-dimensional F_2 vector space and $\mathfrak{X} \simeq S_{2m}(2)$ a subgroup of $GL(\mathfrak{V})$. Then one can define a symplectic scalar product on $\mathfrak{V}$ which is $\mathfrak{X}$-invariant. In particular $\mathfrak{X}$ "acts as a symplectic group" on $\mathfrak{V}$.*

Proof. The case $m = 1$ is trivial and as $A_8 \simeq GL(4, 2)$ has only one conjugacy class of subgroups isomorphic to $\Sigma_6 \simeq S_4(2)$ the assertion is true for $m = 2$.

Choose an elementary abelian subgroup $\mathcal{E}$ in $\mathfrak{X}$ of order 3^m. As F_4 is a splitting field for $\mathcal{E}$ we have a decomposition $\mathfrak{V} = \mathfrak{V}_1 \oplus \cdots \oplus \mathfrak{V}_r$ in irreducible $\mathcal{E}$-invariant subspaces where $\dim \mathfrak{V}_i = 1$ or 2 for $1 \leq i \leq r$. As $\mathcal{E}$ acts faithfully it follows $r = m$ and $\dim \mathfrak{V}_i = 2$ for $1 \leq i \leq m$. For $e, f \in \mathcal{E}$ we introduce an equivalence relation by $e \approx f$ if and only if $\dim [e, \mathfrak{V}] = \dim [f, \mathfrak{V}]$. We have exactly $m + 1$ equivalence classes say $\mathcal{C}_0, \cdots, \mathcal{C}_m$ with $\dim [e, \mathfrak{V}] = 2i$ for $e \in \mathcal{C}_i$. Then

$$|\mathcal{C}_i| = \binom{m}{i} 2^i \quad \text{for} \quad 0 \leq i \leq m.$$

As $m \geq 3$ we have

$$(*) \qquad |\mathcal{C}_0| < |\mathcal{C}_1| < |\mathcal{C}_i| \quad \text{for} \quad 2 \leq i \leq m.$$

If $e \not\approx f$ then e and f can not be conjugate in $\mathfrak{X}$. $\mathfrak{X}$ has exactly m conjugacy classes of elements of order 3 and so $\mathcal{C}_i$ for $1 \leq i \leq m$ are the intersection of these classes with $\mathcal{E}$ and by $(*)$ and the structure of $S_{2m}(2)$ for $r \in \mathcal{C}_1$ we must have

$$C_{\mathfrak{X}}(r) = \langle r\rangle \times \mathcal{L}$$

where $\mathcal{L} \simeq S_{2m-2}(2)$ and $\mathcal{L}$ acts faithfully on $C_{\mathfrak{V}}(r)$ and trivially on $[r, \mathfrak{V}]$. Assume $r_0 \in \mathcal{C}_1$, $r_0 \neq r$ or r^{-1} and $C_{\mathfrak{X}}(r_0) = \langle r_0\rangle \times \mathcal{L}_0$, with $\mathcal{L}_0 \simeq S_{2m-2}(2)$. Then

$$C_{\mathfrak{X}}(r, r_0) = \langle r\rangle \times \langle r_0\rangle \times (\mathcal{L} \cap \mathcal{L}_0)$$

and $\mathfrak{L} \cap \mathfrak{L}_0 \simeq S_{2m-4}(2)$. By induction there is a $\mathfrak{L}_0$-admissible symplectic scalar product on $C_{\mathfrak{V}}(r_0)$ and an $\mathfrak{L}$-admissible symplectic scalar product on $C_{\mathfrak{V}}(r)$ and an $\mathfrak{L} \cap \mathfrak{L}_0$-admissible one on $C_{\mathfrak{V}}(r, r_0)$. We have $C_{\mathfrak{V}}(r) = C_{\mathfrak{V}}(r, r_0) \oplus \mathfrak{K}$ where $C_{\mathfrak{V}}(r, r_0)$ and $\mathfrak{K}$ are regular subspaces, mutually orthogonal, in respect to the scalar product of $C_{\mathfrak{V}}(r)$ which was induced on $C_{\mathfrak{V}}(r)$ by $C_{\mathfrak{X}}(r)$. In the same manner we have a orthogonal decomposition $C_{\mathfrak{V}}(r_0) = C_{\mathfrak{V}}(r, r_0) \oplus \mathfrak{K}_0$. Note that the symplectic scalar product induced on $C_{\mathfrak{V}}(r, r_0)$ by $\mathfrak{L}$, $\mathfrak{L}_0$, $\mathfrak{L} \cap \mathfrak{L}_0$ is always the same. Clearly $\mathfrak{V} = C_{\mathfrak{V}}(r_0, r) \oplus \mathfrak{K} \oplus \mathfrak{K}_0$. Reading this direct sum as a orthogonal sum we define a symplectic scalar product on $\mathfrak{V}$. Certainly this scalar product is $\mathfrak{L}$- and $\mathfrak{L}_0$-admissible. As $\mathfrak{X} = \langle \mathfrak{L}, \mathfrak{L}_0 \rangle$ the assertion follows.

Using (1) we now choose a basis $v_1, v_2, \cdots, v_{2n}$ of V such that $\{v_1, v_{2n}\}$, $\{v_2, v_{2n-1}\}, \cdots, \{v_n, v_{n+1}\}$ are hyperbolic pairs in respect to the action of G/V. In particular there are elements $\tau_{ij} \in G - V$ such that the action of τ_{ij} in respect to our fixed basis is described by matrices of the form t_{ij} (here we have $m = n$) and always $\tau_{ij}^2 \in V$.

Without proof we state:

(2) *Let the t_{ij}'s have their fixed meaning and set*

$$X = \langle t_{ij} \mid i + j \leq 2n + 1 \rangle \simeq S_{2n}(2).$$

Set $s = t_{n-1,n}\, t_{n,n-1}\, t_{n-1,n}\, t_{n+1,n+2}\, t_{n+2,n+1}\, t_{n+1,n+2}$. *Then the classes of involutions in X are represented by*

$$t_{2n,1},\ t_{2n,1}\, t_{2n-1,2},\ \cdots,\ t_{2n,1} \cdots t_{n+1,n}$$

and

$$y,\ t_{2n,1}\, t_{2n-1,2}\, y,\ \cdots,\ t_{2n,1}\, t_{2n-1,2} \cdots t_{n+4,n-3}\, t_{n+3,n-2}^{\alpha}\, y,$$

where $y = (t_{n,n+1}\, s)^2$ *and* $\alpha = 1$ *if n is even and* $\alpha = 0$ *if n is odd.*

(3) *If $\tau \in G - V$ such that $\tau^2 \in V$ then there is a t in τV such that $t^2 = 1$.*

Proof. Choose elements $\rho_1, \cdots, \rho_n$ of order 3 in $G - V$ such that ρ_i normalizes $\langle v_i, v_{2n+1-i} \rangle$ and centralizes

$$\langle v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{2n-i}, v_{2n+2-i}, \cdots, v_{2n} \rangle$$

Set $Y = \langle \rho_i \mid 1 \leq i \leq n \rangle V$. Then Y/V is elementary of order 3^n and Y/V acts fixed-point-free on V. Denote by σ an element in $G - V$ which acts as s on V where s has the meaning as in (2). Then

$$Z = \langle \tau_{2n,1}, \tau_{2n-1,2}, \cdots, \tau_{n+1,n}, \sigma \rangle$$

normalizes Y. Using a Frattini argument it follows that for every $\chi \in Z$, $\chi^2 \in V$ there is an $x \in \chi V$ with $x^2 = 1$. Now using (2) the assertion follows.

(4) *There are involutions $t_{21} \in \tau_{21} V$ and $t_{2n-1,2} \in \tau_{2n-1,2} V$ such that*

$$\langle t_{21}, t_{2n-1,2} \rangle \simeq D_8 \quad \textit{and} \quad \langle t_{21}, t_{2n-1,2} \rangle \cap V = 1.$$

Furthermore there is an involution $t_{2n,1} \in \tau_{2n,1} V$ *such that one of the following possibilities is true*

(i) $[\langle t_{21}, t_{2n-1,2}\rangle, t_{2n,1}] = 1$

(ii) $[t_{2n,1}, t_{2n-1,2}] = v_1$, $[t_{2n,1}, t_{21}] = v_{2n-1}$

and

$$[v_{2n} t_{2n,1}, \langle t_{21}, (t_{21} t_{2n-1,2})^2\rangle] = [v_{2n} t_{2n-1,2}, \langle t_{2n,1}, v_{2n} t_{2n,1}\rangle] = 1$$

Proof. As $\langle \tau_{21}, \tau_{2n-1,2}, \tau_{2n,1}\rangle$ centralizes the nonsingular symplectic subspace $\langle v_3, v_4, \cdots, v_{2n-2}\rangle$ we may restrict our attention to the case $n = 2$. Using the same Frattini argument as in (3) we find involutions t_{21}, t_{32} such that $\langle t_{21}, t_{32}\rangle \simeq D_8$ and $\langle t_{21}, t_{32}\rangle \cap V = 1$. Choose an involution t_{41} in $\tau_{41} V$ and set $\tau = t_{32}$, $\sigma = t_{21}$, $j = t_{41}$ and $\xi = \tau\sigma$. By changing j if necessary by v_3 we may assume $[\tau, j] \in \langle v_1\rangle$.

ξ^2 is an involution and so $[j, \xi^2] \in \langle v_1, v_2\rangle$. Assume $[j, \xi] = v_1^\alpha v_2^\beta v_3^\gamma v_4^\delta$, then $[j, \xi^2] = v_3^\delta w$ where $w \in \langle v_1, v_2\rangle$. So $\delta = 0$ and $[j, \xi^2] = v_1^\beta v_2^\gamma$. Now choose $\rho \in G - V$ such that

$$\xi^2 V \xrightarrow{\rho} \tau j V \xrightarrow{\rho} \tau j \xi^2 V$$

and ρ shall act fixed-point-free on $\langle \xi^2, \tau j\rangle V$. By [6; V, 8.9c)],

$$\xi^2 (\xi^2)^\rho (\xi^2)^{\rho^2} = 1.$$

Set $[\tau, j] = v_1^\chi$. Then

$$(\xi^2)^\rho \in v_4^\chi \, \tau j \langle v_1, v_2\rangle \quad \text{and} \quad (\xi^2)^{\rho^2} \in v_3^{\beta+\chi} v_4^{\gamma+\beta+\chi} \tau j \xi^2 \langle v_1, v_2\rangle.$$

On the other hand

$$(\xi^2)^{\rho^2} \in v_4^\chi \, \tau j \xi^2 \langle v_1, v_2\rangle.$$

Hence $\beta = \chi = \gamma$ and

$$[j, \tau] = v_1^\chi, \qquad [j, \xi] = v_1^\alpha v_2^\chi v_3^\chi, \qquad [j, \xi^2] = v_1^\chi v_2^\chi.$$

As $C_V(\sigma, j) = \langle v_1, v_3\rangle$ we have $[j, \sigma] = v_1^\varepsilon v_3^\delta$. So

$$v_1^\alpha v_2^\chi v_3^\chi = [j, \xi] = [j, \sigma\tau] = v_1^{\chi+\varepsilon} v_2^\delta v_3^\delta$$

which implies $\chi + \varepsilon = \alpha$ and $\delta = \chi$. If we replace j by $v_2^{\alpha+\chi} j$ and denote it now by j we have

$$[j, \tau] = v_1^\chi, \qquad [j, \xi] = v_1^\chi v_2^\chi v_3^\chi, \qquad [j, \xi^2] = v_1^\chi v_2^\chi, \qquad [j, \sigma] = v_3^\chi.$$

For $\chi = 0$ we get case (i) of (4) and $\chi = 1$ implies (ii). As an immediate corollary we have:

(4′) *If* V *is elementary of order* 2^4 *and* $G/V \simeq \Sigma_6$ *and* G *does not split over* V *an* S_2*-subgroup of* G *is uniquely determined.*

(5) *Denote by* H *the centralizer of* v_1 *in* G *and set* $A = O_2(H)$. *Then* A *possesses a* H*-admissible subgroup* X *of index* 2 *such that* $A = X\langle v_{2n}\rangle$ *and* $1 = X \cap \langle v_{2n}\rangle$ *and*

$$A/D(A) = \langle v_{2n}\rangle D(A)/D(A) \times X/D(A)$$

is an H/A-invariant decomposition where $D(A)$ denotes the Frattini subgroup of A.

Further one of the following cases is true:

(i) *X is the direct product of an extra special group Y of width $2n-2$ and type $(+)$ with a group $\langle w\rangle$ of order 2. If $V_0 = \langle v_1, \cdots, v_{2n-1}\rangle$ then $V_0 \subseteq Y$, $Z(Y) = \langle v_1\rangle$ and $Z(X) = \langle v_1, w\rangle$ is elementary abelian of order 4.*

(ii) *$X = Y\langle w\rangle$ where Y is extra special of width $2n-2$, $|w| = 4$, $[Y, \langle w\rangle] = 1$, $Y \cap \langle w\rangle = \langle v_1\rangle$ and $Z(X) = \langle w\rangle$.*

Proof. A/V is elementary abelian of order 2^{2n-1} and H/A acts faithfully on A/V and centralizes $\langle \tau_{2n,1}\rangle V/V$. Choose involutions $w_1 \in \tau_{21} V$, $w_2 \in \tau_{31} V$, $\cdots$, $w_{2n-1} \in \tau_{2n,1} V$. Then $A = \langle v_1, \cdots, v_{2n}, w_1, \cdots, w_{2n-1}\rangle$. Certainly, $A' \subseteq V$. Set $V_0 = \langle v_1, \cdots, v_{2n-1}\rangle$. Then $V_0 \lhd H$. No element in $A - V$ commutes with any element in $v_{2n} V_0$. Therefore if $a \in A$, then $a^2 \in V_0$. But clearly $V_0 \subseteq A'$ and so $A' = D(A) = V_0$ and $Z(A) = \langle v_1\rangle$. We use the "bar convention" for groups and elements in A modulo $Z(A)$. Obviously $\bar{V}_0 \subseteq Z(\bar{A})$. Further

$$C_{A/A'}(H/A) = \langle v_{2n}, w_{2n-1}\rangle A'/A'.$$

Using (4) for $\bar{x} \in (w_1 V_0)^-$ one of the following statements is true:

(i) $[\bar{x}, \bar{w}_{2n-1}] = 1$.
(ii) $[\bar{x}, (v_{2n} w_{2n-1})^-] = 1$.

As $(w_{2n-1} V_0)^-$ and $(v_{2n} w_{2n-1} V_0)^-$ are H/A-invariant cosets and H/A acts transitively on $A/V\langle w_{2n-1}\rangle$ we may assume that for each involution $\bar{x} \in \bar{A} - \bar{V}$ one of the following statements is true:

(i) $[\bar{x}, \bar{w}_{2n-1}] = 1$.
(ii) $[\bar{x}, (v_{2n} w_{2n-1})^-] = 1$.

In case (i) we set $\bar{w} = (w_{2n-1})^-$ in case (ii) we set $\bar{w} = (v_{2n} w_{2n-1})^-$. So

$$Z(\bar{A}) = Z_2(A)/\langle v_1\rangle = (V_0\langle w\rangle)^-$$

and (4) implies that for $\bar{a} \in \bar{A}$ either $\bar{a}^2 = 1$ or $((v_{2n} a)^-)^2 = 1$.

For each pair of involutions $\bar{a}, \bar{b} \in \bar{A} - (V\langle w\rangle)^-$ such that $|\langle \bar{a}, \bar{b}\rangle (V\langle w\rangle)^-/(V\langle w\rangle)^-| = 4$ there is an element ρ of order 3 in H such that ρ permutes the elements in

$$\langle \bar{a}, \bar{b}\rangle^{\#} (V\langle w\rangle)^-/(V\langle w\rangle)^-.$$

Hence $\bar{a}^{\rho} \in \bar{b}(V_0\langle w\rangle)^-$ and $\bar{a}^{\rho^2} \in (ab)^-(V\langle w\rangle)^-$. Assume $(ab)^-$ is not an involution, i.e. $[\bar{a}, \bar{b}] \neq 1$. As $\bar{a}^{\rho^2}$ is an involution we have $\bar{a}^{\rho^2} \in (abv_{2n})^-(V_0\langle w\rangle)^-$ by the above. So $(abv_{2n})^-$ has order 2 and therefore

$$1 = [\bar{a}, \bar{b}][(ab)^-, (v_{2n})^-].$$

Therefore if $\bar{a}, \bar{b} \in \bar{A} - (V\langle w\rangle)^-$ are noncommuting involutions, then

$$[\bar{a}, \bar{b}] = \bar{v}_{2n}^{(ab)^-}\bar{v}_{2n}$$

Assume $\bar{b}$ and $\bar{c}$ are commuting involutions in $\bar{A}$ such that

$$(V\langle w\rangle)^- \neq (bV\langle w\rangle)^- \neq (cV\langle w\rangle)^- \neq (V\langle w\rangle)^-.$$

Assume further that $\bar{a}$ does not commute with $\bar{c}$; then using that $\bar{A}$ has class 2 we conclude

$$[\bar{a}, \bar{b}\bar{c}] = [\bar{a}, \bar{b}][\bar{a}, \bar{c}]$$

and

$$\begin{aligned}[\bar{a}, (bc)^-] &= \bar{v}_{2n}(\bar{v}_{2n})^{(abc)^-} && \text{if} \quad [\bar{a}, (bc)^-] \neq 1 \\ &= 1 && \text{if} \quad [\bar{a}, (bc)^-] = 1\end{aligned}$$

and

$$\begin{aligned}[\bar{a}, \bar{b}][\bar{a}, \bar{c}] &= \bar{v}_{2n}(\bar{v}_{2n})^{(ac)^-}\bar{v}_{2n}(\bar{v}_{2n})^{(ab)^-} && \text{if} \quad [\bar{a}, \bar{b}] \neq 1 \\ &= \bar{v}_{2n}(\bar{v}_{2n})^{(ac)^-} && \text{if} \quad [\bar{a}, \bar{b}] = 1.\end{aligned}$$

So we may conclude, if $\bar{a} \in \bar{A} - (V\langle w\rangle)^-$ is an involution and there is an involution $\bar{b} \in \bar{A} - \langle\bar{a}\rangle(V\langle w\rangle)^-$ which does not commute with $\bar{a}$ then no involution in $\bar{A} - \langle\bar{a}\rangle(V\langle w\rangle)^-$ will commute with $\bar{a}$. Assume this is the case. Let $\bar{a} \in \bar{A} - (V\langle w\rangle)^-$ and $\bar{b} \in \bar{A} - (V\langle w, a\rangle)^-$ be involutions; then $[\bar{v}_{2n}\,\bar{a}, \bar{v}_{2n}\,\bar{b}] = 1$ and it follows that $\bar{A}_0 = \langle Z(\bar{A}), (v_{2n}\,w_1)^-, \cdots, (v_{2n}\,w_{2n-2})^-\rangle$ is an abelian subgroup of index 2 in $\bar{A}$ and $\bar{A}_0$ is of type $(2, 4, 4, \cdots, 4)$. Let A_0 be the counter image of $\bar{A}_0$ in A. Then A_0 has class 2 and $Z(A_0) \subseteq \langle v_1, w\rangle$. Now choose $a \in A_0$ of order 4. Then for every $b \in A_0$ we have $[a^2, b] = [a, b]^2 = 1$. So $\bar{X} = \langle \bar{x}^2 \mid \bar{x} \in \bar{A}_0\rangle \subseteq (Z(\bar{A}_0))^-$. But as $a^2 \in v_{2n}\,v_{2n}^a\langle v_1\rangle$ for each a of order 4 it follows $| X \cap V_0 | \geq 4$, a contradiction.

We have shown that every pair of involutions in $\bar{A} - \bar{V}\langle\bar{w}\rangle$ commutes. It follows that $\bar{X} = \langle Z(\bar{A}), \bar{w}, \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_{2n-2}\rangle$ is elementary abelian of order 2^{4n-3}. Let X be the complete counter image of X. Then $X' = D(X) = \langle v_1\rangle$ and $Z(X) = \langle v_1, w\rangle$.

Set $Y = \langle V_0, w_1, \cdots, w_{2n-2}\rangle$. Then Y is extra special of type $(+)$ as $V_0 \subseteq Y$. (An extra special group X of order 2^{2n+1} is called of type $(+)$ if it contains an elementary abelian group of order 2^{n+1}.) Finally $\bar{X}$ char $\bar{A}$: As for every $\bar{x} \in \bar{v}_{2n}\,Z(\bar{A})$ we have $C_{\bar{A}}(\bar{x}) = \langle\bar{x}\rangle Z(\bar{A})$ it follows that an automorphism α of $\bar{A}$ has the property $\bar{X}^\alpha \cap \bar{v}_{2n}\,Z(\bar{A}) = \emptyset$

As $\bar{A} - (\bar{v}_{2n}\,Z(\bar{A}) \cup \bar{X})$ is the set of elements of order 4 we have shown that X is H-admissible.

The following fact is an easy consequence of the result of Pollatsek [7] and (1).

(6) *Let $\mathfrak{V}$ be a $(2n + 1)$-dimensional F_2 vector space and assume there is a subgroup $\mathfrak{X} \simeq S_{2n}(2)$ of $GL(\mathfrak{V})$ such that $\mathfrak{X}$ centralizes $v \in \mathfrak{V}^\#$ and acts faithfully on $\mathfrak{V}/\langle v\rangle$. Suppose there is no $\mathfrak{X}$-admissible complement of $\langle v\rangle$ in $\mathfrak{V}$. Then $\mathfrak{V}$ has a basis $v, v_1, \cdots, v_{2n}$ such that $\{v_i + \langle v\rangle, v_{2n+1-i} + \langle v\rangle\}$ are hyperbolic pairs with respect to the action of $\mathfrak{X}$ on $\mathfrak{V}/\langle v\rangle$ for $1 \leq i \leq n$. If $\mathfrak{x} \in \mathfrak{X}$ is represented on $\mathfrak{V}/\langle v\rangle$ in respect to the basis $v_i + \langle v\rangle (1 \leq i \leq 2n)$ by the matrix $X = (x_{ij})$*

then the matrix of $\mathfrak{x}$ with respect to the basis of $\mathfrak{V}$ has the form

$$\begin{bmatrix} 1 & 0 \\ K(X) & X \end{bmatrix} \quad \textit{where} \quad K(X) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n} \end{pmatrix} \quad \textit{and} \quad \alpha_i = \sum_{j=1}^{n} x_{ij} x_{i,2n+1-j} .$$

Now (5) implies the existence of an H-admissible subgroup $X \subset A$ such that $X = Y\langle w\rangle$ and we have either $X = Y \times \langle w\rangle$ and $|w| = 2$ or $X = Y\langle w\rangle$, $\langle w\rangle \cap Y = Z(Y)$, $|w| = 4$ where Y is in both cases an extra special 2-group of width $n - 1$ and type $(+)$. We consider $\mathfrak{V} = X/\langle v_1\rangle$ as a F_2 vector space. If we define $q(\alpha) = a^2$ where $\alpha \in \mathfrak{V}$ and $a \in \alpha$ and $(\alpha, \beta) = [a, b]$ for $\alpha, \beta \in \mathfrak{V}$ and $a \in \alpha$, $b \in \beta$, then q is a quadratic form on $\mathfrak{V}$ and $(\ , \ )$ is the symplectic bilinear form belonging to q. We have to distinguish two cases according to the structure of X.

(i) $|w| = 2$, $Y \cap \langle w\rangle = 1$. Then $\mathfrak{V}$ is a orthogonal vectorspace such that rad $\mathfrak{V} = \langle w\langle v_1\rangle\rangle$, $\mathfrak{V}/\langle w\langle v_1\rangle\rangle$ is a regular orthogonal vector space of maximal index and dimension $4n - 4$.
(ii) $|w| = 4$, $Y \cap \langle w\rangle = Z(X)$. Then $\mathfrak{V}$ is a $(4n - 3)$-dimensional, regular orthogonal vector space.

According to (6) and the proof of (5) we have to study the following situation (here $n = m + 1$):
Given a $(4m + 1)$-dimensional F_2 vector space $\mathfrak{V}$ with a basis $w, w_1, \cdots, w_{2m}, v_1, \cdots, v_{2m}$ and an orthogonal form q and a bilinear form $(\ , \ )$ such that either

(i)
$$\begin{gathered} q(w) = 0, \quad q(v_i) = q(w_i) = 0 \quad \text{for} \quad 1 \le i \le 2m \\ (w, v) = 0 \quad \text{for all} \quad v \in \mathfrak{V}, \\ (v_i, w_j) = \delta_{ij} \quad \text{for} \quad 1 \le i,j \le 2m, \\ (v_i, v_j) = (w_i, w_j) = 0 \quad \text{for} \quad 1 \le i,j \le 2m, \\ q(\textstyle\sum_i a_i v_i + \sum_j b_j w_j + cw) = \sum_{i=1}^{2m} a_i b_i \end{gathered}$$

or

(ii)
$$\begin{gathered} q(w) = 1, \quad q(v_i) = q(w_i) = 0 \quad \text{for} \quad 1 \le i \le 2m, \\ (w, v) = 0 \quad \text{for all} \quad v \in \mathfrak{V}, \\ (v_i, w_j) = \delta_{ij} \quad \text{for} \quad 1 \le i,j \le 2m, \\ (v_i, v_j) = (w_i, w_j) = 0 \quad \text{for} \quad 1 \le i,j \le 2m, \\ q(\textstyle\sum a_i v_i + \sum_j b_j w_j + cw) = \sum_i a_i b_i + c. \end{gathered}$$

By (5) it is clear that $H/X \simeq S_{2n-2}(2) \times Z_2$. Thus there is a subgroup $\mathfrak{X} \simeq S_{2m}(2)$ of $GL(\mathfrak{V})$ such that $\mathfrak{X}$ normalizes $\mathfrak{V}_1 = \langle v_1, \cdots, v_{2m}\rangle$ and respects the form q and the scalar product $(\ , \ )$. Note that only in the case $m \geq 3$ the group $\mathfrak{X}$ corresponds to a unique subgroup of H/X.

Furthermore the structure of H tells us that $\mathfrak{X}$ acts reducibly but not completely reducibly on $\mathfrak{V}/\mathfrak{V}_1$ and centralizes in particular $w + \mathfrak{V}_1$. In any case we may assume that we have chosen v_i, w_i for $1 \leq i \leq 2m$ in such a way, that if $\mathfrak{x} \in \mathfrak{X}$ induces the matrix X on $\mathfrak{V}/\langle\mathfrak{V}_1, w\rangle$ with respect to the basis $w_1 + \langle\mathfrak{V}_1, w\rangle, \cdots, w_{2m} + \langle\mathfrak{V}_1, w\rangle$ that the matrix induced by $\mathfrak{x}$ with respect to the basis $w, v_1, \cdots, v_{2m}, w_1, \cdots, w_{2m}$ of $\mathfrak{V}$ has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & S(X) & 0 \\ K(X) & Y(X) & X \end{bmatrix}$$

There $K(X)$ denotes the function described in (6). As $(v_i, w_j) = \delta_{ij}$ for $1 \leq i, j \leq 2m$ we have $S(X) = (X^{-1})^t$ and Y is function such that

$$\text{(A)} \qquad Y(XZ) = Y(X)(Z^{-1})^t + XY(Z).$$

As we have $(w_i, w_j) = 0$ for $1 \leq i, j \leq 2m$ it follows that

$$\text{(B)} \qquad Y(X)X^t = X(Y(X))^t.$$

If we set $Y(X) = (y_{ij})$, $X = (x_{ij})$ and $K(X) = (k_i)$ for $1 \leq i \leq 2m$ implies

$$\text{(C)} \qquad \begin{aligned} 0 &= \textstyle\sum_{l=1}^{2m} y_{il}\, x_{il} && \text{for case (i).} \\ &= \textstyle\sum_{l=1}^{2m} y_{il}\, x_{il} + k_i && \text{for case (ii).} \end{aligned}$$

In other words the diagonal elements of $Y(X)X^t$ are 0 in case (i) and equal k_i in case (ii).

We now determine the function Y in case (i) as well as case (ii). Therefore we set of $1 \leq i, j \leq 2m$,

$$K_{ji} = Y(t_{ij}),$$

$$K_{ij} = (k_{rs}^{ij}) \quad \text{for} \quad 1 \leq r,s \leq 2m,$$

$$\tau_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t_{ij} & 0 \\ K(t_{ij}) & K_{ij} & t_{ji} \end{bmatrix}.$$

Using $\tau_{ij}^2 = 1$ it follows that $0 = K_{ij}\, t_{ij} + t_{ji}\, K_{ij}$. Further $K_{ij}\, t_{ij} = t_{ji}(K_{ij})^t$.

These equations imply for $i + j \neq 2m + 1$,

$$k_{sl}^{ij} = k_{ls}^{ij} \quad \text{for all} \quad 1 \leq s,l \leq 2m,$$

$$k_{il}^{ij} = k_{2m+1-j,l}^{ij} = 0 \quad \text{for all} \quad 1 \leq l \leq 2m, l \neq j, 2m+1-i,$$

$$k_{i,2m+1-i}^{ij} = k_{2m+1-j,j}^{ij}.$$

Finally using equation (C) we have

$$k_{il}^{ij} = 0 \quad \text{for} \quad 1 \leq l \leq 2m \quad \text{and} \quad l \neq j, 2m+1-j,$$

$$k_{ij}^{ij} = k_{jj}^{ij}, \qquad k_{2m+1-j,2m+1-i}^{ij} = k_{2m+1-i,2m+1-i}^{ij}.$$

If $i + j = 2m + 1$ we have again

$$K_{i,2m+1-i} = (K_{i,2m+1-i})^t$$

and

$$k_{il}^{i,2m+1-i} = 0 \quad \text{for all} \quad 1 \leq l \leq 2m, \qquad l \neq 2m + 1 - i$$

$$k_{i,2m+1-i}^{i,2m+1-i} = k_{2m+1-i,2m+1-i}^{i,2m+1-i} + \varepsilon$$

where $\varepsilon = 0$ in case (i) and $\varepsilon = 1$ in case (ii).

Using the equation $[\tau_{ij}, \tau_{rs}] = 1$ for $\{r, s\} \cap \{i, j, 2m + 1 - i, 2m + 1 - j\} = \emptyset$ we get by (A) the equation

$$K_{ij}\, t_{rs} + t_{ji}\, K_{rs} = K_{rs}\, t_{ij} + t_{sr}\, K_{ij}$$

which implies

$$k_{rs}^{ij} = 0 \quad \text{for all} \quad \{r, s\} \cap \{i, j, 2m + 1 - i, 2m + 1 - j\} = \emptyset$$

and $r + s \neq 2m + 1$. Furthermore we have for $\{r, s\} \cap \{i, j, 2m + 1 - i,$ $2m + 1 - j\} = \emptyset$ always $k_{r,2m+1-r}^{ij} = k_{s,2m+1-s}^{ij}$.

Using $[\tau_{rs}, \tau_{i,2m+1-i}] = 1$ for all $\{r, s\} \cap \{i, 2m + 1 - i\} = \emptyset$ we get

$$K_{rs}\, t_{i,2m+1-i} + t_{sr}\, K_{i,2m+1-i} = t_{2m+1-i,i}\, K_{rs} + K_{i,2m+1-i}\, t_{rs}$$

and it follows that

$$k_{rs}^{i,2m+1-i} = 0 \quad \text{for all} \quad \{r, s\} \cap \{i, 2m + 1 - i\} = \emptyset$$

and $r + s \neq 2m + 1$. Furthermore we have $k_{r,2m+1-r}^{ij} = k_{s,2m+1-s}^{ij}$ for $\{r, s\} \cap \{i, 2m + 1 - i\} = \emptyset$.

Therefore we can write for $r + s \neq 2m + 1$,

$$\begin{aligned} K_{rs} = {} & \textstyle\sum_{l \neq s} \alpha_l(r, s)(E_{sl} + E_{ls}) + \sum_{l \neq 2m+1-r} \beta_l(r, s)(E_{2m+1-r,l} \\ & + E_{l,2m+1-r}) + \alpha_r(r, s)E_{ss} + \beta_{2m+1-s}(r, s)E_{2m+1-r,2m+1-r} \\ & + \gamma(r, s) \textstyle\sum_{k \neq r,s,2m+1-s,2m+1-r} E_{k,2m+1-k} \end{aligned}$$

Note that the entry for the index $(2m + 1 - r, s)$ and $(s, 2m + 1 - r)$ is $\alpha_{2m+1-r}(r, s) + \beta_s(r, s)$. We will later denote this entry by $\varepsilon(r, s)$. Further for all $1 \leq r \leq 2m$,

$$\begin{aligned} K_{r,2m+1-r} = {} & \textstyle\sum_{k \neq 2m+1-r} \alpha_k(r, 2m + 1 - r)(E_{2m+1-r,k} + E_{k,2m+1-r}) \\ & + \tilde{\alpha}_r(r, 2m + 1 - r)E_{2m+1-r,2m+1-r} \\ & + \gamma(r, 2m + 1 - r) \textstyle\sum_{k \neq r,2m+1-r} E_{k,2m+1-k} \end{aligned}$$

Here $\tilde{\alpha}_r(r, 2m + 1 - r) = \alpha_r(r, 2m + 1 - r)$ in case (i) and

$$\tilde{\alpha}_r(r, 2m + 1 - r) = \alpha_r(r, 2m + 1 - r) + 1 \qquad \text{in case (ii).}$$

The equation $[\tau_{rs}, \tau_{rj}] = 1$ for $j \neq 2m + 1 - r, s$ implies

$$\beta_r(r, j) = \gamma(r, j) + \alpha_{2m+1-j}(r, s) \tag{1.1}$$

and $[\tau_{rs}, \tau_{js}] = 1$ for $j \neq 2m+1-s, r$ gives us

$$(2.1) \qquad \beta_{2m+1-s}(j, s) = \beta_{2m+1-s}(r, s).$$

The equation $[\tau_{rs}, \tau_{r,2m+1-r}] = 1$ leads to

$$(3.1) \qquad \alpha_r(r, s) + \alpha_r(r, 2m+1-r) = \gamma(r, 2m+1-r)$$

and since $[\tau_{r,2m+1-s}, \tau_{s,2m+1-s}] = 1$ it follows that

$$(4.1) \qquad \alpha_s(s, 2m+1-s) + \beta_s(r, 2m+1-s) = \gamma(s, 2m+1-s).$$

For $m \geq 3$ and $j \neq r, s, 2m+1-r, 2m+1-s$ the equation $[\tau_{rs}, \tau_{j,2m+1-j}] = 1$ implies

$$(5.1) \qquad \alpha_r(j, 2m+1-j) = \alpha_j(r, s),$$

$$(5.2) \qquad \alpha_{2m+1-s}(j, 2m+1-j) = \beta_j(r, s).$$

The equation $[\tau_{j,2m+1-j}, \tau_{r,2m+1-r}] = 1$ for $j \neq 2m+1-r$ gives us

$$(6.1) \qquad \alpha_r(j, 2m+1-j) = \alpha_j(r, 2m+1-r).$$

For $j + s \neq 2m+1 \neq r+s$ and $j \neq r$ we have $[\tau_{rs}, \tau_{sj}] = \tau_{rj}$ which implies

$$K_{rs}\, t_{sj}\, t_{rs} + t_{sr}\, K_{sj}\, t_{rs} + t_{sr}\, t_{js}\, K_{rs} = K_{rj}\, t_{sj} + t_{jr}\, K_{sj}.$$

Computing both sides of this equation yields

$$(7.1) \qquad \varepsilon(r, s) + \varepsilon(r, j) + \varepsilon(s, j) + \beta_s(r, j) + \alpha_{2m+1-j}(s, j) = \gamma(r, s),$$

$$(7.2) \qquad \alpha_r(r, s) + \alpha_r(s, j) = \alpha_s(r, j),$$

$$(7.3) \qquad \alpha_{2m+1-j}(r, s) + \beta_r(s, j) = \gamma(r, j),$$

$$(7.4) \qquad \beta_{2m+1-j}(r, s) + \beta_{2m+1-s}(r, j) = \beta_{2m+1-j}(r, j).$$

To obtain these equations we must have $m \geq 3$. If $m \geq 4$, then we also obtain $\gamma(r, j) = 0$. The equation $[\tau_{ik}, \tau_{k,2m-1-k}] = \tau_{i,2m-1-k}\, \tau_{i,2m-1-i}$ implies

$$\begin{aligned} K_{ik}\, t_{k,2m+1-k}\, t_{ik} &+ t_{ki}\, K_{k,2m+1-k}\, t_{ik} + t_{ki}\, t_{2m+1-k,k}\, K_{ik} \\ &= K_{i,2m+1-k}\, t_{i,2m+1-i}\, t_{k,2m+1-k} + t_{2m+1-k,i}\, K_{i,2m+1-i}\, t_{k,2m+1-k} \\ &\quad + t_{2m+1-k,i}\, t_{2m+1-i,i}\, K_{k,2m+1-k}. \end{aligned}$$

And therefore we have

$$\begin{aligned} (8.1) \quad & \varepsilon(i, k) + \alpha_{2m+1-k}(i, k) + \varepsilon(i, 2m-1-k) + \tilde{\alpha}_k(k, 2m+1-k) \\ &= \beta_k(i, 2m+1-k) + \alpha_{2m+1-k}(i, 2m+1-i) + \alpha_k(i, 2m+1-i) \\ &\quad + \alpha_i(k, 2m+1-k) + \alpha_k(k, 2m+1-k) + \gamma(i, 2m+1-i), \end{aligned}$$

$$\begin{aligned} (8.2) \quad & \alpha_i(i, k) + \alpha_k(i, 2m+1-k) \\ &= \alpha_i(k, 2m+1-k) + \gamma(i, 2m+1-i), \end{aligned}$$

$$(8.3)\quad \begin{aligned} \varepsilon(i, k) + \tilde{\alpha}_k(k, 2m+1-k) + \varepsilon(i, 2m+1-k) + \alpha_{2m+1-k}(i, k) \\ + \alpha_i(i, 2m+1-k) + \alpha_i(i, 2m+1-i) \\ = \gamma(k, 2m+1-k) + \alpha_{2m+1-k}(i, 2m+1-i), \end{aligned}$$

$$(8.4)\quad \begin{aligned} \alpha_i(i, k) + \alpha_k(k, 2m+1-k) + \alpha_i(k, 2m+1-k) \\ + \beta_k(i, 2m+1-k) + \alpha_k(i, 2m+1-i) \\ = \gamma(k, 2m+1-k). \end{aligned}$$

Also if $m \geq 3$ we obtain $\gamma(i, 2m+1-i) = \gamma(i, 2m+1-k)$. First we assume $m \geq 3$. Using our fixed basis we define $\varphi \in \mathrm{Aut}\,(\mathfrak{V})$ by

$$\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{2m} & 0 \\ 0 & S & I_{2m} \end{bmatrix}.$$

S is a $2m \times 2m$ matrix with entries $s_{ij} = \alpha_i(j, 2m+1-j)$ for $i \neq j$ and $i + j \neq 2m+1$. Further set $s_{ii} = s_{i,2m+1-i} = 0$. By (6.1), S is well defined. We replace τ_{ij} by $\varphi^{-1}\tau_{ij}\varphi$ and denote these elements again by τ_{ij}. (This operation is nothing else then replacing the basis $w, v_1, \cdots, v_{2m}, w_1, \cdots, w_{2m}$ by $w^\varphi, v_1^\varphi, \cdots, v_{2m}^\varphi, w_1^\varphi, \cdots, w_{2m}^\varphi$ which has the same properties as the old one.) We have

$$\begin{aligned} K_{i,2m+1-i} = \alpha_i(i, 2m+1-i)(E_{2m+1-i,i} + E_{i,2m+1-i}) \\ + \tilde{\alpha}_i(i, 2m+1-i)E_{2m+1-i,2m+1-i} \\ + \gamma(i, 2m+1-i)\textstyle\sum_{k\neq i,2m+1-i} E_{k,2m+1-k}. \end{aligned}$$

Using (5.1) and (5.2) we have

$$\begin{aligned} K_{rs} = {} & \alpha_r(r, s)(E_{sr} + E_{ss} + E_{rs}) \\ & + \beta_{2m+1-s}(r, s)(E_{2m+1-r,2m+1-s} + E_{2m+1-s,2m+1-r} + E_{2m+1-r,2m+1-r}) \\ & + \varepsilon(r, s)(E_{s,2m+1-r} + E_{2m+1-r,s}) \\ & + \gamma(r, s)\textstyle\sum_{k\neq r,s,2m+1-r,2m+1-s} E_{k,2m+1-k} \end{aligned}$$

If $m \geq 4$ then at once $\gamma(k, 2m+1-k) = \gamma(r, s) = 0$, but also (7.3) does imply this equation. Combining (7.2), (7.4), and (8.4) we get finally

$$(+)\quad \begin{aligned} K_{i,2m+1-i} &= \tilde{\alpha}_i(i, 2m+1-i)E_{2m+1-i,2m+1-i}, \\ K_{ik} &= \varepsilon(i, k)(E_{2m+1-i,k} + E_{k,2m+1-i}) \end{aligned}$$

Looking in the proof of (5) and using the terminology of (5) we have

$$H/X \simeq S_{2n-2}(2) \times Z_2$$

where Z_2 corresponds to the coset $v_{2n}X$. So in the case of $m \leq 2$ we may choose $\mathfrak{X} \simeq S_{2m}(2)$ suitably such that $\gamma(k, 2m+1-k) = 0$ by using $(t_{k,2m+1-k}\, t_{2m+1-k,k})^3 = 1$. So we get the equations (+) in the case

$m = 1$. In the case $m = 2$ again we may assume

$$K_{i,2m+1-i} = \alpha_i(i, 2m+1-i)(E_{2m+1-i,i} + E_{i,2m+1-i}) + \tilde{\alpha}_i(i, 2m+1-i)E_{2m+1-i,2m+1-i}.$$

The equation $(\tau_{rs}\,\tau_{sr})^3 = 1$ implies

$$\varepsilon(r, s) + \varepsilon(s, r) = \alpha_{5-s}(r, s) = \alpha_{5-r}(s, r) \quad \text{and} \quad \beta_{5-r}(s, r) = \beta_{5-s}(r, s).$$

Then (6.1), (3.1), (4.1), (8.2), and (8.4) again imply finally the equations (+). Now set

$$\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{2m} & 0 \\ 0 & \Delta & I_{2m} \end{bmatrix} \in \operatorname{Aut}(\mathfrak{V})$$

where $\Delta^t = \Delta$ and $\Delta = \sum_{l=1}^{2m} a_{l,2m+1-l}\, E_{l,2m+1-l}$. Choose $a_{i,2m+1-i}$ in such a way that $a_{2m,1} + a_{s,2m+1-s} = \varepsilon(2m, s)$ for $m + 1 \leq s \leq 2m - 1$.

We replace now τ_{ij} by τ_{ij}^{γ} and denote these elements again by τ_{ij}. So we may assume that $\varepsilon(2m, s) = 0$ for $m + 1 \leq s \leq 2m - 1$.

(I) Assume that we are in case (i). We have $\tilde{\alpha}_i(i, 2m+1-i) = \alpha_i(i, 2m+1-i) = 0$ and so $K_{i,2m+1-i} = 0$. Using equation (8.3) we have $\varepsilon(i, k) = \varepsilon(i, 2m+1-k)$. With (7.1) we get $\varepsilon(s, \mu) = \varepsilon(2m, s) + \varepsilon(2m, \mu) = 0$ if $2 \leq s, \mu \leq 2m - 1$. For $2 \leq k \leq 2m - 1$ we have further

$$\begin{aligned} 0 = \varepsilon(2m, k) &= \varepsilon(2m+1-k, 1) = \varepsilon(2m+1-k, 2m+1-1) \\ &= \varepsilon(2m+1-k, 2m). \end{aligned}$$

So for all possible k, r we have $\varepsilon(r, k) = 0$

(II) Assume that we are in case (ii). We have $\tilde{\alpha}_i(i, 2m+1-i) = 1$ and (8.3) implies $\varepsilon(i, k) = \varepsilon(i, 2m+1-k) + 1$. Hence $\varepsilon(2m, l) = 1$ for $2 \leq l \leq m$. (7.1) implies $\varepsilon(s, \mu) = \varepsilon(2m, s) + \varepsilon(2m, \mu)$. Hence $\varepsilon(s, \mu) = 1$ for

$$(s, \mu) \in \{2, \cdots, m\} \times \{m+1, \cdots, 2m-1\} \cup \{m+1, \cdots, 2m-1\} \times \{2, \cdots, m\}$$

Finally $\varepsilon(2m, k) = \varepsilon(2m+1-k, 1) = \varepsilon(2m+1-k, 2m) + 1$ and so

$$\begin{aligned} \varepsilon(j, 2m) &= 1 \quad \text{for} \quad 2 \leq j \leq m \\ &= 0 \quad \text{for} \quad m+1 \leq j \leq 2m-1 \end{aligned}$$

If we summarize the results of (I) and (II) we can state:

(7) *There is a subgroup K of H of index 2 with $K \cap A = X$. If we are in case* (i) *of* (5) *then there is an elementary group W of A such that $WV = A$, $W \cap V = Z(X)$ and W is K/A-admissible.*

If we are in case (ii) *of* (5) *K/X acts reducibly but not completely reducibly on X/X'. The action of K/X on $V/\langle v_1\rangle$ is uniquely determined.*

(8) *If we are in case* (i) *of* (5) *H and G splits over V.*

Proof. First we assume that we are in the situation of case (i) of (5). Keeping the same notation we have $A = \langle v_{2n}\rangle X$ where X is a H/X-admissible group which is isomorphic to the direct product of a group $\langle w\rangle$ of order 2 with an extra special 2-group Y of width $n - 1$ and type $(+)$.

As every involution in $H/A \simeq S_{2n-2}(2)$ has a pre-image which is an involution too (see for instance (3)), we have by [4] a subgroup $H_0 \subset H$ such that $H_0 A = H$ and $H_0 \cap A = X$. $X/\langle w\rangle$ is an extra special group and the situation of (7) applies further to X/X'. Hence there is an elementary abelian group $W \subseteq A$ such that $V_0 W = X$, $V_0 \cap W = Z(X)$ and W is H_0/X-admissible where V_0 has the same meaning as in the proof of (5). By the structure of $GL(2n-1, 2)$ we can find a subgroup H_1 of H_0 such that $H_1 X = H_0 H_1 \cap A = W$ and H_1/W acts on W in such a way that $\langle v_1\rangle$ has a H_1/X-invariant complement W_0, with $W_0 \times \langle v_1\rangle = W$. On the other hand there is a subgroup H_2 of H such that $H_2 X = H_0$ and $H_2 \cap X = V_0$.

Hence with the modular law

$$H_1 = H_1 \cap H_0 = H_1 \cap H_2 W = (H_1 \cap H_2)W.$$

Hence $H_1 \cap H_2 = Z(X)$ and $(H_1 \cap H_2)/Z(X) \simeq S_{2n-2}(2)$. As every involution in $(H_1 \cap H_2)/Z(X)$ has a pre-image which is an involution we have subgroup $H_3 \subset H_1 \cap H_2$, $H_3 \simeq S_{2n-2}$, $H_3 A = H$ and H_3 normalizes W_0. So $W_0 H_3 \cap V = 1$ and hence we get the assertion for the case (i) by a result of Gaschütz [6; I, 17.4].

From now on we have only to handle the situation described in case (ii) of (5). For $1 \leq i, j \leq 2n$ and $i + j \neq 2n + 1$ we choose involutions t_{ij} which act as it is suggested by the notation (use (3)). For $1 \leq i \leq 2n$ choose elements $t_{i,2n+1-i}$ of order 4 such that $t_{i,2n+1-i}$ acts on V in the way suggested by the notation. By (3) it follows $(t_{i,2n+1-i})^2 = v_{2n+1-i}$.

For $1 \leq i \leq 2m$ we set $H_i = C_G(v_i)$ and $A_i = O_2(H_i)$. With this notation we have

$$H_i = \langle t_{rs} \mid 1 \leq r, s \leq 2n; \{s, r\} \cap \{i, 2n+1-i\} = \emptyset\rangle . V . \langle t_{ri} \mid 1 \leq r \leq 2n; i \neq r\rangle$$

and

$$A_i = V\langle t_{ri} \mid 1 \leq r \leq 2n; i \neq r\rangle.$$

In the course of the following argument we are going to modify the t_{ij} by elements in V step by step. We say the k-th component of t_{ij} is determined if we do not change t_{ij} in the course of the argument by v_k any more. We always make use of the action of H_i/X_i on X_i/X_i' as it was developed in the proof of (7) where X_i corresponds to the subgroup X of A. As we will not change the order of the t_{ij} the i- and the $(2n + 1 - j)$-component of these elements are already determined.

First we consider H_{2n}. We may assume that we have choosen $t_{1,2n}, t_{2,2n}, \cdots, t_{2n-1,2n}$ in such a way that $\langle v_2, t_{2,2n}\rangle, \cdots, \langle v_{2n-1}, t_{2n-1,2n}\rangle$ are dihedral groups of order 8 and $\langle t_{1,2n}\rangle$ is of order 4 and X_{2n} is the central product of these groups, where X_{2n} corresponds to the group X of (5).

As $\langle t_{1,2n}\rangle$ commutes with all the dihedral groups, it follows that all components with exception of the $2n$-component of $t_{1,2n}$ are determined.

Further there is a skew symmetric matrix $\emptyset = (\varphi_{ij})$ with $2 \leq i, j \leq 2n - 1$ and numbers $\alpha(i, j; k, 2n)$ for $j \neq k, 1; i \neq 2n + 1 - k, 2n$ such that

$$[t_{ij}, t_{k,2n}] = v_j^{\varphi_{ki}} v_{2n+1-i}^{\varphi_{k,2n+1-j}} v_{2n}^{\alpha(i,j;k,2n)}$$

for $k \neq 1, 2n$ and $i + j \neq 2n + 1$ and $[t_{i,2n+1-i}, t_{k,2n}] = v_{2n+1-i}^{\varphi_{ki}} v_{2n}^{\alpha(i,2n+1-i;k,2n)}$

The proof of (7) tells us that we have if necessary to change t_{ij} by v_1 to obtain these equations. Therefore the 1-component for t_{ij} $2 \leq i, j \leq 2n - 1$ is determined. Moreover we replace $t_{k,2n}$ by $v_2^{\varphi_{k,2}} \cdots v_{2n-1}^{\varphi_{k,2n-1}} t_{k,2n}$ for $2 \leq k \leq 2n - 1$ and denote again this element by $t_{k,2n}$. By the proof of (8) it follows that X_{2n} is still the central product of $\langle t_{1,2n}\rangle, \langle v_2, t_{2,2n}\rangle, \cdots, \langle v_{2n-1}, t_{2n-1,2n}\rangle$. In this way all components of $t_{k,2n}$ for $1 \leq k \leq 2n - 1$ but the $2n$-component are determined and $\varphi_{ij} = 0$ for $2 \leq i, j \leq 2n - 1$.

We have, by the above, numbers $\alpha(i, j; i, 2n)$ with

$$[t_{ij}, t_{i,2n}] = v_{2n}^{\alpha(i,j;i,2n)}$$

As $t_{ij}, t_{i,2n} \in A_{2n+1-i}$ there exist elements in $t_{ij}V$ and $t_{i,2n}V$ of the same order as t_{ij} and $t_{i,2n}$ respectively which do commute. Hence $\alpha(i, j; i, 2n) = 0$.

Further we have numbers $\gamma(i, 2n + 1 - i, 2n)$ and $\gamma(i, j, 2n)$ such that

$$[t_{i,2n+1-i}, t_{2n+1-i,2n}] = v_{2n+1-i} v_{2n}^{\gamma(i,2n+1-i,2n)} t_{i,2n} t_{1,2n} \quad \text{for} \quad 2 \leq i \leq 2n - 1.$$

If $i + j \neq 2n + 1$

$$\begin{aligned}
[t_{ij}, t_{j,2n}] &= v_{2n+1-i} v_{2n}^{\gamma(i,j,2n)} t_{i,2n} && \text{for} \quad 2 \leq i, j \leq n \\
& && \text{or} \quad n + 1 \leq i, j \leq 2n - 1 \\
&= v_{2n}^{\gamma(i,j,2n)} t_{i,2n} && \text{for} \quad 2 \leq i \leq n; \quad n + 1 \leq j \leq 2n - 1 \\
& && \text{or} \quad n + 1 \leq i \leq 2n - 1; \quad 2 \leq j \leq n
\end{aligned}$$

We proceed now by induction and assume that we have shown the following for $k \geq 1$.

(i) $\langle t_{i,2n+1-l} \mid 1 \leq i \leq 2n; i \neq 2n + 1 - l\rangle$ is an abelian group of type $(4, 2, \cdots, 2)$ for $1 \leq l \leq k$.

(ii)
$$\begin{array}{l}
t_{k,2n}, \cdots, t_{3,2n}, t_{2,2n} \\
t_{k,2n-1}, \cdots, t_{3,2n-1} \\
\vdots \\
t_{k,2n-k+2}
\end{array}$$

are completely determined in all components. The $t_{i,2n+1-l}$ for $i \neq 2n+1-l$ are either completely determined if they are an element listed above or they are completely determined up to their $(2n+1-l)$-component for $1 \leq l \leq k$.

(iii) For $1 \leq l \leq k$ the following relations hold: There are numbers

$$\alpha(i,j;k,2n+1-l), \quad \gamma(i,2n+1-i,2n+1-l), \quad \gamma(i,j,2n+1-l)$$

such that

$$[t_{ij}, t_{k,2n+1-l}] = v_{2n+1-l}^{\alpha(i,j;k,2n+1-l)}$$

for $\quad j \neq k, \quad 2n+1-i \neq k, \quad l \neq j, \quad l \neq 2n+1-i.$

$$[t_{i,2n+1-i}, t_{2n+1-i,2n+1-l}] = v_{2n+1-i}\, v_{2n+1-l}^{\gamma(i,2n+1-i,2n+1-l)} t_{i,2n+1-l}\, t_{l,2n+1-l}.$$

$$\begin{aligned}[t_{ij}, t_{j,2n+1-l}] &= v_{2n+1-i}\, v_{2n+1-l}^{\gamma(i,j,2n+1-l)} t_{i,2n+1-l} && \text{for } i,j \leq n \\ &= v_{2n+1-l}^{\gamma(i,j,2n+1-l)} && \text{for } i \leq n \text{ and } j > n \\ & && \text{or } i > n \text{ and } j \leq n\end{aligned}$$

(iv) $\alpha(i,2n+1-l;r,2n+1-f) = 0$ for all $1 \leq l < f \leq k$ and all possible i, r. $\alpha(i,j;i,2n+1-l) = 0$ for all possible i, j and $1 \leq l \leq k$. $\alpha(r,s;i,2n+1-l) = 0$ for all possible r, s and $1 \leq i, l \leq k$.

(v) If t_{ij} is not an element listed under (ii) then the l-component of t_{ij} is determined for $1 \leq l \leq k$ and $l \neq j, 2n-1-i$.

We have $A_{2n-k} = \langle t_{i,2n-k} \mid 1 \leq i \leq 2n \rangle V$. By (iii) we know for $i \leq k$ and $j \neq 2n+1-i$ that $[t_{i,2n-k}, t_{j,2n-k}] = 1$. By changing if necessary $t_{2n+1-i,2n-k}$ by v_i and $t_{i,2n-k}$ by v_{2n+1-i} we may assume

$$[t_{j,2n-k}, t_{i,2n-k}] = 1 \quad \text{for} \quad 1 \leq i \leq k \quad \text{and all } j.$$

In this way all components of $t_{k+1,2n+1-l}$ $(1 \leq l \leq k)$ are determined and we will see that $t_{k+1,2n+1-l}$ $(1 \leq l \leq k)$ is not being changed in the course of the argument.

Changing $t_{j,2n-k}$ $(k+1 \leq j \leq 2n)$ by elements in $\langle v_{k+2}, \cdots, v_{2n} \rangle$ we may assume that (i) is true.

We have further by (iii) and (iv),

$$[t_{s,2n+1-l}, t_{i,2n-k}] = v_{2n+1-l}^{\alpha(i,2n-k;s,2n+1-l)} \quad \text{for} \quad 1 \leq l \leq k \quad \text{and} \quad i \neq k+1$$

and $[t_{s,2n+1-l}, t_{k+1,2n-k}] = 1$. Set $\alpha(i,2n-k;s,2n+1-l) = \varphi_{is}$.

We only have to change t_{rs} for $r \geq k+2$ and $s \leq 2n+2-k$ by v_{k+1} if necessary in order to get with help of (7) the fact

$$[t_{rs}, t_{i,2n-k}] = v_s^{\varphi_{ir}} v_{2n+1-r}^{\varphi_{i,2n+1-s}} v_{2n-k}^{\alpha(r,s;i,2n-k)},$$

$$[t_{r,2n+1-r}, t_{i,2n-k}] = v_{2n+1-r}^{\varphi_{ir}} v_{2n-k}^{\alpha(r,2n+1-r;i,2n-k)}$$

for $r \neq 2n-k, 2n+1-i; i \neq s \neq k+1$.

In this way the 1-, $\cdots$, $(k+1)$-components of t_{rs} are determined. Moreover by (iv), $\alpha(i, 2n - k; s, 2n + 1 - l) = 0$ $(1 \leq l \leq k)$ if $1 \leq i \leq k$. So $\varphi_{is} = 0$ for $i \leq k$ and all s.

If $1 \leq i \leq k$ then the determination of the $(k + 1)$-component of t_{rs} forces $\alpha(r, s; k + 1, 2n + 1 - i) = 0$ and so

$$[t_{rs}, t_{i,2n-k}] = 1$$

We now replace $t_{i,2n-k}$ by $v_1^{\varphi i1} \cdots v_{2n}^{\varphi i,2n} t_{i,2n-k}$. As $\varphi_{is} = \varphi_{si} = 0$ for all s and $i \leq k$, it follows that $t_{i,2n-k}$ stays unchanged for $1 \leq i \leq k$ and $t_{i,2n-k}$ is only changed in the t-component where $t \geq k + 2$ as desired.

In this way we have determined all components of $t_{s,2n-k}$ but the $(2n - k)$-component for $s \geq k + 2$.

Moreover we have for $1 \leq l < s \leq k + 1$; $l \neq 2n + 1 - j, s$: $i \neq 2n + 1 - j, s$,

$$[t_{i,2n+1-l}, t_{j,2n+1-s}] = 1$$

and as $t_{ij}, t_{i,2n-k} \in A_{2n+1-i}$ we have

$$[t_{ij}, t_{i,2n-k}] = 1$$

By (7) we have furthermore numbers $\gamma(i, 2n + 1 - i, 2n - k)$, $\gamma(i, j, 2n - k)$ such that

$$\begin{aligned}
[t_{ij}, t_{j,2n-k}] &= v_{2n+1-i}\, v_{2n-k}^{\gamma(i,j,2n-k)} t_{i,2n-k} && \text{for} \quad 1 \leq i, j \leq n \\
& && \text{or} \quad n + 1 \leq i, j \leq 2n \\
&= v_{2n-k}^{\gamma(i,j,2n-k)} && \text{for} \quad 1 \leq i \leq n; \quad n + 1 \leq j \leq 2n \\
& && \text{or} \quad n + 1 \leq i \leq 2n; \quad 1 \leq j \leq n.
\end{aligned}$$

$$[t_{i,2n+1-i}, t_{2n+1-i,2n-k}] = v_{2n+1-i}\, v_{2n-k}^{\gamma(i,2n+1-i,2n-k)} t_{i,2n-k}\, t_{k+1,2n-k}.$$

And for $i \neq 2n - k, 2n + 1 - s; j \neq s, k + 1$ we have

$$[t_{ij}, t_{s,2n-k}] = v_{2n-k}^{\alpha(i,j;s,2n-k)}$$

where

$$\alpha(i, j; i, 2n - k) = 0 \quad \text{for all } i \text{ and } j,$$

$$\alpha(i, 2n + 1 - l; s, 2n - k) = 0 \quad \text{for all} \quad i, s \quad \text{and} \quad 1 \leq l \leq k,$$

and finally

$$\alpha(r, s; f, 2n + 1 - d) = 0 \quad \text{for all} \quad r, s \quad \text{and} \quad 1 \leq f, \quad d \leq k + 1.$$

As we have not changed results obtained by the induction step $i \to i + 1$ for $i \leq k$ it follows that (i)–(v) are verified for the induction step $k \to k + 1$.

Therefore we end up finally with

(i) For $1 \leq i, j \leq 2n$, t_{ij} is completely determined in all its components if $i + j \neq 2n + 1$.

(ii) $t_{i,2n+1-i}$ is completely determined in all its components but the $(2n+1-i)$-component for $1 \leq i \leq 2n$.

(iii) $(t_{i,2n+1-i})^2 = v_{2n+1-i}$, $t_{ij}^2 = 1$ for $1 \leq i, j \leq 2n$ and $i + j \neq 2n + 1$.

(iv) $[t_{ij}, t_{rs}] = 1$ if $\{i, j\} \cap \{r, s, 2n+1-r, 2n+1-s\} = \emptyset$ or $i = r$ or $j = s$ and $1 \leq i, j \leq 2n$.

$$[t_{i,2n+1-i}, t_{2n+1-i,s}] = v_{2n+1-i}\, v_s^{\gamma(i,2n+1-i,s)} t_{is}\, t_{2n+1-s,s}$$

for $1 \leq i, s \leq 2n$ and $i + s \neq 2n + 1$.

$$\begin{aligned} [t_{ij}, t_{js}] &= v_{2n+1-i}\, v_s^{\gamma(i,j,s)} t_{is} && \text{for} \quad 1 \leq i, j \leq n \\ & && \text{or} \quad n+1 \leq i, j \leq 2n \\ &= v_s^{\gamma(i,j\,s)} t_{is} && \text{for} \quad 1 \leq i \leq n; \quad n+1 \leq j \leq 2n \\ & && \text{or} \quad 1 \leq j \leq n; \quad n+1 \leq i \leq 2n \end{aligned}$$

where $i + j \neq 2n + 1 \neq j + s$.

Using that $t_{ij} = t_{2n+1-j,2n+1-i}$ and $t_{js} = t_{2n+1-s,2n+1-j}$ we conclude that

$$\begin{aligned} [t_{ij}, t_{js}] &= v_{2n+1-i}\, t_{is} && \text{if} \quad 1 \leq i, j \leq n; n+1 \leq s \leq 2n \\ &= v_{2n+1-i}\, v_s\, t_{is} && \text{if} \quad 1 \leq i, j, s \leq n \\ &= t_{is} && \text{if} \quad n+1 \leq i, s \leq 2n; 1 \leq j \leq n \\ &= v_s\, t_{is} && \text{if} \quad 1 \leq j, s \leq n; n+1 \leq i \leq 2n \\ &= t_{is} && \text{if} \quad 1 \leq i, s \leq n; n+1 \leq j \leq 2n \\ &= v_{2n+1-i}\, v_s\, t_{is} && \text{if} \quad n+1 \leq i, j, s \leq 2n. \end{aligned}$$

Clearly $(t_{ij}\, t_{ji})^3 \in V$ and $t_{ij}\, t_{ji}$ commutes with every t_{rs} for

$$\{r, s\} \cap \{i, j, 2n+1-i, 2n+1-j\} = \emptyset.$$

So

$$(t_{ij}\, t_{ji})^3 \in \langle v_i, v_j, v_{2n+1-i}, v_{2n+1-j} \rangle = B_{ij}.$$

But as $t_{ij}\, t_{ji}$ acts fixed-point-free on B_{ij} we conclude

$$(t_{ij}\, t_{ji})^3 = 1 \quad \text{for all} \quad 1 \leq i, j \leq 2n.$$

Therefore we have determined our multiplication table up to the $(2n+1-i)$-component of $t_{i,2n+1-i}$ and the numbers $\gamma(i, 2n+1-i, j)$. If we set $\varepsilon(i, j) = 0$ for $1 \leq i, j \leq n$ or $n+1 \leq i, j \leq 2n$ and $i + j \neq 2n+1$, $i \neq j$ and $\varepsilon(i, j) = 1$ for $1 \leq i \leq n$, $n+1 \leq j \leq 2n$ or $1 \leq j \leq n$, $n+1 \leq i \leq 2n$ and $i + j \neq 2n + 1$ we can set

$$[t_{ij}, t_{js}] = v_s^{\varepsilon(2n+1-s,2n+1-j)} v_{2n+1-i}^{\varepsilon(i,j)}\, t_{is}.$$

Further

$$[t_{i,2n+1-i},\ t_{2n+1-i,s}] = v_{2n+1-i}\, v_s^{\gamma(i,2n+1-i,s)} t_{is}\, t_{2n+1-s,s}$$

and

$$t_{sr}^{[t_{i,2n+1-i},t_{2n+1-i,s}]}$$
$$= v_{2n+1-i}^{1+\varepsilon(2n+1-i,s)} v_r^{\gamma(i,2n+1-i,r)+\varepsilon(2n+1-s,i)+\varepsilon(2n+1-r,2n+1-i)} v_s^{1+\varepsilon(2n+1-i,s)+\varepsilon(2n+1-s,i)} \cdot t_{ir}\, t_{sr}\, t_{2n+1-s,r}\, t_{2n+1-r,r}$$

and

$$t_{sr}^{q} = v_r^{\gamma(2n+1-s,s,r)+\varepsilon(2n+1-r,2n+1-s)+\gamma(i,2n+1-i,s)} v_{2n+1-i}^{\varepsilon(i,s)}\, v_s\, t_{ir}\, t_{sr}\, t_{2n+1-s,r}\, t_{2n+1-r,r}$$

where $q = v_{2n+1-i}\, v_s^{\gamma(i,2n+1-i,s)} t_{is}\, t_{2n+1-s,s}$. This implies

$$(*)\quad \begin{aligned} \gamma(i, 2n+1-i, r) = {} & \gamma(2n+1-s, s, r) + \gamma(i, 2n+1-i, s) \\ & + \varepsilon(2n+1-s, i) \\ & + \varepsilon(2n+1-r, 2n+1-i) \\ & + \varepsilon(2n+1-r, 2n+1-s). \end{aligned}$$

By changing $t_{i,2n+1-i}$ if necessary by v_{2n+1-i} we may assume that

$$\gamma(i, 2n+1-i, 1) = 0 \quad \text{for} \quad 2 \le i \le 2n-1,$$
$$\gamma(1, 2n, 2) = \gamma(2n, 1, 2) = 0.$$

Then $(*)$ determines all other $\gamma(i, 2n+1-i, j)$.

So we can state:

(9) *If G is a nonsplit extension of V by $S_{2n}(2)$, then G is uniquely determined. Moreover G is generated by elements t_{ij} for $1 \le i, j \le 2n$, $i + j \le 2n + 1$, $i \ne j$ which satisfy the relations listed above.*

Using (9) and (8) and a result of Griess [5] it follows that if G is a nonsplit extension, that G is uniquely determined and that there are such nonsplit extensions.

References

1. Emil Artin, *The orders of the classical simple groups*, Comm. Pure Appl. Math., vol. 8 (1955), pp. 455–472.
2. Ulrich Dempwolff, *On second cohomology of GL(n, 2)*, Austral. J. Math., vol. 16 (1973), pp. 207–209.
3. ———, *On extensions of an elementary abelian group of order 2^5 by GL(5, 2)*, Rend. Sem. Padova, vol. 63 (1972), pp. 359–364.
4. Robert Griess Jr., *Schur multipliers of the known finite simple groups*, Bull. Amer. Math. Soc., vol. 78 (1972), pp. 68–71.
5. ———, *Automorphisms of extra special groups and nonvanishing degree 2-cohomology*,
6. Bertram Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.
7. Harriet Pollatsek, *First cohomology groups of some linear groups over fields of characteristic two*, Illinois J. Math., vol. 15 (1971), pp. 393–417.

The Ohio State University
Columbus, Ohio