ASSEMBLING COMPACT RIEMANN SURFACES WITH GIVEN BOUNDARY CURVES AND BRANCH POINTS ON THE SPHERE

BY

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1. Introduction

The conformal structure of a Riemann surface is determined by any nonconstant meromorphic function on the surface. Every closed Riemann surface may be presented as an n-sheeted covering of the Gaussian sphere branched over w points a_i on the sphere [2, p. 47]. Over eighty years ago, A. Hurwitz [11] showed how to associate with such a covering a system of w permutations H_i on n symbols whose product $H_1 H_2 \cdots H_w$ is the identity and which generate a transitive group on the n symbols. Draw w rays α_i from the a_i to the common reference point ∞. Lifting the α_i to the surface decomposes it into a finite cell complex with n faces covering the same slit region $S^2 - \bigcup \alpha_i$ on the sphere. The $2w$ edges cover the interiors of the arcs α_i and n of the vertices, each of degree w, lie over ∞. The remaining vertices, $\hat{a}_{ij}$, of degrees δ_{ij}, $i = 1, \ldots, w$, $j = 1, \ldots, k_i$, lie over the a_i. The permutation H_i has k_i cycles of lengths δ_{ij}, each permuting the faces incident to $\hat{a}_{ij}$ in cyclic order. The Euler characteristic of the surface satisfies the so-called Hurwitz-Riemann relation $\chi = 2n - \mu$, where $\mu = nw - \sum k_i = \sum (\delta_{ij} - 1)$ is the branching number of the covering. Note that the covering projection is locally $\delta_{ij}{:}1$ in a deleted neighborhood of $\hat{a}_{ij}$. Conversely, given a system of Hurwitz permutations H_i for the points a_i, the surface may be reassembled by identifying edges of n polygonal cells of $2w$ sides each according to the information contained in the H_i.

A related but considerably more difficult problem is to develop an analogous combinatorial characterization of bordered Riemann surfaces. For present purposes, let a *compact Riemann surface* be a pair (M, F), where M is a compact, connected topological 2-manifold with border ∂M, and F a continuous map from M to S^2 which is locally $1:1$ at all but a finite number of so-called *critical points* lying in $M - \partial M$. For each critical point $p \in M$ there is a nonnegative integer $\mu(p)$ so that F is locally topologically equivalent to the complex power function $w = z^{\mu(p)+1}$. (We shall discuss our terminology and justify the invocation of Riemann's name in Section 5.) Extend μ to vanish at noncritical points of M. For $q \in S^2$, set $\mu(q) = \tilde{\mu}(F^{-1}(q))$, where $\tilde{\mu}$ is the obvious numerical measure on the subsets of M induced by μ. If ∂M is empty we say that M is a *closed*

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surface. Otherwise M is said to be a *bordered surface*. If F has no critical points we say that F is an *immersion*. Otherwise F is said to be a *polymersion*. A point $q \in S^2$ where μ is positive is called a *branchpoint* with *multiplicity* $\mu(q)$. The *branching number* of F is $\mu = \tilde{\mu}(M)$. A polymersion of a closed surface is, of course, a *branched covering* of the sphere. A surface that admits a polymersion is necessarily orientable. We use a fixed orientation on S^2, orient M so that F is sensepreserving and orient the finite number of border circles so that M lies to their left. We shall abbreviate $F \mid \partial M$ by ∂F. If q is the initial point of an arc λ in S^2, $p \in F^{-1}(q)$ and p is not a critical point of F, then the *lift of λ from p* is the maximal arc $\hat{\lambda}$ in $F^{-1}(\lambda)$ beginning at p and having no critical points on its interior.

A *cellulation* of M is a decomposition of a compact surface M into a finite number of *vertices*, *edges*, and *faces* (mutually disjoint open cells of dimensions 0, 1, 2) so that the closure of a cell is the union of the cell with its adherent lower dimensional cells. The number of edges incident to a vertex, counted with multiplicity, is the degree of the vertex. For example, $\{z = 1\}$, $\{|z| = 1, z \neq 1\}$, $\{|z| < 1\}$ is a cellulation of the unit disc with one vertex of degree 2. The cellulation is *faithful* to a polymersion F of M if all critical points of F lie on vertices and F is 1:1 on each cell. It is *nearly faithful* if critical points lie on vertices and faces and F is 1:1 on every face that carries no critical points. Note that in a nearly faithful cellulation the degree of a critical vertex p is one more than its multiplicity $\mu(p)$. Thus we may *erase* a vertex of degree 1 and its adherent edge (that is, include the point set carrying them in the adherent face, and delete the vertex and edge from the cell structure). This erasure does not affect the (nearly) faithful character of a cellulation. For a cellulation without vertices of degree 1 there is a (sensepreserving) parametrization of the closure of each face by the closed unit disc, which is 1:1 on its interior and locally 1:1 on the boundary.

The problem to be solved is the following. Given finitely many points a_i and oriented closed curves f_j on S^2, determine all bordered surfaces M and polymersions F of M branched over the a_i only and such that $F \mid \partial M$ parametrizes the f_j. Our choice of definition requires that the f_j must be locally simple. If we ignore the intended location and multiplicity of the branchpoints, we may appeal to a result of Morse and Heins [16], who showed that there always exist polymersions F from a plane region (genus zero) extending a finite collection of closed sensed locally simple curves. Theirs remains the only fully general result where no further assumption is made about the *a priori* position of the curves. For reasons discussed in Section 5, we shall assume that the curves, and points when specified, lie in "general position," a concept we shall make more precise below. Now, the idea is simple enough. With Hurwitz, draw a set of rays from the a_i to a common reference point ∞ and lying in "general position" with respect to the f_j. Given the polymersion (F, M), lift the rays by F^{-1}. This will certainly decompose M faithfully for F, but we have no guarantee that the connected complementary components of the one-skeleton are simply connected.

For this purpose, sufficiently many additional rays are drawn, depending only on the position of the curves f_j, not on the map F. An *assemblage* for F will consist in a certain set of permutations on the crossings of the rays and curves (a finite set by general position) and the poles (points in $F^{-1}(\infty)$, also finite by compactness). The assemblage determines a cellulation of M faithful to F. The genus of M, the branching number of F, in short, all topological invariants of F are determined by the assemblage. That is, two polymersions, branched over the same points on S^2 and with the same boundary image, differ by a homeomorphism of their domains if and only if they have identical assemblages. Thus we have a solution to the problem in the "generic" case. For particular cases, particular methods would have to be developed.

The problem was more or less posed in this way by C. Löwner and H. Hopf around 1948 [19]. Seventy years after Hurwitz, C. J. Titus [18] gave the first algorithm, based on the succession of nodes on a single normal curve f in R^2 (its Gauss-Whitney-Titus intersection sequence), which determines whether or not f extends to a polymersion F of the disc D^2 to R^2. In this form, Titus solved Picard's problem of determining when a closed polygonal curve is the boundary image of a holomorphic map of the upper half plane. In the following decade rapid progress was made on other special cases for M, notably on the possible location and multiplicity of branchpoints [13], [15]. S. J. Blank [3] constructed the first assemblage (in the above sense) for F an immersion (no branchpoints) of D^2 to R^2. M. L. Marx [14] developed this method for polymersions of D^2 to R^2. [4] deals with immersions of D^2 to S^2. K. Bailey [1] found assemblages for immersions in R^2 of surfaces of arbitrary genus but one border circle. S. Troyer [20] did the same for surfaces of arbitrary border, but of genus zero. A general solution synthesizing these methods was announced in [6]. However, several stimulating conversations with W. Magnus [12] in 1973, who suggested the connection with Hurwitz' paper, led to the present simplification in the definition of an assemblage. While our exposition is selfcontained, it owes much to its predecessors and the reader is encouraged to consult these earlier papers, especially for the many examples and detailed examination of special cases. The author acknowledges the helpful correspondence with his anonymous referee, the valuable stylistic advice of his editor, and many illuminating discussions with W. Abikoff concerning conformal structures.

2. Normal curves, rayings and assembling permutations

By an (oriented) curve on a surface we shall mean an equivalence class of continuous functions from a real interval or the unit circle to the surface such that two representatives differ by a (sense-preserving) homeomorphism of their domains. By a parametrization $f(s)$ of f we shall mean a particular choice representing f. Let f be a finite family of oriented, closed, locally simple curves in $R^2 = S^2 - \{\infty\}$ considered as an immersion of a finite set of oriented circles. A point $q \in R^2$ is a *node* [8] of f if $f^{-1}(q) = \{p_1, p_2\}$, $p_1 \neq p_2$, and there are arcs λ_i (homeomorphs of a closed interval) centered on p_i so that both $f(\lambda_i)$ are simple, $f(\lambda_1) \cap f(\lambda_2) = \{q\}$ and $f(\lambda_1)$ separates the exiting from the entering semiarc of $f(\lambda_2)$. That is, f literally *crosses* itself once in a neighborhood of q. This is Morse's [16] C^0-version of Whitney's [21] C^1-notion of a *normal crossing*. That is, if f is a C^1-immersion (the curves admit regular parameters) then the two tangent vectors at a node are independent. If every point $q \in R^2$ on f (i.e., f passes through q) is either a node or a simple point (i.e., card $f^{-1}(q) = 1$) we call f a *normal family of curves*.

Let $f(s)$ be an immersion of the unit circle in R^2. Then there exists a continuous (for example, constant) function $0 < e(s) < \pi/4$ so that $f \mid [s, s + e(s)]$ is 1:1 for each s. Set

$$\Delta f(s) = f(s + e(s)) - f(s)/|f(s + e(s)) - f(s)|.$$

The Hopf degree of this map is independent of the choice of parameter s and gauge $e(s)$. We shall call it the *turning number* $\tau(f)$ of f. For any continuous deformation f_r (i.e., $f_r(s)$ is continuous in r, s) through immersions, τ is constant, provided some gauge $e(r, s)$ remains bounded away from 0 over the range of r.

This number was called the "amplitude" of f by Gauss [8], the "angular order" by Morse [16]. In the regular case, where $\Delta f(s) = df(s)/ds$, it has been called the "tangent or normal winding or rotation number or index" by other authors since. We extend this turning number over several curves by addition. Now let q be a node of a curve family f in the plane, and let f^q denote the curve family obtained from f by *cutting through* q [8]. That is, we exchange the two exiting subarcs of f at q. It is not difficult to convince oneself that the turning number is the same for both f and f^q. Now, if every node of a normal curve family is cut through, Gauss observed that the new curve family $f^{\oplus}$ consists of a finite collection of mutually noncrossing closed oriented Jordan curves, and that $\tau(f) = \tau(f^{\oplus})$ is the algebraic sum of their orientations. We shall call these curves the *Gaussian circles* of f. They are also known as "Seifert circuits" in knot theory.

Let α be a finite family of *rays* on S^2 concurrent at a point ∞. That is, α is an embedding (1:1 continuous map) of a finite set of closed radii of a disc into S^2, with center going to ∞. We call any such α a *raying* for a normal curve family f provided they lie in general position. That is, if $x = f(s) = \alpha(t)$ is a common point of f and α, then $x \neq \infty$, x is not a node of f, x is not the initial point of the ray through x, and the two arcs consisting of f, α restricted to small intervals about s, t, cross in the sense used above to define the nodes of f. Let $X(f, \alpha)$ denote the set of these *crossings* of f and α. Any raying α of f is *sufficient* for our purposes provided there is at least one crossing on each member curve of f, and at least one crossing on each Gaussian circle in $f^{\oplus}$ which is negatively oriented in the plane $S^2 - \{\infty\}$.

It will prove notationally convenient to express geometric objects on S^2 in terms of the crossings. For $x \in X(f, \alpha)$, α_x is to be the ray through x with *initial point* a_x, and f_x is to be the closed curve in the family f that passes through

x. If $f_x = f_y$, let $[x, y]f$ denote that so called *section* of f, which runs from x to y on the closed oriented curve f_x. A section of f may, of course, intersect itself at nodes. We agree that $[x, x]f = f_x$. If $\alpha_x = \alpha_y$, $x \neq y$, let $[x, y]\alpha$ denote the so called *segment* of α, which is the subarc of α_x, oriented from x to y.

Among the permutations on $X = X(f, \alpha)$, let S denote the *successor permutation* that takes each crossing x into the next succeeding crossing x^S on f_x in the sense of f_x. Note that if $x^S = x$ (f_x is crossed only once by α), then $[x, x^S]f = f_x$. Each crossing has a positive (negative) sign if f_x crosses from the right to the left side of α_x oriented to ∞ (respectively, vice versa). By a *pair* on X we shall mean a transposition (xy) that exchanges two crossings of opposite sign but on the same ray, with the negative crossing separating the positive crossing from the initial point. We frequently write a pair as $(\bar{x}x)$, where $\bar{x}$ is the negative partner. By a *fan* $(x_1x_2 \cdots x_n)$, $n \geq 2$, we shall mean a cyclic permutation of positive crossings sharing the same ray. An *assembling permutation* P on X is a product of disjoint pairs and fans. If there are no fans in P, we frequently call P a *pairing*. Thus the crossings on a single ray are invariant under P, and crossings not appearing in a pair or fan of P are left fixed.

An assembling permutation P on $X = X(f, \alpha)$ serves to rearrange the sections of f and segments of α into a new (not necessarily normal) family f^P of closed oriented curves. It also serves to define a cellular 2-manifold M_P. Let $R = SP$ be the product of the two given permutations. That is

(*) if $y = x^S$ and $z = y^P$ then $z = x^R$.

It is convenient to use our labeling by elements of X. Thus, if Q is a permutation of X, let Q_x denote that (unique) cycle of Q wherein x occurs (i.e., which moves or fixes x).

First, each cycle of R, say R_x, determines a "section" $[x, x^R]f^P$ of the member curve f_x^P of f^P as follows. In the notation (*), there are three cases to consider, depending on whether P_y is the trivial cycle (y), the pair (yz), or a fan $(yz \cdots)$. In the first case, set $[x, z]f^P = [x, y]f$; in the second case, set $[x, z]f^P = [x, y]f + [y, z]\alpha$, where $+$ stands for concatenation; in the third case, $[x, z]f^P = [x, y]f + [y, a_y]\alpha + [a_y, z]\alpha$. Note that in the last case the section $[x, x^R]f^P$ is not an immersed interval because it is not locally 1:1 at the ray initial a_y at the end of the "slit" along α_y.

Next, to each cycle R_x of R associate a (positively oriented closed) disc D_x: Label distinct points on the border ∂D_x by letters in R_x and interpolate points corresponding to crossings and ray-initials in the (cyclic) order encountered by the closed curve f_x^P. Now assemble the D_x by identifying certain edges and vertices (the edge identification is, as usual, in the contrary sense). A hat distinguishes the point on M_P from its namesake. Note that under our labeling, $R_x = R_z$ so that $D_x = D_z$, x, z related as in (*). If P_y is a pair, identify $[\hat{y}, \hat{z}]\partial D_z$ with $[\hat{z}, \hat{y}]\partial D_y$. If P_y is a fan, identify $[\hat{y}, \hat{a}_y]\partial D_z$ with $[\hat{a}_y, \hat{y}]\partial D_y$ and $[\hat{a}_y, \hat{z}]\partial D_z$ with $[\hat{z}, \hat{a}_y]\partial D_y$. The example below should settle these ideas.

Thus M_P is a compact, oriented cellular 2-manifold and it is connected precisely if S and P generate a group that is transitive on X. A practical way of

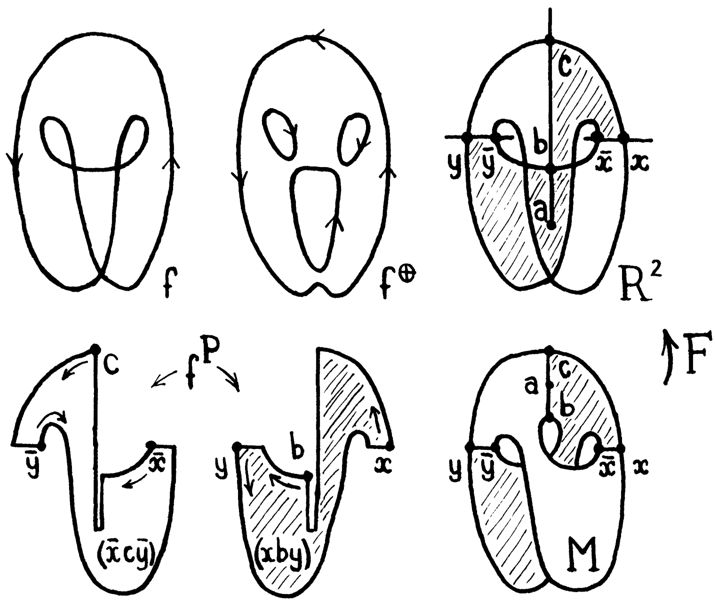

FIGURE 1

checking this condition is to write S as a disjoint product of its cycles and connect all letters in S that occur in the same cycle of P. If this process connects up all cycles of S, we shall say that P is *transitive on S*. Note that M_P has ρ border circles where ρ is the number of cycles in S. ρ is also the number of curves in f, by the first condition for the sufficiency the raying α. The number ζ of cycles in $R = SP$ is also the number of *faces* (2-cells) of M_P. If P has m cycles and card $(X) = n$ then it is not difficult to check directly that M_P has Euler characteristic $\chi = \zeta + m - n$. We shall, however, obtain this as part of our inductive arguments later.

For each fan, say P_x, of P there is an interior vertex $\hat{a}_x$. Since its adherent edges (its star) all end on the border vertices corresponding to the crossings enumerated by P_x in clockwise cyclic order about $\hat{a}_x$, the degree of $\hat{a}_x$ equals the length of P_x. Note that in the event that there are two distinct fans P_x and P_y permuting crossings on the same ray, $a_x = a_y$, the vertices $\hat{a}_x$, $\hat{a}_y$ are distinct. For each pair, say (xy), of P there is one interior edge connecting $\hat{x}$ to $\hat{y}$. It is clear that we can define a "polymersion" $\tilde{F}_P$ of a neighborhood U_P of the one-skeleton of M_P so that $\tilde{F}_P \mid \partial M_P = f$ and with critical points the vertices $\hat{a}_x$ for which P_x is a fan of P. The real problem is to determine those additional properties of P necessary and sufficient for $\tilde{F}_P$ to be extended over the faces, so as to be 1:1 on each face.

Example. Let f be Titus' pretzel curve [17], [19]. See Figure 1. Its turning number τ is 0 because $f^{\oplus}$ (second frame, the corners have been smoothed for visibility) has two negative and two positive Gaussian circles. A convenient, sufficient raying consist of three rays (3rd frame, ∞ is out of the picture). Label the crossings and the ray initial as shown. Crossings $\bar{x}$, $\bar{y}$ are negative, b, c are positive. The successor permutation is $S = (xcy\bar{x}b\bar{y})$. If we take $P = (\bar{x}x)(\bar{y}y)(bc)$, with two pairs and one fan of length 2, then the resultant $R = (\bar{x}c\bar{y})(xby)$ has two cycles ($\zeta = 2$). The two curves of f^P are drawn next, with the doubly covered "slit" ab pulled apart for visibility (only the essential points are labeled). If we assemble the two discs (same part of the figure) as prescribed, we obtain the surface M (last frame, the hats are omitted) which is a torus with a disc removed. Thus f extends to a polymersion F of M into the plane with a single simple branchpoint at a ($\mu(F) = 1$).

2. Assemblages for polymersions into R^2

PROPOSITION 1. *If f is a single normal curve with turning number τ and α is a sufficient raying with no negative crossings, then there is a polymersion F with branching number μ from a disc M to the plane $R^2 = S^2 - \{\infty\}$ such that $\partial F = f$ and $\mu = \tau - 1$.*

Proof. Using different methods, and in general, many more rays than required under the two conditions of sufficiency, Marx first obtained this result in [14]. We use induction on the number N of nodes of f. If $N = 0$, then f is simple. The first condition of sufficiency and the absence of negative crossings guarantees that f is positively oriented in R^2. By the plane separation theorem for closed Jordan curves, the bounded complementary component of f is a topological disc and the identity map F on its closure M is the required polymersion. Since F is a homeomorphism, $\mu = 0$, and since f is a positively oriented closed Jordan curve, $\tau = 1$.

For N positive, choose a node q of f adherent to the unbounded complementary component of f. By the second condition of sufficiency and the absence of negative crossings, all Gaussian circles are positive, in particular the two through q. Hence ∞ lies to the right of both subarcs of f crossing at q. Cutting through q only, produces two normal curves f_i, $i = 1, 2$, each of which has fewer than N nodes. By induction, we have polymersions F_i of discs M_i into the plane with $\partial F_i = f_i$ and $\mu_i = \tau_i - 1$. Select a point a to the left of both subarcs of f through q connected to q by an arc aq which meets f only at q. Lift aq to arcs a_iq_i on M_i where $q_i \in \partial M_i$. Using the conventional model for mapping the unit disc under $w = z^2$ (middle row of Figure 2) we join the M_i into a disc M (top row) and so obtain the polymersion F which has one additional simple branchpoint ($\mu(a) = 1$) at a (bottom row). Since $\partial F = f$, $\mu = \mu_1 + \mu_2 + 1$ and $\tau = \tau_1 + \tau_2$, the induction step is complete. $\square$

Before proceeding to the next proposition we define a topological and a combinatorial tool. Let α be a raying for a normal family f with crossings X. By a *normal neighborhood* T of α we shall mean a sense preserving homeomorph of the polar (r, θ)-plane (the origin corresponds to ∞ and a finite number of radial segments correspond to the rays of α) such that T meets f only along a system

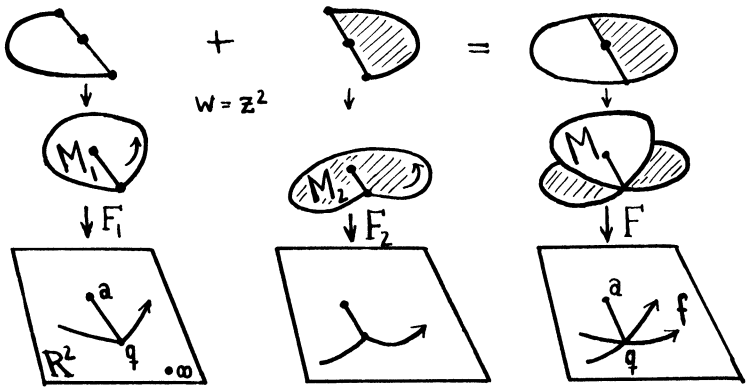

FIGURE 2

of card (X) simple (open) arcs. These should correspond to disjoint (open) arcs of circles centered at the origin in the model (r, θ)-plane. Let (r_x, θ_x) correspond to the crossing $x \in X$. There is a positive e, such that for $x \neq y$, $|r_x - r_y|$ and $|\theta_x - \theta_y|$ exceed e. Thus θ becomes a local parameter along f_x and r along α_x near x. Note that at a positive crossing x, θ turns clockwise at θ_x. On occasion we shall also use the Cartesian (r, θ)-plane as a model when the discussion does not involve ∞.

Let Q be a permutation on a finite set X considered as the disjoint product of its cycles written as words on the letters of X separated by parentheses. By an *expansion* of Q we shall mean a permutation Q' on a finite set X' containing X as a proper subset such that Q is obtained from Q' by deleting those letters in Q' belonging to $X' - X$. We further require that every cycle of Q' contains at least one letter of X so that both permutations have the same number of cycles on their respective domains (trivial cycles are included in this count). On occasion we shall also consider Q a permutation on X' which leaves the members of $X' - X$ pointwise fixed.

PROPOSITION 2. *Let f be a normal family of ρ curves with turning number τ. Let α be a sufficient raying with ν negative crossings in $X(f, \alpha)$. Let P be a pairing which is transitive on the successor permutation S and let $R = SP$ have ζ cycles. If P consists of exactly ν pairs, then there is a polymersion F with branching number μ from the connected surface $M = M_P$ of genus γ into the plane such that $\partial F = f$ and*

$$\mu = \nu + \tau - \zeta \tag{3.1}$$

$$2\gamma = 2 + \nu - \rho - \zeta. \tag{3.2}$$

Moreover, the cellulation on M is nearly faithful to F.

Proof. We use induction on the number of negative crossings ν. We see that if there are none, P is the identity, $R = S$ and $\zeta = \rho$. Transitivity of P

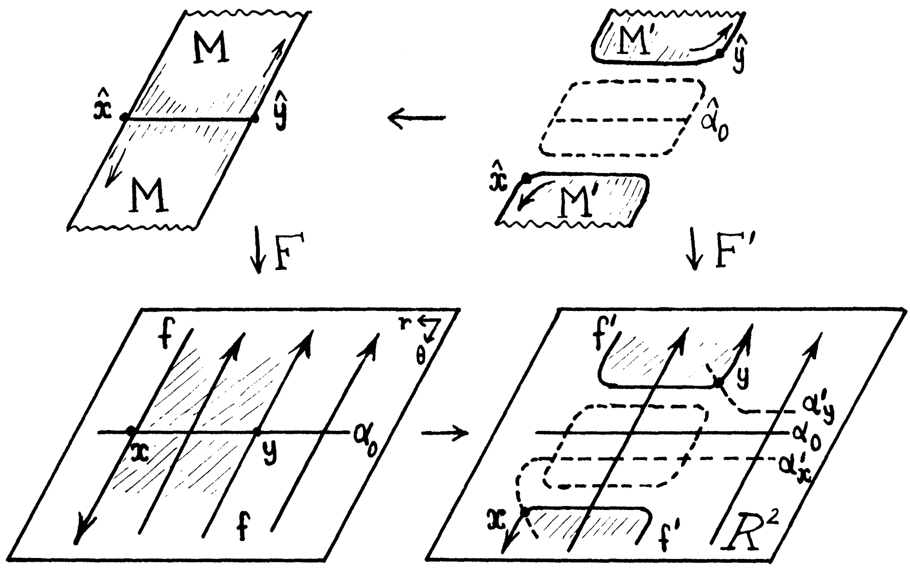

FIGURE 3

and sufficiency of α require that S be a single cycle. So $\rho = 1$ and Proposition 1 applies. (3.1) reduces to $\mu = \tau - 1$, and (3.2) follows from the fact that a disc has genus zero. The cellulation is trivially nearly faithful.

For v positive, let x be an *ultimate negative crossing.* That is, any crossing on $[x, \infty]\alpha$, other than x, is positive. By our hypothesis on P, there is one such, namely $y = x^P$. Consider the family of closed oriented curves obtained from f by replacing sections $[x, x^S]f$ by $[x, y]\alpha + [y, y^S]f$ and $[y, y^S]f$ by $[y, x]\alpha + [x, x^S]f$. This family is not normal, the arcs $[x, y]\alpha$ and $[y, x]\alpha$ occupy the same pointset. We normalize (in a normal neighborhood of α with respect to f) to produce the normal family f', i.e., retract the curves slightly away from the ray. See the successive pictures on the Cartesian (r, θ)-plane in Figure 3.

For combinatorial reasons it is convenient to relabel the ray through x, y by α_0, and assign the labels x, y to points on f' as shown. Draw two new rays, α_x', α_y', so as to cross f' at x, y as shown and which run parallel to α_0 to ∞. Note that since the original x was an ultimate negative crossing, all crossings of the new rays with f' are positive, including the new x and y. Consequently, the set $X'(f', \alpha')$, where α' is α augmented by α_x' and α_y', has one fewer negative crossings than $X(f, \alpha)$. Let us check that α' is sufficient for f'. The first condition of sufficiency is guaranteed by our drawing the two new rays. However, all new nodes on f', i.e., those that are not already nodes of f, correspond, one on each side, to the crossings of α_0 with f between the old x, y. Since these were all positive, every Gaussian circle of f' that passes through one of these nodes is positive. Thus any negative Gaussian circle of f' is a negative Gaussian circle of f and is already crossed by α.

Consider the relation between the permutation $S_0 = S(xy)$ on $X = X(f, \alpha)$ and the successor permutation S' of f' in $X' = X'(f', \alpha')$. Each cycle of S' is obtained by taking a cycle of S_0 and, possibly, inserting a few of the new crossings. Every new crossing is so inserted in some cycle of S_0. Thus S' is an expansion of S_0. Regard the permutation $P' = (xy)P$ as acting on X'. It is a pairing, with one fewer pairs than P. The resultant $R' = S'P'$ is an expansion of $R = SP$, and so the cycle numbers ζ' and ζ are equal. Each cycle of R' expands a corresponding cycle of R.

While P' has as many pairs as there are negative crossings in X', it need not be transitive on S'. However, the family f' consists in at most two subfamilies, f'_i, $i = 1, 2$, with successor permutations S'_i, so that $S' = S'_1 S'_2$, $P' = P'_1 P'_2$, and P'_i is transitive on S'_i. Here we may apply induction, since $\nu' = \nu'_1 + \nu'_2 = \nu - 1$. Thus $M' = M_{P'}$ has at most two components, and F' is a polymersion of M' into the plane with $\partial F' = f'$. Note that since α'_x, α'_y carry no negative crossings of X', they play no part in the cell structure on M', which, by induction is nearly faithful. Since $\tau' = \tau + 1$, we obtain (3.1) as soon as we can embed M' in a connected (oriented) surface M and extend F' to F on M with $\partial F = f$ so as to introduce no new branching. Attach a rectangular strip to $\partial M'$ which embeds in R^2 so as to fill the rectangular strip marked by dashed lines in the figure. Since the Euler characteristics are related as $\chi' = \chi + 1$ and $\chi = 2 - 2\gamma - \rho$ for a connected (compact, oriented) surface, we have (3.2). The attachment to $\partial M'$ is made along disjoint interior subarcs of border edges of M'. (It could be the same edge if M' is already connected.) Subdivide the bridging strip by the lift $\hat{\alpha}_0$ of α_0 via F^{-1}. Erase the two subarcs in $\partial M'$ (leaving their endpoints, of course), to obtain the cellulation M_P for M. The face(s) of M' adherent to the deleted subarcs have been enlarged slightly and F' has been extended so as to be 1:1 on the additions. So the critical points of F and F' coincide. Moreover, it is clear from our construction that if F' is 1:1 on one of these faces in M' then F is 1:1 on its enlargement. Hence M_P is faithful to F. □

Observe that if we eliminate ζ from (3.1–2) we obtain a Hurwitz-Riemann relation for polymersions into the plane:

$$(3.3) \qquad \chi = \tau - \mu \quad \text{or} \quad 2 + \mu = 2\gamma + \rho + \tau.$$

This formula, for the case that $\gamma = 0$, follows from the central formula of Morse and Heins [16, p. 67]. If $\zeta = \nu + \tau$, hence $\mu = 0$, then F is an immersion, and the cellulation is faithful (in particular, the immersion is 1:1 on each cell). In this case, (3.3) becomes Haefliger's formula, proved in [10] for the case that F and M are smooth. Let us briefly sketch that (3.3) is indeed a necessary condition also for continuous immersions into the plane.

PROPOSITION 3. *If F is an immersion of a compact, bordered surface M into the plane then the turning number τ of $f = \partial F$ equals the Euler characteristic χ of M.*

Proof. To compute τ we must orient f, so that for a neighborhood U of $p \in \partial M$ on which F is 1:1, $F(U - \partial M)$ lies to the left of $f \mid U \cap \partial M$. Since both χ, τ are additive, we assume that M is connected. Consider an arc joining two border points through the interior of M that does not disconnect M. Let M' be the surface obtained by performing the associated Riemann cut and F' the immersion of M' into the plane which factors into the immersion of M' in M followed by F. Since $f' = \partial F'$ makes four additional pairwise supplementary turns, $\tau' = \tau + 1$. Since $\chi' = \chi + 1$, the difference $\chi' - \tau' = \chi - \tau$ is constant under the finitely many Riemann cuts required to reduce the surface to a disc.

So let M be a disc ($\chi = 1$) with polar coordinates (r, θ), $0 \leq r \leq 1$, and F a (sense preserving) immersion of M into R^2. As $r \to 0$, the restrictions f_r of F to circles of radius r deform $f = f_1$ into a continuous succession of oriented, locally simple, closed curves such that for each sufficiently small interval $r_1 \leq r \leq r_2$, f_r is simple on arcs of length $e > 0$. So $\tau(f_r)$ is locally constant on $0 < r \leq 1$. Since F is locally 1:1 on M, there is some small $r > 0$, so that f_r is a positively oriented closed Jordan curve in the plane. Here $\tau = 1$. $\square$

Let us call an assembling permutation on $X(f, \alpha)$ *effective* if it has exactly as many pairs as there are negative crossings. (By definition it cannot have more.) Consider an immersion F of a compact, connected, bordered surface M so that $F(M)$ does not contain some point, say ∞, on S^2. We take R^2 to be $S^2 - \{\infty\}$, as usual. Finally, assume also that $\partial F = f$ is a normal family of curves. Let α be a sufficient raying for f. For each negative crossing $x \in X(f, \alpha)$, the lift of $[x, \infty]\alpha$ from the unique point $\hat{x} \in F^{-1}(x) \cap M$, must terminate at some border point $\hat{y}$ where y is a positive crossing. Thus F uniquely determines an effective pairing P. To establish that the effective, transitive pairings on $X(f, \alpha)$ for which $\zeta = \nu + \tau$ constitute the assemblages for all such immersions, it remains to demonstrate the following.

PROPOSITION 4. *Let F be an immersion in R^2 of a compact connected, bordered surface M so that $f = \partial F$ is a normal family of ρ curves of turning number τ. Let α be a sufficient raying for f with ν negative crossings in $X(f, \alpha)$. Let S be the successor permutation induced by f and P the effective pairing induced by F. Then P is transitive on S and there is a (sense-preserving) homeomorphism $H\colon M_P \to M$, so that $F_P = F \circ H$, where (M_P, F_P) is the cellular pair constructed in Section 2.*

Proof. We use induction on ν. For $\nu = 0$ we first apply Proposition 1 to each of the ρ member curves of f to see that $\tau \geq \rho$. By Proposition 3, $2 = 2\gamma + \rho + \tau \geq 2(\gamma + \rho) \geq 2$. Hence $\rho = \tau = 1$ and $\gamma = 0$. Thus M is a disc and f a closed Jordan curve. (For, if f were not simple, by normality there would be a node. Through each node pass two distinct Gaussian circles, which are positive by assumption that $\nu = 0$ and that α is sufficient. Hence, by Gauss' observation, $\tau \geq 2$.) Although P is the identity, S is already transitive, since it consists of a single cycle. Hence P, S generate a transitive group. M_P is the closure of the bounded component of the complement of f and F_P is the identity.

Finally, F is 1:1, being an immersion of the disc in the plane with border mapping to a Jordan curve. Set $H = F^{-1}$.

For $\nu > 0$, let x be an ultimate negative crossing, and let $y = x^P$, which is determined by F. Let us denote by λ the arc on M over $[x, y]\alpha$ to distinguish it from $\hat{\alpha}_0$ in M_P. Since F is an immersion, its restriction to some neighborhood U of λ is a homeomorphism. Let T be a sufficiently narrow normal neighborhood of α so that $F(U)$ covers the relevant portion of T about $[x, y]\alpha$. Proceed with the modification described in the proof of Proposition 2, thereby obtain the subsurface M' of M. (M' could have two components.) Apply induction to $F' = F \mid M'$ (or its two components F_i') to obtain a homeomorphism $H'\colon M_{P'} \to M'$ so that $F_{P'} = F' \circ H'$. On $F_{P'}^{-1}(F(U) \cap T)$ set $H = F^{-1} \circ F_P$. For a point p in the domain of H and in $M_{P'}$, we see that

$$H(p) = F^{-1}(F_{P'}(p)) = F^{-1}(F' \circ H'(p)) = (F^{-1} \circ F)H'(p) = H'(p).$$

So we may extend H to the rest of M_P by H'. Since M is connected and M_P is cellular, P is transitive on S. $\square$

Now let P be an arbitrary assembling permutation on $X(f, \alpha)$. Let the *partial branching number* π be the branching number of $\tilde{F}_P$ as defined on M_P at the end of Section 2. That is, π equals the total number of crossings occurring in all fans of P minus the number of fans. We have a generalization of Proposition 2.

PROPOSITION 5. *Under the hypotheses of Proposition* 2, *except that P now is an assembling permutation with partial branching number π, the same conclusions hold except that in the formulas* (3.1–2) *π is to be added to the right-hand sides.*

Proof. We use induction on π. For $\pi = 0$, we have Proposition 2 itself. For $\pi > 0$, let $P_0 = (x_1 \cdots x_n)$, $n \geq 2$, be a fan of P permuting positive crossings on ray α_0 with initial point a_0. Let f'' be a normal closed curve (with n nodes), located in the complementary component of f containing a_0, which winds n times clockwise about a_0 and which crosses the ray α_0 at n new negative crossings in the succession $S'' = (\bar{x}_1 \cdots \bar{x}_n)$. See Figure 4. (For precision, take f'' as the image of a suitably situated and negatively oriented closed Jordan curve about the origin under $w = a_0 + z^n$.) Note that α is still sufficient for the normal curve family f' consisting of f and f''. f' has $\rho' = \rho + 1$ curves of turning number $\tau' = \tau - n$ such that $S' = SS''$ is its successor permutation on $X' = X \cup \{\bar{x}_1, \ldots, \bar{x}_n\}$.

The permutation $P' = (\bar{x}_1 x_1) \cdots (\bar{x}_n x_n) P_0^{-1} P$ is an assembling permutation for f' with one fewer fans and n additional pairs. Thus $\pi' = \pi - n + 1$ and $\nu' = \nu + n$. Algebraically:

$$\begin{aligned} R' &= S'P' \\ &= S(\bar{x}_1 \bar{x}_2 \cdots \bar{x}_n)(\bar{x}_1 x_1)(\bar{x}_2 x_2) \cdots (\bar{x}_n x_n)(x_n x_{n-1} \cdots x_1)P \\ &= S(\bar{x}_1 x_1)(\bar{x}_2 x_2) \cdots (\bar{x}_n x_n)P \\ &= SP(\bar{x}_1 x_2)(\bar{x}_2 x_3) \cdots (\bar{x}_n x_1). \end{aligned}$$

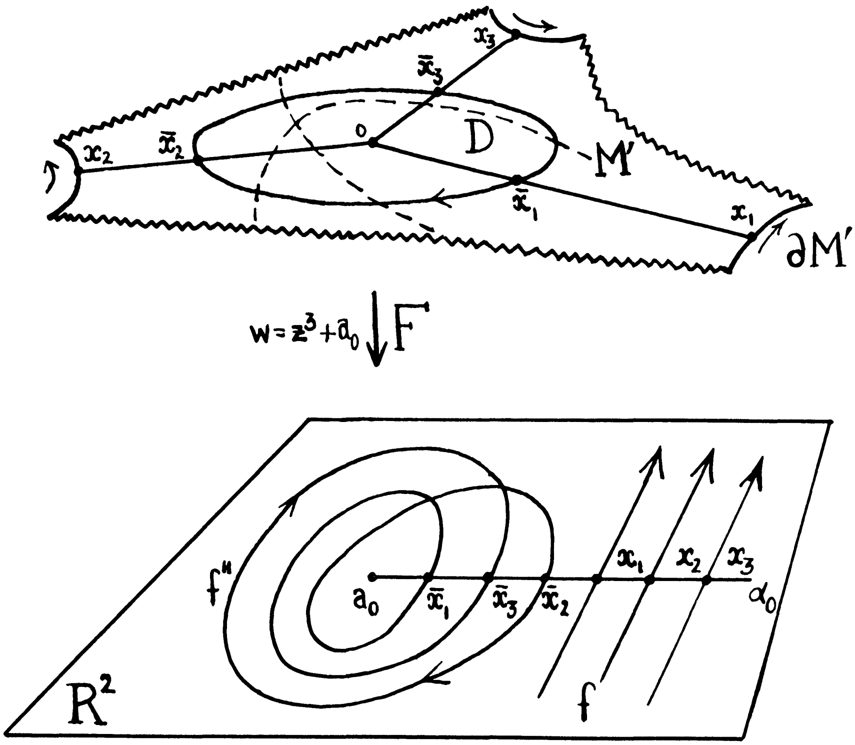

FIGURE 4

Hence R' is an expansion of R and $\zeta' = \zeta$. Apply induction to f' and P', where $f' = \partial F'$, and F' is the cellular polymersion on $M' = M_{P'}$. Attach a disc D to M' along the border circle corresponding to f''. The disc is to be mapped by the extension F of f' across D so as to have a single critical point of multiplicity $n - 1$ over a_0. (The figure shows the case $n = 3$, hats have been omitted and the dashed curves are the "counterimages" of f'' in $M' \cup D$, for reference.) Thus $\mu = \mu' + n - 1$. We check that

$$(\mu - n + 1) = (\nu + n) + (\tau - n) - \zeta + (\pi - n + 1).$$

Whence the generalization of (3.1) goes through the induction step. The genus is unchanged and we have

$$2\gamma = 2 + (\nu + n) - (\rho + 1) - \zeta + (\pi - n + 1),$$

whence the second formula goes through. Note that we do get a nearly faithful cellulation on M for F simply by erasing the edges on ∂D. Since M is connected, P is transitive on S. $\square$

Observe that $\mu \geq \pi$ and if equality holds then all branching occurs on the interior vertices on M_P. Hence $\zeta = \nu + \tau$ if and only if the cellulation is faithful to F. Now suppose that F is a given polymersion in S^2 which does not cover the sphere and which is branched over $a_1, \ldots, a_w$. We shall require that $f = \partial F$ and the a_i lie in general position, i.e., f is a normal family of curves not passing through any of the a_i. We then can choose a sufficient raying α for f such that each a_i is the initial point of some ray, and the reference point $\infty \notin F(M)$. Let $\hat{a}_0$ be a critical preimage of some branchpoint a_0 of F, i.e., $\mu(a_0) = n - 1 \geq 1$. There are n distinct lifts of $[a_0, \infty]\alpha$ emanating from $\hat{a}_0$ and these arcs terminate at n points $\hat{x}_j$ on ∂M where x_j is a positive crossing. Let $P_0 = (x_1 \cdots x_n)$ permute the x_j clockwise. This is a fan of the effective assembling permutation P uniquely determined by F. (The pairs are obtained as before.) Under these circumstances the analogous generalization of Proposition 4 is essentially obvious.

PROPOSITION 6. *Under the hypotheses of Proposition* 4, *except that F is a polymersion as described above, and P is the associated assembling permutation, the same conclusion holds.*

Proof. Induction on μ. For $\mu = 0$ we have Proposition 4. For $\mu > 0$ choose a critical point $\hat{a}_0$ of F and a chart (U, h) centered at $\hat{a}_0$ so that $F \circ h^{-1}(z) = a_0 + z^n$ restricted to $h(U)$. Choose a clockwise oriented closed Jordan curve enclosing a disc D in $h(U)$ so that $F \circ h^{-1}(\partial D)$ is the curve f used in the proof of Proposition 5. (The reader familiar with function theory will recognize that our construction is the topological analog of choosing a local parameter z at $\hat{a}_0$.) The rest of the proof proceeds in straightforward analogy to that of Proposition 4. $\square$

Before proceeding with the fully general case, let us pause for a review. Given the normal curve family f and the intended set of branch points A_0 in the complement of f, to determine all polymersions F extending the data (f, A_0) we choose any convenient but sufficient raying α with the set A of initial points containing A_0. ("Convenient" means few crossings and a visually or combinatorially practical placement of the rays.) As we have seen, the F without poles (i.e., $\infty \notin F(M)$) are in 1:1 correspondence with the assembling permutations P on $X(f, \alpha)$ that have four properties: P and S generate a transitive group, P has a maximal number of pairs (effective), $R = SP$ has a maximal number of cycles ($\zeta = \nu + \tau$ or $\mu = \pi$), and the fans of P only permute crossings on rays initiating in A_0. We now see the meaning of the Löwner-Titus conjecture [19] concerning curves of nonnegative circulation. (The *circulation* about a point q not on f is the sum of the degrees of

$$S^1 \to S^1 : s \to f_i(s) - q/|f_i(s) - q|,$$

over the members f_i of f.) If the circulation about each point in R^2 of f is nonnegative it is always possible to obtain an effective pairing. If the closure of the set of positive circulation is connected, it is always possible to find enough fans, if needed, to make the assembling permutation transitive. Hence f necessarily extends to some polymersion into the plane. This was the approach followed in this author's first proof of the conjecture [5]. The short proof based on the construction used in Proposition 1 (applied only to a single normal curve for simplicity) appeared in [7], [9]. But the associated cellulation of the domain of F is only nearly faithful. With some effort (drawing more rays from selected complementary components of f), one can in principle search for and find an assemblage for which $\zeta = \nu + \tau$ as well, and thus obtain a faithful cellulation for F. Efficient ways of constructing assemblages for this problem (i.e., given f only, no also A_0) are yet to be developed.

Unlike polymersions (into S^2, of course) of closed surfaces, for which the minimum number of faces of a faithful cellulation of the surface is given by the degree of the map, polymersions of bordered surfaces do not have a constant topological degree. A second unsolved problem is to determine the minimum number of faces a faithful cellulation, however obtained, can have, and then to construct it. It can be shown that an upper bound for this number equals the number of positive Gaussian circles of f, computed in any plane $R^2 = S^2 - \{\infty\}$ in which f has nonnegative circulation.

4. Polymersion to S^2

Let F be a polymersion of M to S^2 with branchpoints a_i, $i = 1, \ldots, w$, such that $f = \partial F$ is a normal curve family not passing through the a_i. Let α be a sufficient raying for f with reference point ∞ and let A be the set of m initial points of the rays. We assume that the a_i belong to A so that F is not branched over ∞ and $m \geq w$. The $\beta = \text{card } F^{-1}(\infty)$ points in $F^{-1}(\infty)$ are called the *poles* of F. Let B be an abstract set of β elements, and let $F^{-1}(\infty)$ be labeled by B: $b \in B$ is assigned to pole $\hat{b}$. For $a \in A$, the lift of $[a, \infty]\alpha$ from $\hat{a} \in F^{-1}(a)$ (or of $[x, \infty]\alpha$ from $\hat{x} = F^{-1}(x) \cap \partial M$, if x is a negative crossing in $X = X(f, \alpha)$) may now terminate at a pole. (Indeed, for a closed surface, this is invariably the case.) Let U_b be a neighborhood of $\hat{b} \in F^{-1}(\infty)$, separating $\hat{b}$ from the closed subset $F^{-1}(A \cup F(\partial M))$ so that $F \mid U_b$ is a homeomorphism. In a normal neighborhood of α choose β concentric discs D_b about the origin in the (r, θ)-plane, so that their homeomorphs in S^2 (also denoted by D_b) lie in $\bigcap F(U_b)$. Let g be the family of β disjoint simple closed curves g_b, where g_b is ∂D_b oriented clockwise about ∞, i.e., counterclockwise in $R^2 = S^2 - \{\infty\}$. Then $M^* = M - \bigcup F^{-1}$ (Int D_b) is a connected surface of characteristic $\chi^* = \chi - \beta$, $F^* = F \mid M^*$ is a polymersion of M^* to R^2 and $f^* = \partial F^*$ is the normal family of $\rho^* = \rho + \beta$ curves $f \cup g$, of turning number $\tau^* = \tau + \beta$ in R^2. Note that α is still sufficient for f^*. All $m\beta$ new crossings in $X^* = X(f^*, \alpha)$ are positive and separate the crossings in X from ∞.

Applying our theory to F^*, we obtain the effective assembling permutation P^* on X^*, which is transitive on the successor permutation S^* so that $R^* = S^*P^*$ has $\zeta = \nu + \tau^* = \nu + \tau + \beta$ cycles. The branching number of F^* equals that of P^*, i.e., $\mu = \pi^*$. Thus we obtain the Hurwitz-Riemann relation for polymersions to S^2:

$$\chi = \tau - \mu + 2\beta \quad \text{or} \quad 2 + \mu = 2\beta + 2\gamma + \rho + \tau. \tag{4.1}$$

It is clear how to obtain a surface M_Π from M_{P*} by attaching β discs (each radially subdivided into m sectors) to ∂M_{P*} along $(F \circ H^*)^{-1}(g)$, and how to extend the polymersion F_{P*} to F_Π, and the homeomorphism H^* to H, so that $F_\Pi = H \circ F$. (To obtain a faithful cellulation on M_Π with ζ faces, we must erase the superfluous edges corresponding to the g_b. As usual we erase the vertices of degree 1 and their adherent edges.) The only remaining problem is to give meaning to the symbol Π which is independent of the choice of g and the associated labeling of X^*. Note that for each $b \in B$ the successor permutation of g_b, $S^* \mid X(g_b, \alpha)$, is a labeled copy of the same cyclic permutation K on A that cycles the rays clockwise about ∞. For each $a \in A$, $P^* \mid X(f^*, \alpha_a)$ is a labeled copy of a permutation H_a on $X_a + B$ (disjoint union of $X_a = X(f, \alpha_a)$ and B). Thus the entire information, modulo a choice of g and labeling of X^*, contained on S^* and P^* is contained in the permutations S on X, K on A, and the m permutations H_a on $X_a + B$. That is, under identification of X^* with $X + A \times B$, we may recover S^*, P^* as follows. (To avoid unprintable symbols, we write xP for x^P.) For $x \in X$, set $xS^* = xS$ and $xP^* = xH_a$ where $a = a_x$, and for $(a, b) \in A \times B$, set $(a, b)S^* = (aK, b)$ and $(a, b)P^* = (a, bH_a)$ or just bH_a if this is in X_a. Since S and K are fixed for given f and α, we shall call $\Pi = \{H_a: a \in A\}$ an *assembling system* for $X(f, \alpha)$ of *degree* β at ∞. (For $\beta = 0$, the product of the H_a is just an assembling permutation.) It is clear what the adjectives "effective" and "transitive" should mean for an assembling system in terms of the associated S^*, P^* on X^*.

Let us describe the cell structure on M_Π more explicitly. (We drop the subscript Π.) For $x \in X$ we have the border vertices $V(x) = \hat{x}$, on ∂M. For each fan Φ_a of H_a we have an interior vertex $V(\Phi_a)$, which is critical for F, and which has degree equal to the length of the cycle Φ_a. For $\beta > 0$ and $b \in B$, there is an interior vertex $V(b)$, which is a pole of F. (Those of degree 1 may be erased later.) For each $x \in X$, there is the oriented border edge $E^\partial(x)$ between $V(x)$ and $V(xS)$ on ∂M. Let $\hat{\alpha} = F^{-1}(\alpha)$ denote the network of arcs on M. For each $a \in A$ we have the following interior edges. For $\bar{x} \in X_a$ a negative crossing, we have the edge $E^0(\bar{x})$ on $[V(\bar{x}), V(\bar{x}H_a)]\hat{\alpha}$. For a positive $x \in X_a$ which occurs in a fan Φ_a of H_a we have $E^0(x)$ on $[V(\Phi_a), V(x)]\alpha$. For $(a, b) \in A \times B$, with b, occurring in some fan Φ_a of H_a, we have the edge $E^0(a, b)$ on $[V(\Phi_a), V(b)]\hat{\alpha}$. For each of the ζ cycles in $R^* = S^*P^*$ there is one face. For these it is convenient to use the labeling $C(u)$ for R^*_u, where $u \in X^* = X + A \times B$. Note that at each vertex V, P^* defines a clockwise cyclic permutation of the adherent faces that takes $C(u)$ to $C(uP^*)$. Once we erase the interior vertices of degree 1 and their adherent edges, we can use the cycles R^*_u to describe the boundary curve on $\partial C(u)$ very nicely.

Thus, as the reader may check, we have proved the following.

THEOREM. *Let f be a normal family of ρ curves on S^2. Let α be a sufficient raying for f with reference point ∞ and set A of initial points. Let τ be the turning number of f in $R^2 = S^2 - \{\infty\}$ and $X(f, \alpha)$ have ν negative crossings in it.*

Let $\Pi = \{H_a: a \in A\}$ be an assembling system for $X(f, \alpha)$ of degree β, with associated permutations S^*, P^* on X^*. Let π be the partial branching number of P^* and $R^* = S^*P^*$ have ζ cycles. If S^*, P^* generate a transitive group on X^* and P^* has ν pairs, then there is a compact, connected, bordered (if $\rho > 0$), cellular surface M_Π of genus γ, and a polymersion F_Π of M to S^2 with $f = \partial F_\Pi$ and with branching number μ where

$$\mu = \nu + \tau - \zeta + \pi + \beta, \tag{4.2}$$

and

$$2\gamma = 2 + \nu - \rho - \zeta + \pi - \beta. \tag{4.3}$$

Moreover, the cellulation on M_Π is nearly faithful to F_Π. It is faithful if and only if

$$\zeta = \nu + \tau + \beta \quad \text{or} \quad \mu = \pi. \tag{4.4}$$

Conversely, let F be a polymersion of a compact, connected surface M to S^2 with $\partial F = f$, with β poles and branched over a subset A_0 of A. Let Π be the effective assembling system of degree β induced by F and S^*, P^*, $R^* = S^*P^*$ the associated permutations on X^*. Then P^* is transitive on S^*, (4.4) *holds, and only those H_a have fans for which $a \in A_0$. Moreover, there is a homeomorphism H from M_Π to M so that $F_\Pi = F \circ H$.* $\square$

Suppose F is an immersion, then in the induced effective, transitive assembling system $\Pi_F = \{H_a: a \in A\}$ each H_a is a pairing on $X_a + B$, and we may simplify matters as follows. Let us define a *wheel* on $X(f, \alpha)$ to be a cyclic W on negative crossings in X, each of which lies on a distinct ray and so that the cycle K on A is an expansion of W acting on A. Recall that $S^* = SK^*$ where K^* is the product of β distinct copies of K. We state, without proof, that there is a unique permutation P on X with β wheels (counting fixed negative crossings too) and pairs on X such that K^*P^* is an expansion of P under the adjunction of $A \times B$ to X. Furthermore, $R^* = S^*P^*$ is an expansion of $R = SP$. Thus R has the same number of cycles as R^*.

Conversely, if P is a product of (disjoint) pairs and β wheels on $X(f, \alpha)$, such that $R = SP$ has $\nu + \tau + \beta$ cycles, and S, P generate a transitive group on X, then there is a unique transitive, effective assembling system so related to P. However, rayings are not very efficient means for constructing assemblages for immersions of bordered surfaces to S^2; in general there are too many crossings to consider. In a forthcoming study we shall develop better assemblages for immersions in connection with a similar combinatorial classification of proper, stable maps of surfaces to S^2 and R^2. The above assertions will constitute a special case there.

In conclusion, we return to Hurwitz' original case of an n-sheeted branched covering (M, F) of S^2 with w branchpoints. Here $n = \beta = \text{card}(\beta)$, $w = \text{card}(A)$, H_a is an arbitrary permutation on B for each $a \in A$, and K is a cyclic permutation of A. Further, our associated permutations S^*, P^*, and $R^* = S^*P^*$ on $X^* = A \times B$ satisfy

$$(*) \qquad (a, b)S^* = (aK, b),\ (a, b)P^* = (a, bH_a),\ (a, b)R^* = (aK, bH_{aK}).$$

From this it follows that S^*, P^* are transitive on B (recall that K is transitive on A). Iterating R^*, we notice that the length $\lambda(a, b)$ of the cycle $R^*_{(a,b)}$ must satisfy $\lambda(a, b) \geq w$ (w is the order of K). Since R^* acts on nw elements and has ζ cycles, we have $nw \geq \zeta w$ or $n \geq \zeta$. Thus equality (4.4), $n = \beta = \zeta$, holds if and only if each $\zeta(a, b) = w$ which is equivalent, by (*), to Hurwitz' condition

$$H_{aK}H_{aK^2}\cdots H_{aK^w} = 1.$$

5. Discussion

Hurwitz' tacit identification of a closed Riemann surface with a branched covering (M, F) of S^2 induced by a meromorphic function is historically valid. Our use of Riemann's name in connection with our generalization to bordered surfaces is no more frivolous, since Riemann is a totem in many disciplines, including combinatorial topology. A polymersion F of a surface M, as we have defined it, induces a conformal structure on M in terms of which F becomes a meromorphic function. There is no question here of classifying conformal structures on bordered surfaces by means of assemblages. While it is true that two polymersions that have the same assemblage (with reference to the same set of curves and branchpoints) induce conformally equivalent structures, the converse is false. Milnor's paisley curves [3, 6] admit many different assemblages for immersions of the disc into the plane. Which assemblages belong to the same conformal class is still to be solved. What we have classified by assemblages is the polymersions that induce the structures.

Our placing of the curves in general position with respect to the intended branchpoints and with respect to themselves (normality) deserves a brief explanation. A local power map which is not critical on the border points of its domain is what Morse [16] called an "interior" map satisfying his first boundary condition. (We have avoided the term "interior" because it more often means just "open." By Stoilow's theorem, it is an open and light map that is a local power map.) Permitting branchpoints to lie on the curves leads to a more complicated Hurwitz-Riemann relation and to more complicated assemblages. Consider, for example, the restrictions of the meromorphic map $w = (z^2 - i)/(z^2 + i)$ to the (compactified) quadrant on the one hand and to the union of the first three quadrants on the other. Both domains are topological discs, both maps are locally 1:1 on the interior and 1:1 on the border, which is mapped to the unit circle in both cases. To obtain a faithful cellulation for the second map, however, the domain has to be subdivided so that each sector is a face.

It is difficult to say anything about the bounding possibilities of a nonnormal curve. Consider the figure eight in R^2, but oriented so as to have turning number 1, not 0. That is, the double point is not a node, but a "tangency."

There is no polymersion of a surface to R^2 whose border maps to this curve. By remaining in the C^0-context, we have departed from the tradition established in the studies cited in the introduction, except for [16], all of which assume the given curve(s) f to be C^1-regular. While this regularity condition simplifies the discussion of nodes and turning numbers, it complicates modifications by "cutting and pasting" because one has forever to "smooth corners." Further, the polymersions extending regular curves should locally be smooth (i.e., the Jacobian vanishes only at critical points over branchpoints). For this one has to use versions of the smooth Schoenflies theorem.

On the other hand, in C^0, locally simple closed curves do not constitute an open subset, even in the Frechet topology used by Morse. (Witness the deformation that smoothly shrinks one loop of the smooth figure eight, through a cusp, to a simple curve.) Using supplementary smooth arguments, our results hold also for smooth normal curves and polymersions. For this reason we have retained the "smooth" term "immersion," and coined the neologism "polymersion" to recall the "polynomial" structure at branch points. Since our maps of bordered surfaces do not have constant "degree" (even counting multiplicity), we avoided calling our maps "branched coverings."

There are three directions one could go from here. The first is to develop assemblages for polymersions whose target is a surface of characteristic $\chi < 1$. The second is to investigate further the "surgery" on polymersions, which is expressed in multiplying the successor permutations of a sufficiently rayed normal family of curves in R^2, by successive pairs and fans. The third is to observe how suitable deformations of curves from one general position into another is reflected in "morphisms" between their assemblages.

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