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# MOODY'S INDUCTION THEOREM[1]

BY

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**Dedicated to the Memory of Irving Reiner**

## 1. Introduction

Our purpose is to give a proof of the recent remarkable induction theorem of John Moody [1], a proof that is straightforward and more or less self contained. Let $\Gamma$ be a finitely generated abelian by finite group, and let $S * \Gamma$ be a crossed product of a left noetherian ring $S$ with $\Gamma$. Let $G_0(S * \Gamma)$ denote the Grothendieck group of the category of all finitely generated $S * \Gamma$-modules. For any subgroup $F$ of $\Gamma$, there is a map $G_0(S * F) \to G_0(S * \Gamma)$ given by sending the class $[M]$ of an $S * F$-module $M$ to the class $[S * \Gamma \otimes_{S * F} M]$ of the induced module.

MOODY'S THEOREM. *Let $\alpha$ be the sum of the maps from $\Sigma G_0(S * F)$ to $G_0(S * \Gamma)$, where $F$ varies over all finite subgroups of $\Gamma$. Then $\alpha$ is surjective.*

As an application to $G_0$ of group rings, let $H$ be a polycyclic by finite group, and let $k$ be a noetherian ring.

MOODY'S THEOREM FOR POLYCYCLIC BY FINITE GROUPS. *The map from $\Sigma G_0(kF)$ to $G_0(kH)$, given by the sum of inductions from finite subgroups $F$ of $H$, is surjective.*

To prove this, let $H_1$ be a normal subgroup of $H$ of smaller Hirsch length than $H$, such that $H/H_1 = \Gamma$ is abelian by finite, and write the group ring $kH$ as a crossed product $(kH_1) * (H/H_1)$. Then use induction on the Hirsch length.

Here is an outline of our proof of Moody's Theorem. Let $A$ be a finitely generated free abelian normal subgroup of $\Gamma$ of finite index, and let $G$ denote

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the factor group $\Gamma/A$. Suppose that $\mathbf{Q} \otimes_{\mathbf{Z}} A$ is a free $\mathbf{Q}G$-module. Then $A$ is contained as a subgroup of finite index $n$ in a group $B$ which is a free $\mathbf{Z}G$-module. In Section 2 we show that the matrix ring $M_n(S * A)$ is graded by $B$, in a way which is compatible with the action of $G$; then, picking a positive cone $B_+$ in $B$, we define a certain subring $R$ of $M_n(S * \Gamma)$ generated by $G$ and $B_+$, and $R$ is then graded by the non-negative integers. Moreover we can identify $R_0$ as a direct sum of full matrix rings over certain finite subgroups of $\Gamma$. That $G_0(R) \cong G_0(R_0)$ follows from work of Quillen [3]. In Section 3 we show that our map $\alpha\colon \Sigma G_0(S * F) \to G_0(S * \Gamma)$ is the composition of four other maps, two of which come from Morita equivalences, one from Quillen's theorem, and one from localization. In Section 4 we give a proof of Quillen's theorem; we are able to avoid the use of Quillen's topological machinery, since we are only interested in $G_0$, and not in higher $K$-theory. At this point, Moody's theorem will follow, under the assumption that $\mathbf{Q} \otimes_{\mathbf{Z}} A$ is a free $\mathbf{Q}G$-module; if not, we form a semi-direct product of $\Gamma$ with a free abelian group $N$ to get a group $\Gamma_1$ for which Moody's theorem will have been proved, and then show in Section 5 that we can reduce back to $\Gamma$. In Section 6 we deal with the Goldie rank problem for the group ring of a polycyclic by finite group over a division ring $k$.

We would like to thank John Moody for sending us a copy of his thesis.

## 2. The grading on $M_n(S * \Gamma)$

Let $S$ be a ring and $\Gamma$ a group. (All rings here are associative with 1.) Suppose that for each $\gamma \in \Gamma$ there is an automorphism of $S$, denoted $s \mapsto {}^{\gamma}s$ for $s \in S$. A ring is called a *crossed product* of $S$ with $\Gamma$, denoted $S * \Gamma$, if it has a basis as a left $S$-module $\{\bar{\gamma}\colon \gamma \in \Gamma\}$ indexed by $\Gamma$, with multiplication given by $\bar{\gamma}s = {}^{\gamma}s\bar{\gamma}$ for $s \in S$ and $\gamma \in \Gamma$, and $\bar{\gamma}\bar{\delta} = f(\gamma, \delta)\overline{\gamma\delta}$ for $\gamma, \delta \in \Gamma$, where $f(\gamma, \delta)$ is some unit of $S$.

Let $A$ be a finitely generated free abelian group, contained as a normal subgroup of finite index in the group $\Gamma$, and let $G$ denote the factor group $\Gamma/A$. Then $A$ is a $\mathbf{Z}G$-module. Suppose, for now, that $\mathbf{Q} \otimes_{\mathbf{Z}} A$ is a free $\mathbf{Q}G$-module. Let $B$ be a free $\mathbf{Z}G$-module containing $A$ as a submodule of finite index $n$. (Explicitly, one may take a $\mathbf{Q}G$-basis of $\mathbf{Q} \otimes_{\mathbf{Z}} A$ contained in $1 \otimes A$ and let $A_1$ be the $\mathbf{Z}G$ span of this basis; then multiply $A_1$ by a rational number so that it contains $1 \otimes A$, letting the result be $B$, and identify $1 \otimes A$ with $A$.) The extension $\Gamma$ of $G$ by $A$ leads to an extension $\Delta$ of $G$ by $B$. Since $B$ is a free $\mathbf{Z}G$-module, the extension $\Delta$ splits, and we shall regard $G$ as a subgroup of $\Delta$.

Let $X$ be a set of representatives of right cosets of $A$ in $B$. Then $X$ has cardinality $n$, and is also a set of representatives of the right cosets of $\Gamma$ in $\Delta$. Let $V$ be a free right $S * \Gamma$-module with basis $\{v_x\colon x \in X\}$ indexed by $X$. Let $\mathscr{E} = \operatorname{End}_{S*\Gamma}(V)$. Then using the basis $\{v_x\colon x \in X\}$, $\mathscr{E}$ is isomorphic to

$M_n(S * \Gamma)$, and is therefore an $(S * \Gamma, S * \Gamma)$-bimodule. We shall show that $\mathscr{E}$ is a $\mathbf{Z}$-graded ring.

For $x, y \in X$, let $\sigma_{x,y} \in \mathscr{E}$ be the map which sends $v_x$ to $v_y$ and which sends $v_z$ to 0 for $z \in X$, $z \neq x$, so $\mathscr{E}$ is a free $S * \Gamma$-module with basis $\{\sigma_{x,y}\}$. Let $e_x = \sigma_{x,x}$. For $\delta \in \Delta$, define $\phi(\delta) \in \mathscr{E}$ as follows:

$$\phi(\delta)(v_x) = v_y\bar{\gamma}, \text{ where } \delta x = y\gamma \text{ for some } y \in X,\ \gamma \in \Gamma.$$

From this definition, it follows that

$$e_y\phi(\delta) = \phi(\delta)e_x. \tag{1}$$

For $x, y \in X$, let $\delta = yx^{-1} \in \Delta$; then $\delta x = y$, and $\sigma_{x,y}\bar{1} = \phi(\delta)e_x$, so it follows that

$$\{\phi(\delta)e_x\colon \delta \in \Delta,\ x \in X\}$$

is an $S$-basis of $\mathscr{E}$.

For $\delta_1, \delta_2 \in \Delta$, to form the product $\phi(\delta_1)\phi(\delta_2)$, take $x \in X$ and write $\delta_2 x = y\gamma_1$, for some $y \in X$ and $\gamma_1 \in \Gamma$; then write $\delta_1 y = z\gamma_2$ for some $z \in X$ and $\gamma_2 \in \Gamma$. From the definition of $\phi$, we find that

$$\phi(\delta_1)\phi(\delta_2)(v_x) = v_z\overline{\gamma_2}\,\overline{\gamma_1}, \quad \phi(\delta_1\delta_2)(v_x) = v_z\overline{\gamma_2\gamma_1},$$

which implies that

$$\phi(\delta_1)\phi(\delta_2)e_x = \phi(\delta_1\delta_2)se_x \quad \text{for some } s \in S \text{ which depends on } x. \tag{2}$$

Let $\mathscr{B} = \{b_1, b_2, \dots, b_m\}$ be a $G$-invariant basis of the free abelian group $B$. Define

$$d\left(\prod b_i^{n_i}\right) = \sum n_i,$$

so it follows that for $b, b' \in B$ we have $d(bb') = d(b) + d(b')$ and for $g \in G$ we have $d(gbg^{-1}) = d(b)$. For $\delta \in \Delta$, we may write $\delta$ uniquely in the form $\delta = bg$, for some $b \in B$, $g \in G$. Then define

$$\deg(\phi(bg)e_x s) = d(b), \quad b \in B,\ g \in G,\ x \in X,\ s \in S.$$

It follows from formulas (1) and (2) that this makes $\mathscr{E}$ into a $\mathbf{Z}$-graded ring. Let

$$B_+ = \left\{b = \prod b_i^{n_i} \in B\colon n_i \geq 0,\ i = 1, \dots, m\right\}$$

and let $R$ be the subring of $\mathscr{E}$ given by

$$R = \left\{ \sum s\phi(bg)e_x \colon s \in S,\ b \in B_+,\ g \in G,\ x \in X \right\}.$$

Then $R$ is **N**-graded. Let

$$\mathscr{T} = \left\{ \sum s\phi(b)e_x \colon s \text{ a unit of } S,\ b \in B_+,\ x \in X \right\}.$$

Then $\mathscr{T}$ is a multiplicatively closed set of elements of $R$ invertible in $\mathscr{E}$, and is an Ore set by formulas (1) and (2). Moreover every element of $\mathscr{E}$ is of the form $t^{-1}r$ for some $t \in \mathscr{T}$ and $r \in R$.

Let us now consider the degree 0 part $R_0$ of $R$. From the definition of the grading, $R_0$ has $S$-basis

$$\{\phi(g)e_x \colon g \in G,\ x \in X\}.$$

It follows from (1) and (2) that $G$ permutes the set of orthogonal idempotents $\{e_x \colon x \in X\}$ via $\phi$. For $x \in X$, let $G_x$ denote the stabilizer of $e_x$ in $G$ and let $T_x$ be a set of representatives of the left cosets of $G_x$ in $G$. Let $\varepsilon_x = \sum_{g \in T_x}\phi(g)e_x\phi(g)^{-1}$ be the sum of the idempotents in the $G$-orbit of $e_x$. Then $R_0$ is the direct sum of the two-sided ideals $R_0\varepsilon_x$ as $x$ varies over a set $\mathscr{X}$ of representatives of the distinct $G$-orbits of $X$. From the definition of $\phi$, if $g \in G_x$ then $gx = x\gamma_g$ for some $\gamma_g \in \Gamma$; let $F_x$ denote the set of all the resulting elements $\gamma_g$ as $g$ varies over $G_x$. Then $\overline{x^{-1}G_xx} = F_x$, so $F_x$ is a finite subgroup of $\Gamma$. Moreover, since $\phi(g)e_x = e_x\overline{\gamma_g}$ for $g \in G_x$, it follows that $R_0e_x$ is closed under right multiplication by $S * F_x$, and $R_0e_x$ is an $(R_0\varepsilon_x, S * F_x)$-bimodule. We have $\{\phi(\underline{g})e_x \colon g \in G\}$ as a basis of $R_0e_x$ as a right $S$-module, and since $\phi(g)e_x = e_x\overline{\gamma_g}$ for $g \in G_x$, we see that

$$\{\phi(g)e_x \colon g \in T_x\}$$

is a basis of $R_0e_x$ as a right $S * F_x$-module. Then left multiplication by $R_0\varepsilon_x$ on the $(R_0\varepsilon_x, S * F_x)$-bimodule $R_0e_x$ shows that $R_0\varepsilon_x$ is isomorphic to $\mathrm{End}_{S * F_x}(R_0e_x)$ which in turn is isomorphic to the full matrix ring of degree $|G \colon G_x|$ over $S * F_x$.

Returning to $R$, we see that $R$ is finitely generated as an $R'$-module over the subring $R'$ generated over $S$ by

$$\{\phi(b)e_x \colon b \in B_+,\ x \in X\}.$$

Then $R'$ is a skew polynomial ring over $R'_0 = \sum_{x \in X}Se_x$ in the variables $\{\phi(b_1), \ldots, \phi(b_m)\}$, so $R'$ and hence $R$ are noetherian. We shall need to know that $R_0$ has finite projective dimension as a right $R$-module. Using a skew version of Hilbert's syzygy theorem, (see [2, 13.4.4]) we see that $R'_0$ has finite projective dimension as a right $R'$-module. Take a finite projective right $R'$-resolution $\{P_i\}$ of $R_0'$ and apply the functor $-\otimes_{R'} R$, which is exact since $R$ is a free left $R'$-module, having basis $\mathscr{G}=\{\phi(g)\colon g \in G\}$. Since $\mathscr{G}$ is also a left basis of $R_0$ over $R_0'$, then $R$ is a crossed product $R' * G$ and $R_0$ is a crossed product $R_0' * G$. Then

$$R_0' \otimes_{R'} R \cong R_0' \otimes_{R'} (R' * G) \cong R_0' * G = R_0$$

so $R_0$ has finite projective dimension as a right $R$-module, as desired.

## 3. The commutative diagram

We shall keep the same notation as in the previous section.

We recall that $G_0$ of a ring $R$ is defined by taking the free abelian group on the isomorphism classes $[M]$ of finitely generated $R$-modules, and factoring out the relations $[M]=[M']+[M'']$ for any short exact sequence $0 \to M \to M' \to M'' \to 0$. In this section we consider the following diagram.

$$\begin{array}{ccccc}
\sum_{x\in\mathscr{X}} G_0(S * F_x) & & \xrightarrow{\quad\alpha\quad} & & G_0(S*\Gamma) \\
\beta\big\downarrow & & & & \big\uparrow\varepsilon \\
G_0(R_0) & \xrightarrow[\gamma]{} & G_0(R) & \xrightarrow[\delta]{} & G_0(\mathscr{E})
\end{array}$$

We first define the maps. In this section, we only deal with the generators of $G_0$, so we shall suppress the brackets around our modules. The top horizontal map $\alpha$ comes from sending a left $S * F_x$-module $M$ to $S * \Gamma \otimes_{S * F_x} M$, and is well defined since $S * \Gamma$ is free over $S * F_x$. The left vertical map $\beta$ comes from Morita equivalence, but we need a precise version. For $x \in \mathscr{X}$ we have the $(R_0, S * F_x)$-bimodule $R_0 e_x$, which is free as a right $S * F_x$-module, and we define $\beta$ by sending a left $S * F_x$-module $M$ to $R_0 e_x \otimes_{S * F_x} M$. The ring $R$ is a free right $R_0$-module with basis $\{\phi(b)\colon b \in B_+\}$, and we get the map $\gamma$ by sending a left $R_0$-module $M$ to $R \otimes_{R_0} M$. The ring $\mathscr{E}$ is gotten from $R$ by localizing at the Ore set $\mathscr{T}$, so $\mathscr{E}$ is flat as a right $R$-module, and $\delta$ is defined by sending a left $R$-module $M$ to $\mathscr{E} \otimes_R M$. Fix an element $y$ of $X$. Then $e_y\mathscr{E}$ is an $(S * \Gamma, \mathscr{E})$-bimodule, and since $e_y$ is idempotent, then $e_y\mathscr{E}$ is projective as a right $\mathscr{E}$ module, so we get the map $\varepsilon$ by sending a left $\mathscr{E}$-module $M$ to the left $S * \Gamma$-module $e_y\mathscr{E} \otimes_{\mathscr{E}} M$.

Next we prove that the diagram commutes. Starting with the left $S * F_x$-module $M$, $\beta$ sends $M$ to $R_0 e_x \otimes_{S * F_x} M$ and $\gamma$ sends this to

$$R \otimes_{R_0} R_0 e_x \otimes_{S * F_x} M \cong R e_x \otimes_{S * F_x} M.$$

Then $\delta$ sends this to $\mathscr{E} \otimes_R Re_x \otimes_{S*F_x} M \cong \mathscr{E}e_x \otimes_{S*F_x} M$ and $\varepsilon$ maps this to

$$e_y\mathscr{E} \otimes_{\mathscr{E}} \mathscr{E}e_x \otimes_{S*F_x} M \cong e_y\mathscr{E}e_x \otimes_{S*F_x} M.$$

Since $e_y\mathscr{E}e_x \cong S*\Gamma$ as an $(S*\Gamma, S*F_x)$-bimodule, then

$$e_y\mathscr{E}e_x \otimes_{S*F_x} M \cong S*\Gamma \otimes_{S*F_x} M.$$

We have therefore proved that the diagram commutes.

To prove Moody's Theorem, we must show that $\alpha$ is surjective. To do this, we shall show that $\beta$, $\gamma$, $\delta$, and $\varepsilon$ are surjective. Indeed $\beta$ and $\varepsilon$ are isomorphisms since they come from Morita equivalences. We shall prove that $\gamma$ is an isomorphism in the next section. For $\delta$, let $M$ be a finitely generated left $\mathscr{E}$-module, with a finite set of generators $Y$. Then let $M'$ be the $R$-submodule of $M$ generated by $Y$, and it is clear that $\mathscr{E} \otimes_R M' \cong M$.

## 4. Quillen's Theorem

In this section we prove the following result.

THEOREM. *Let $R$ be a left noetherian graded ring such that $R$ is flat as a right $R_0$-module and such that for each left $R$-module $M$ there exists a positive integer $m$ such that* $\operatorname{Tor}_i^R(R_0, M) = 0$ *for all $i \geq m$. Then the map $\gamma$: $G_0(R_0) \to G_0(R)$ given by sending the class $[M]$ of a left $R_0$-module $M$ to $[R \otimes_{R_0} M]$ is an isomorphism.*

This is a special case of Quillen's Theorem 7 in [3]. Quillen considers all higher $K$ groups of the category of finitely generated $R$-modules, not just $G_0$. For Moody's Theorem, we only need surjectivity of $\gamma$; the ring $R$ in the previous section satisfies the Tor hypothesis above since $R_0$ has finite projective dimension as a right $R$-module.

Before giving the proof, we shall consider two lemmas, the first of which will also be needed in the next section.

LEMMA 1. *Let $R_0$ be a subring of a ring $R$, such that $R$ is flat as a right $R_0$-module. Further, let $R'$ be another ring and let $\phi$. $R \to R'$ be a ring homomorphism, so $R'$ is then a right $R$-module. If $M$ is a left $R_0$-module, then* $\operatorname{Tor}_i^R(R', R \otimes_{R_0} M) \cong \operatorname{Tor}_i^{R_0}(R', M)$ *for all $i > 0$.*

*Proof.* Take a projective left $R_0$-resolution $\{P_i\}$ of $M$. To compute $\operatorname{Tor}^{R_0}(R', M)$, apply the functor $R' \otimes_{R_0} -$ obtaining the complex $\{R' \otimes_{R_0} P_i\}$ and take homology. Since $R$ is flat as an $R_0$-module, $\{R \otimes_{R_0} P_i\}$ is a projective $R$-resolution of $R \otimes_{R_0} M$. To compute

$$\operatorname{Tor}^R\left(R', R \otimes_{R_0} M\right),$$

apply the functor $R' \otimes_R -$ to this resolution, obtaining

$$\left\{R' \otimes_R R \otimes_{R_0} P_i \cong R' \otimes_{R_0} P_i\right\}.$$

It is now clear that $\operatorname{Tor}_i^R(R', R \otimes_{R_0} M) \cong \operatorname{Tor}_i^{R_0}(R', M)$ for all $i > 0$, and the proof is complete.

LEMMA 2. *Let $R$ be a left noetherian graded ring, and let $M$ be a finitely generated graded left $R$-module. Suppose that there is an integer $j$ such that $M$ is generated by its $j$-th homogeneous component, i.e., $M = RM_j$. Suppose further that* $\operatorname{Tor}_1^R(R_0, M) = 0$. *Then $M \cong R \otimes_{R_0} M_j$.*

*Proof.* Let $I = \sum_{i>0} R_i$ be the ideal of $R$ generated by the elements of positive degree. Then $R_0 \otimes_R M$ is naturally isomorphic to $M/IM$. We have a graded map $\psi$ from $R \otimes_{R_0} M_j$ onto $M$ given by $\psi(r \otimes m) = rm$ for $r \in R$ and $m \in M_j$, hence an exact sequence

$$0 \to \ker \psi \to R \otimes_{R_0} M_j \to M \to 0.$$

Applying $R_0 \otimes_R -$ yields

$$0 \to \ker \psi / I \ker \psi \to M_j \to M/IM \to 0$$

since $\operatorname{Tor}_1^R(R_0, M) = 0$ and $R_0 \otimes_R R \otimes_{R_0} M_j \cong M_j$. Since $M = \sum_{i=j}^{\infty} M_i$, then $M_j \cap IM = 0$, so we deduce that $M_j \cong M/IM$ and therefore $\ker \psi / I \ker \psi = 0$. Then $\ker \psi = I \ker \psi$, from which it follows that $\ker \psi = 0$, since $\ker \psi$ is graded and finitely generated (because $R$ is noetherian.) Then $R \otimes_{R_0} M_j \cong M$. This completes the proof.

*Proof of Theorem.* Let $M$ be a finitely generated left $R$-module. We first assume that $M$ is graded, and we shall prove that $[M]$ is in the image of $\gamma$. We have a positive integer $i$ such that $\operatorname{Tor}_i^R(R_0, M) = 0$. If $i > 1$, let $\sigma$ be a graded homomorphism from a finitely generated free $R$-module $F$ onto $M$, and let $M'$ be the kernel of $\sigma$, giving us the short exact sequence

$$0 \to M' \to F \to M \to 0.$$

Since $[F]$ is in the image of $\gamma$, in order to prove that $[M]$ is in the image of $\gamma$, it suffices to prove that $[M']$ is. But $\operatorname{Tor}_{i-1}^R(R_0, M') = \operatorname{Tor}_i^R(R_0, M) = 0$. Then by induction, we may assume that $\operatorname{Tor}_1^R(R_0, M) = 0$.

Next, write $M = \sum_{i=j}^{\infty} M_i$ for some integer $j$, with $M_j \neq 0$. If $M = RM_j$, then Lemma 2 tells us that $[M]$ is in the image of $\gamma$. If $M \neq RM_j$, define $M(l) = \sum_{i=j}^{l} RM_i$. Since $M$ is finitely generated, there is an integer $l > j$ with the property that $M = M(l)$ but $M \neq M(l-1)$. We shall show that $[M]$ is in the image of $\gamma$, for graded $R$-modules $M$ satisfying $\mathrm{Tor}_1^R(R_0, M) = 0$, by induction on $l - j$. We have the exact sequence

$$0 \to M(l-1) \to M \to M/M(l-1) \to 0. \tag{3}$$

Let $N = M/M(l-1)$. Apply $R \otimes_{R_0}$ , obtaining the exact sequence

$$\to \mathrm{Tor}_2^R(R_0, N) \to \mathrm{Tor}_1^R(R_0, M(l-1)) \to \mathrm{Tor}_1^R(R_0, M) \to \mathrm{Tor}_1^R(R_0, N) \\ \to M(l-1)/IM(l-1) \to M/IM \to N/IN \to 0. \tag{4}$$

We claim that

$$M(l-1)/IM(l-1) \to M/IM$$

is injective. To prove this, we must show that if $x \in M(l-1) \cap IM$ then $x \in IM(l-1)$. We have

$$M(l-1) = \sum_{i=j}^{l-1} RM_i = \sum_{i=j}^{l-1} (R_0 + I)M_i = \sum_{i=j}^{l-1} M_i + IM(l-1).$$

Then $x = y + z$ where $y \in \sum_{i=j}^{l-1} M_i$ and $z \in IM(l-1)$. But $x$ and $z$ are in $IM$, so $y$ is too, hence $y \in \sum_{i=j}^{t} IM_i$ for some $t$. Since $y \in \sum_{i=j}^{l-1} M_i$ it follows that $t < l - 1$, so

$$y \in \sum_{i=j}^{l-1} IM_i,$$

and the claim holds. Since $M(l-1)/IM(l-1) \to M/IM$ is injective and $\mathrm{Tor}_1^R(R_0, M) = 0$, it follows from (4) that $\mathrm{Tor}_1^R(R_0, N) = 0$. Since $M = M(l)$ and $N = M/M(l-1)$, we have $N = RN_l$. Then Lemma 2 implies that $N \cong R \otimes_{R_0} N_l$, and Lemma 1, with $R' = R_0$, gives

$$\mathrm{Tor}_2^R(R_0, N) \cong \mathrm{Tor}_2^{R_0}(R_0, N_l) = 0$$

(since $R_0$ is a flat $R_0$-module.) Since $\mathrm{Tor}_1^R(R_0, M) = 0$, it now follows from (4) that

$$\mathrm{Tor}_1^R(R_0, M(l-1)) = 0,$$

so the induction hypothesis applies to $M(l-1)$. Therefore $[M(l-1)]$ is in the image of $\gamma$. Since $[N]$ is as well, we deduce from (3) that $[M]$ is in the image of $\gamma$, as desired.

For a non-graded module $M$, we shall proceed (as does Quillen) as in Swan [4, p. 131]. Let $z$ be an indeterminate, and consider the polynomial ring $R[z]$, which is graded by assigning the monomial $r_i z^j$ the degree $i + j$, for $r_i \in R_i$, so $(R[z])_0 = R_0$. We shall check that $R[z]$ satisfies the hypotheses of the theorem. It is noetherian by Hilbert's basis theorem, and it is free over $R$, hence flat over $R_0$. Let $L$ denote a left $R[z]$-module. We have the short exact sequence

$$0 \to R[z] \xrightarrow{\kappa} R[z] \xrightarrow{\lambda} R \to 0$$

where $\kappa$ is left multiplication by $z$ and $\lambda$ sends $z$ to 0. We then have the short exact sequence

$$0 \to R[z] \otimes_R L \to R[z] \otimes_R L \to L \to 0.$$

Therefore, in order to prove that $\mathrm{Tor}_i^{R[z]}(R_0, L) = 0$ for all sufficiently large $i$, it suffices to show that

$$\mathrm{Tor}_i^{R[z]}(R_0, R[z] \otimes_R L) = 0.$$

But Lemma 1 tells us that $\mathrm{Tor}_i^{R[z]}(R_0, R[z] \otimes_R L) \cong \mathrm{Tor}_i^R(R_0, L)$, which is 0 for large $i$ by hypothesis. Thus $R[z]$ satisfies the hypotheses of the theorem.

Let us return to our $R$-module $M$, and let $F$ be a free $R$-module of finite rank which maps onto $M$, with $K$ being the kernel of this map. Then $F$ is a graded $R$-module, by assigning the free generators any convenient degree. Fix a finite set $Y$ of generators of $K$. Take $y \in Y$ and write $y$ in the form

$$y = \sum_{i=j}^{l} f_i, \quad f_i \in F_i.$$

Then define $\hat{y} \in F[z]$ by

$$\hat{y} = \sum_{i=j}^{l} f_i z^{l-i}$$

so $\hat{y}$ is a homogeneous element of the graded $R[z]$-module $F[z]$. Let $\hat{K}$ be the graded $R[z]$-submodule of $F[z]$ generated by the set $\{\hat{y}\colon\ y \in Y\}$. If $L$ is a graded $R[z]$-module, then left multiplication on $L$ by $1 - z$ is injective, so the functor $\Phi$ which assigns $L$ the $R$-module $L/R[z](1 - z)L$ is exact. It follows that $\Phi(F[z]/\hat{K}) \cong M$. We have already proved that $F[z]/\hat{K}$ is in the image of the map $G_0(R_0) \to G_0(R[z])$. Then apply the functor $\Phi$, to see that $\gamma$ is surjective.

To show that $\gamma$ is injective, we construct a left inverse. For a finitely generated left $R$-module $L$, define $\tau[L] = \sum_{i=0}^{\infty}(-1)^i[\mathrm{Tor}_i^R(R_0, L)]$. Using the fact that $R$ is noetherian, it follows that $\mathrm{Tor}_i^R(R_0, L)$ is finitely generated as an $R_0$-module. The long exact Tor sequence shows that $\tau$ respects the relations of $G_0$, and the sum is finite by hypothesis, so $\tau$ is indeed a homomorphism. For a finitely generated left $R$-module $M$, it follows from Lemma 1 that

$$\mathrm{Tor}_i^R\left(R_0, R \otimes_{R_0} M\right) = 0 \quad \text{for } i > 0,$$

so $\tau\gamma[M] = [R_0 \otimes_R R \otimes_{R_0} M] = [M]$. Thus $\tau$ is a left inverse for $\gamma$, and the proof is complete.

## 5. The second commutative diagram

We have completed the proof of Moody's Theorem under the assumption that $\mathbf{Q} \otimes_{\mathbf{Z}} A$ is a free $\mathbf{Q}G$-module. We now discuss the general case. Since $\mathbf{Q} \otimes_{\mathbf{Z}} A$ is projective as a $\mathbf{Q}G$-module, there exists a finitely generated $\mathbf{Z}G$-module $N$ such that $\mathbf{Q} \otimes_{\mathbf{Z}} (A \oplus N)$ is a free $\mathbf{Q}G$-module. Let $\Gamma_1$ denote the semidirect product $N \rtimes \Gamma$. Then we have the crossed product $S * \Gamma_1$, which may be considered as the crossed product $(SN) * \Gamma$, where $SN$ denotes the group ring of $N$ over $S$. Moreover Moody's Theorem has been proved for $S * \Gamma_1$. We have the following diagram:

$$\begin{array}{ccc} \sum G_0(S * F) & \xrightarrow{\alpha_1} & G_0(S * \Gamma_1) \\ \zeta\downarrow & & \downarrow\eta \\ \sum G_0(S * FN/N) & \xrightarrow[\alpha]{} & G_0(S * \Gamma) \end{array}$$

In the upper left corner, $F$ varies over finite subgroups of $\Gamma_1$, and $\alpha_1$ is the sum of inductions, which we have proved surjective. Since $N$ is torsion-free, we have $FN/N \cong F$; then if $M$ is an $S * F$-module, we define $\zeta[M] = [M]$, where the $M$ on the right is considered as an $S * FN/N$-module. The map $\alpha$ is the sum of inductions. For an $S * \Gamma_1$-module $M$, we define $\eta[M]$ to be

$$\sum_{i=0}^{\infty} (-1)^i \left[\mathrm{Tor}_i^{S * \Gamma_1}(S * \Gamma, M)\right],$$

analogous to the left inverse map defined in the proof of Quillen's Theorem. Since $S * \Gamma_1$ is noetherian, in order to show that $\eta$ is well defined we must only check that this sum of Tors is a finite sum. By Hilbert's syzygy theorem, $S$ has finite projective dimension as an $SN$-module. Then it follows by inducing that

$S * \Gamma$ has finite projective dimension as an $SN * \Gamma$-module, and $SN * \Gamma = S * \Gamma_1$. Thus $\eta$ is well defined.

We now prove that the diagram commutes. Let $M$ be a finitely generated $S * F$-module, where $F$ is a finite subgroup of $\Gamma_1$. We apply Lemma 1 of the previous section, with $R = S * \Gamma_1$, $R_0 = S * F$, and $R' = S * \Gamma$, with the ring homomorphism $\phi$ coming from the natural homomorphism $\Gamma_1 \to \Gamma$. We find that $\mathrm{Tor}_i^{S * \Gamma_1}(S * \Gamma, S * \Gamma_1 \otimes_{S * F} M) = 0$ for all $i > 0$, so

$$\eta\alpha[M] = [S * \Gamma \otimes_{S * F} M] = \alpha\zeta[M],$$

and the diagram commutes.

Since we have proved that $\alpha_1$ is surjective, and since $\zeta$ clearly is, in order to prove that $\alpha$ is surjective, we must prove that $\eta$ is. Let $M$ be a left $S * \Gamma$-module; then we claim that $[M] = \eta[S * \Gamma_1 \otimes_{S * \Gamma} M]$. This follows from Lemma 1 once more, this time with $R' = S * \Gamma$ and $R$, $R_0$, and $\phi$ as before. This completes the proof of Moody's Theorem.

## 6. Goldie ranks

In this section we use Moody's Theorem to solve the Goldie rank problem for the group ring $kH$ of a polycyclic by finite group $H$ over an arbitrary division ring $k$. (See also Rosset [4].) We assume that $H$ has no finite normal subgroup. Then $kH$ is a prime noetherian ring, and therefore has a classical (left) ring of quotients, which we shall denote by $k(H)$, which is a simple artinian ring, isomorphic to the full matrix ring $M_n(D)$ over some division ring $D$. (This is proved in [2] for commutative $k$.) The size $n$ of the matrix ring is called the Goldie rank of $H$.

THEOREM. *The Goldie rank $n$ of $kH$ is equal to the least common multiple of the orders of the finite subgroups of $H$.*

*Proof.* If $M$ is a finitely generated left $kH$-module, then $k(H) \otimes_{kH} M$ is isomorphic to a direct sum of a certain number, say $m$, copies of the simple $k(H)$-module; let $r_H(M) = m/n$, so $r_H(kH) = 1$. Then $r_H$ gives rise to a $\mathbf{Q}$-valued function on $G_0(kH)$. Let $H_1$ be a torsion-free normal subgroup of $H$ of finite index $l$; then $kH$ is a free left $kH_1$-module of rank $l$, and $k(H)$ is a free $k(H_1)$-module of rank $l$. It follows that

$$r_H(M) = r_{H_1}(M_{H_1})/|H : H_1|,$$

where $M_{H_1}$ denotes the restriction of $M$ to $kH_1$. Let $F$ be a finite subgroup of

$H$ and let $L$ be a finitely generated left $kF$-module. From Mackey's formula,

$$(kH \otimes_{kF} L)_{H_1} \cong \sum_x kH_1 \otimes_{k(xFx^{-1} \cap H_1)} xL$$

where the sum is over representatives of the $(H_1, F)$-double cosets of $H$. Since $F$ is finite and $H_1$ is torsion-free, $xFx^{-1} \cap H_1 = 1$, and since $H_1$ is normal, $H_1xF = xH_1F$. Then each summand on the right is isomorphic to $\dim_k L$ copies of $kH_1 \otimes_k k$, and there are $|H : H_1F|$ such summands. It follows that

$$\begin{aligned} r_H(kH \otimes_{kF} L) &= \frac{1}{|H : H_1|} r_{H_1}((kH \otimes_{kF} L)_{H_1}) \\ &= \frac{|H : H_1F|}{|H : H_1|} \dim_k L \\ &= \frac{\dim_k L}{|F|}. \end{aligned}$$

Pick $L$ to have $k$-dimension 1; then $r_H(kH \otimes_{kF} L) = 1/|F|$, and this is some multiple of $1/n$, so $n$ is divisible by the order of each finite subgroup. Let $M$ be a $kH$-module with the property that $k(H) \otimes_{kH} M$ is a simple $k(H)$-module, so $r_H(M) = 1/n$. Moody's Theorem implies that

$$r_H(M) = \sum_i \pm r_H(kH \otimes_{kF_i} L_i),$$

for certain $kF_i$-modules $L_i$, where the $F_i$ are finite subgroups of $H$. Then

$$1/n = \sum_i \pm \dim_k L_i / |F_i|,$$

and it follows that $n$ divides $\operatorname{lcm}|F_i|$. This completes the proof.

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