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RNP AND CPCP IN LEBESGUE-BOCHNER FUNCTION SPACES

ZHIBAO HU AND BOR-LUH LIN

In this paper we study the extremal structure of the unit ball of a Lebesgue Bochner function space. Throughout, X will denote a Banach space, B_X the unit ball, S_X the unit sphere, X^* the dual space of X, (Ω, Σ, μ) a positive measure space, and $1 < p,\ q < \infty$ with $1/p + 1/q = 1$.

Let K be a subset of X. A point x in K is a point of sequential continuity of K if for every sequence (x_n) in K, weak-$\lim_n x_n = x$ implies $\lim_n \|x_n - x\| = 0$. The point of sequential continuity is a generalization of the point of continuity. A space X has the Kadec-Klee property if every point x in S_X is a point of sequential continuity of B_X.

It is well-known that if (Ω, Σ, μ) is not purely atomic, then $L^p(\mu, X)$ with the Kadec-Klee property must be strictly convex. This result, due to M. Smith and B. Turett [ST], is one of the most surprising results in the theory of Lebesgue-Bochner function spaces. Our first main result (Theorem 2.2) asserts that if (Ω, Σ, μ) is atom-free, then every point of sequential continuity of $B_{L^p}(\mu, X)$ must be an extreme point of $B_{L^p(\mu, X)}$. This gives a local version of the result of Smith and Turett.

Theorem 2.2 has several interesting consequences; for example, it implies that if (Ω, Σ, μ) is not purely atomic then:

(i) The Radon-Nikodym Property (RNP) and the Convex Point of Continuity Property (CPCP) are equivalent for $L^p(\mu, X)$ and $L^p(\mu, X)^*$.

(ii) The super-RNP and the super-CPCP are equivalent for $L^p(\mu, X)$ and $L^p(\mu, X)^*$.

Recall that the RNP implies the PCP (Point of Continuity Property) which, in turn, implies the CPCP, and that RNP, PCP, and CPCP are distinct [BR], [GMS1]. It follows that if X has the PCP but fails the RNP, and if (Ω, Σ, μ) is not purely atomic, then $L^p(\mu, X)$ does not have the CPCP. Consequently, neither the PCP nor the CPCP can be "lifted" from X to $L^p(\mu, X)$. We would like to mention (1) it is still an open problem whether the super-RNP and the super-CPCP are equivalent in general, (2) the RNP and the CPCP are equivalent for Banach spaces with the Krein-Milman Property [Sc], and

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(3) the RNP and the PCP are equivalent for Banach lattices not containing isomorphic copies of c_0 [GM].

Let f be a norm one element in $L^p(\mu, X)$. The condition that for almost all t in the support of f such that $f(t)/\|f(t)\|$ is an extreme point of B_X is strictly stronger than the condition that f is an extreme point of the unit ball of $L^p(\mu, X)$ [G]. We do not know whether the conclusion of Theorem 2.2 can be strengthened so that $f(t)/\|f(t)\|$ is an extreme point of B_X for almost all t in supp f. It is shown, however, that if (Ω, Σ, μ) is atom-free and that f is a $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-point of sequential continuity of $B_{L^p(\mu, X)}$, then $f(t)/\|f(t)\|$ is a strongly extreme point of B_X for almost all t in supp f, thus f is in fact a strongly extreme point of $B_{L^p(\mu, X)}$ in this case.

Another generalization of the point of continuity is the point of small combination of slices (SCS-points, for short). It is known [GGMS] that X is strongly regular if and only if every non-empty bounded closed convex set K in X is contained in the norm-closure of SCS(K). Schachermayer [Sc] proved that a Banach space has the RNP if and only if it is strongly regular and it has the Krein-Milman Property. We will show that the "point-version" of this result is also true; i.e., if K is a closed convex set in X and $x \in K$, then x is a denting point of K if and only if x is both a SCS-point and an extreme point of K. An example is given to show that we can not replace the point of sequential continuity by the SCS-point in Theorem 2.2.

The main tool used in the proof of Theorem 2.2 is developed in Section I, where we study the weak-convergence of sequences of vector-valued Rademacher functions. The major part of Section II is devoted to the proof of Theorem 2.2 and its consequences.

Section I

The usual Rademacher functions are associated with the dyadic partitions of the unit interval. To define our "Rademacher functions" we use countable partitions of Ω and a special index set.

Let T be the set consisting of all the finite sequences of positive integers with the natural partial order; i.e., $(i_1, \dots, i_m) \leq (j_1, \dots, j_n)$ if and only if $m \leq n$ and $i_k = j_k$, $k = 1, \dots, m$, and with the empty set ϕ as the smallest element in T. For $\alpha \in T$, let $|\alpha|$ be the cardinality of P_α where $P_\alpha = \{\beta\colon \beta \in T, \beta < \alpha\}$ and let $T_n = \{\alpha\colon \alpha \in T, |\alpha| = n\}$, $n \geq 0$. If $\alpha = (i_1, \dots, i_m)$ and i is a natural number, then we also use αi to denote $(i_1, \dots, i_m, i)$.

We call a "subset" $\{E_\alpha\}_{\alpha \in T}$ of Σ a Rademacher tree of measurable sets if it satisfies the following conditions:

For all $k \geq 0$ and $\alpha \in T_k$, $\{E_{\alpha n}\}_{n \geq 1}$ is a partition of E_α and $\mu(E_{\alpha 2n-1}) = \mu(E_{\alpha 2n})$, and $\mu(E_\phi) < \infty$.

We say that a sequence $\{f_k\}$ of functions from Ω to X is Rademacher if there are a Rademacher tree $\{E_\alpha\}_{\alpha \in T}$ in Σ and $\{x_\alpha\}_{\alpha \in T}$ in X, $\alpha \in T$ such that for $k \geq 0$,

$$f_k = \sum_{\alpha \in T_k} x_\alpha \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}.$$

Each f_k is called a Rademacher function, and $\{E_\alpha\}_{\alpha \in T}$ is called a Rademacher tree associated to $\{f_k\}$, and $\{f_k\}$ is said to be determined by $\{E_\alpha, x_\alpha\}_{\alpha \in T}$. We use $\Sigma(T)$ to denote the sub-σ-algebra of Σ generated by the tree $\{E_\alpha\}_{\alpha \in T}$. It is obvious that each f_k is $\Sigma(T)$-measurable.

PROPOSITION 1.1. *Every bounded Rademacher sequence in $L^p(\mu, X)$ is null with respect to the $\sigma(L^p(\mu, X), L^q(\mu, X^*))$ topology. In particular, if X is an Asplund space, then every bounded Rademacher sequence in $L^p(\mu, X)$ is weakly null.*

Proof. Suppose $\{f_k\}$ is a bounded Rademacher sequence in $L^p(\mu, X)$. Let $\{E_\alpha\}_{\alpha \in T}$ be a Rademacher tree associated to $\{f_k\}$. For x^* in X^*, $\tau \in T_m$, and $k \geq m$, we have

$$\int_\Omega \left(x^* \chi_{E_\tau}, f_k(t) \right) d\mu(t) = 0.$$

Since span$\{x^* \chi_{E_\tau}: x^* \in X$ and $\tau \in T\}$ is dense in $L^q(\mu, \Sigma(T), X^*)$,

$$\sigma(L^p(\mu, X), L^q(\mu, \Sigma(T), X^*)) - \lim_k f_k = 0.$$

Let P be the conditional expectation projection from $L^q(\mu, X^*)$ onto $L^q(\mu, \Sigma(T), X^*)$ (see e.g. [Bi]), and suppose $g \in L^q(\mu, X^*)$. Since f_k is $\Sigma(T)$-measurable,

$$\int_\Omega (g(t), f_k(t))\, d\mu(t) = \int_\Omega (Pg(t), f_k(t))\, d\mu(t).$$

Hence $\{f_k\}$ is $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-null. Finally if X is an Asplund space, then $L^q(\mu, X^*)$ is the dual of $L^p(\mu, X)$ [DU], so $\{f_k\}$ is weakly null.
QED

In general, it is not true that every bounded Rademacher sequence in $L^p(\mu, X)$ is weakly null as shown by Example 1.2. In Theorem 1.3, we give a sufficient condition for a Rademacher sequence in $L^p(\mu, X)$ to be weakly null.

Example 1.2. Let X be the space l^1 with the usual norm, and μ the Lebesgue measure on $[0, 1)$. If $\{r_k\}$ is the usual Rademacher sequence on

$[0,1)$, and $\{e_k\}$ is the canonical basis for l^1. Define the X-valued sequence $\{f_k\}$ by $f_k(t) = r_k(t)e_{k+1}$ for t in $[0,1)$ and $k \geq 0$. Then $\{f_k\}$ is a bounded Rademacher sequence in $L^2(\mu, X)$. It is easy to check that $\mathrm{co}\{f_k\}$ is a subset of the unit sphere. So the weak closure of $\mathrm{co}\{f_k\}$ still lies in the unit sphere. Therefore $\{f_k\}$ is not weakly null.

THEOREM 1.3. *Suppose $\{f_k\}$ is an X-valued Rademacher sequence determined by $\{E_\alpha, x_\alpha\}_{\alpha \in T}$. If $\{x_\alpha\}$ is bounded and there is $\varepsilon_k > 0$ such that*

$$\lim_k \varepsilon_k = 0 \quad \textit{and} \quad \|x_\beta - x_\alpha\| < \varepsilon_k \quad \textit{for } k > 0,\ \alpha \in T_k, \textit{ and } \beta \geq \alpha,$$

then $\{f_k\}$ is weakly null in $L^p(\mu, X)$, $1 < p < \infty$.

Proof. Let Q_k be the natural projection from $\bigcup_{i \geq 0} T_{k+i}$ to T_k, i.e., for each $\alpha \in T_{k+i}$, $Q_k(\alpha)$ is the unique element in T_k such that $Q_k(\alpha) \leq \alpha$.

Claim. For all $t \in \Omega$, $k \geq 1$ and $i \geq 0$,

$$\left\| f_{k+i}(t) - \sum_{\alpha \in T_{k+i}} x_{Q_k(\alpha)} \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}(t) \right\| < \varepsilon_k \chi_{E_\phi}(t).$$

We only need to prove this for $t \in E_\phi$. Note that $\{E_{\alpha n}\colon \alpha \in T_{k+i}$, and $n \geq 1\}$ is a partition of E_ϕ, so if $t \in E_\phi$, then $t \in E_{\gamma s}$ for some $\gamma \in T_{k+i}$ and $s \geq 1$. Thus $f_{k+i}(t) = (-1)^s x_\gamma$ and

$$\sum_{\alpha \in T_{k+i}} x_{Q_k(\alpha)} \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}(t) = (-1)^s g(t_{Q_k(\gamma)}).$$

So we have

$$\begin{aligned}\left\| f_{k+1}(t) - \sum_{\alpha \in T_{k+i}} x_{Q_k(\alpha)} \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}(t) \right\| &= \left\|(-1)^s x_\gamma - (-1)^s x_{Q_k(\gamma)}\right\| \\ &= \|x_\gamma - x_{Q_k(\gamma)}\| < \varepsilon_k \\ &= \varepsilon_k \chi_{E_\phi}(t).\end{aligned}$$

Assume that $\{f_k\}$ does not converge weakly to 0. Then there exists $F \in L^p(\mu, X)^*$ with $\|F\| = 1$, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $\delta > 0$, such that $F(f_{n_k}) > \delta$ for $k \geq 1$. It follows that for every $h \in \mathrm{co}\{f_{n_k}\colon k \geq 1\}$, $\|h\| \geq F(h) > \delta$.

For $k \geq 1$, let $h_k = \sum_{\alpha \in T_k} \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}$. Then $\{h_k\}$ is a bounded Rademacher sequence in $L^p(\mu)$. By Proposition 1.1, w-$\lim_k h_k = 0$, so w-$\lim_k h_{n_k} = 0$. Choose $M \geq \mu(E)^{1/p}$ such that $\|x_\alpha\| \leq M$ for all $\alpha \in T$. Then choose $k_0 > 1$ with $\varepsilon_{n_{k_0}} < \delta/3M$. Since $\{h_{n_k}\}$ is weakly null,there exist

$\lambda_i \geq 0$, $1 \leq i \leq m$, with $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$\left\| \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\| < \frac{\delta}{3M}.$$

Then

$$\left\| \sum_{i=1}^{m} \lambda_i f_{n_{k_0+i}} \right\| > \delta.$$

On the other hand,

$$\begin{aligned}
\left\| \sum_{i=1}^{m} \lambda_i f_{n_{k_0+i}} \right\| &\leq \left\| \sum_{i=1}^{m} \lambda_i \left(f_{n_{k_0+i}} - \sum_{\alpha \in T_{n_{k_0+i}}} x_{Qn_{k_0}(\alpha)} \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}} \right) \right\| \\
&\quad + \left\| \sum_{i=1}^{m} \lambda_i \sum_{\alpha \in T_{n_{k_0+i}}} x_{Qn_{k_0}(\alpha)} \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}} \right\| \\
&\leq \sum_{i=1}^{m} \lambda_i \| \varepsilon_{n_{k_0}} \chi_{E_\phi} \| + \left\| \sum_{\alpha \in T_{n_{k_0}}} x_\alpha \chi_{E_{\alpha n}} \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\| \\
&= \sum_{i=1}^{m} \lambda_i \varepsilon_{n_{k_0}} \mu(E)^{1/p} + \left\| \sum_{\alpha \in T_{n_{k_0}}} x_\alpha \chi_{E_{\alpha n}} \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\| \\
&\leq \varepsilon_{n_{k_0}} \mu(E)^{1/p} + \left(\max_{\alpha \in T_{n_{k_0}}} \|x_\alpha\| \right) \left\| \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\| \\
&< \frac{\delta}{3} + M\left(\frac{\delta}{3M} \right) < \delta,
\end{aligned}$$

which is impossible. Therefore $\{f_k\}$ does converge weakly to 0. QED

Next we consider a special construction of Rademacher tree of measurable sets.

LEMMA 1.4 [D, p. 154]. *Suppose (Ω, Σ, μ) is atom-free. Then for any E in Σ with $\mu(E) < \infty$, there exists a partition $\{E_1, E_2\}$ of E such that $\mu(E_1) = \mu(E_2)$.*

Recall that an atom in Σ is a measurable set E in Σ such that for any measurable subset F of E, either $\mu(F) = 0$ or $\mu(F) = \mu(E)$. We say that (Ω, Σ, μ) is atom-free if Σ does not contain any atoms of positive finite measure.

LEMMA 1.5. *Suppose that* (Ω, Σ, μ) *is finite and that* f_i *is a separably valued measurable function from* Ω *to Banach space* X_i *for* $1 \leq i \leq k$. *Then for any* $\varepsilon > 0$, *there is a partition* $\{E_n\}$ *of* Ω *such that* $\operatorname{diam} f_i(E_n) < \varepsilon$, $1 \leq i \leq k$, $n \geq 1$. *If, in addition,* (Ω, Σ, μ) *is atom-free, then we may also require that* $\mu(E_{2n-1}) = \mu(E_{2n}) > 0$.

Proof. The first conclusion is obvious. To prove the second one, first we choose a partition $\{F_n\}$ of Ω such that $\mu(F_n) > 0$ and $\operatorname{diam} f_i(F_n) < \varepsilon$, $1 \leq i \leq k$, $n \geq 1$, then by Lemma 1.4, we choose for each $n \geq 1$ a partition $\{E_{2n-1}, E_{2n}\}$ of F_n such that $\mu(E_{2n-1}) = \mu(E_{2n})$. Then $\{E_n\}$ is the partition of Ω we wanted. QED

Using Lemma 1.5, it is easy to prove the following result.

PROPOSITION 1.6. *Suppose that* (Ω, Σ, μ) *is atom-free and* f_i *is a separably valued measurable function from* Ω *to Banach space* X_i *for* $1 \leq i \leq m$. *Then for any* $\varepsilon_k > 0$, $k \geq 0$, *and* E *in* Σ *with* $0 < \mu(E) < \infty$, *there is a Rademacher tree of measurable sets* $\{E_\alpha\}_{\alpha \in T}$ *in* Ω *such that*

$$E_\phi = E, \qquad \mu(E_\alpha) > 0, \qquad \operatorname{diam} f_i(E_\alpha) < \varepsilon_k$$
$$\text{for } 1 \leq i \leq m,\ k > 0, \text{ and } \alpha \in T_k.$$

Section II

Recall that X is said to have the Schur property if every weakly convergent sequence in X is norm convergent. It is obvious that X has the Schur property if and only if 0 is a point of sequential continuity of B_X. If K is a subset of X, we use $\operatorname{psc} K$ (resp. $\operatorname{ext} K$) to denote the set of points of sequential continuity (resp. extreme points) of K.

LEMMA 2.1. *Suppose that* K *is a bounded closed convex set in* X *and that* $x \in \operatorname{psc} K$. *If* $x = \frac{1}{2}(y+z)$ *for some* y *and* z *in* K, *then both* y *and* z *are points of sequential continuity of* K. *Thus if* X *fails the Schur property and* x *is a point of sequential continuity of* B_X, *then* $\|x\| = 1$.

Proof. We only need to show that $y \in \operatorname{psc} K$. So let (y_n) be a sequence in K which converges weakly to y. Then $\text{w-lim}_n \frac{1}{2}(y_n + z) = x$ and $\frac{1}{2}(y_n+z) \in K$, thus $\lim_n \frac{1}{2}(y_n + z) = x = \frac{1}{2}(y+z)$. It follows that $\lim_n y_n = y$. Hence $y \in \operatorname{psc} K$. QED

THEOREM 2.2. *Suppose (Ω, Σ, μ) is atom-free. Then every point of sequential continuity of $B_{L^p(\mu, X)}$ is an extreme point of $B_{L^p(\mu, X)}$.*

Proof. Let $f \in \operatorname{psc} B_{L^p(\mu, X)}$. Since $L^p(\mu, X)$ contains a copy of $L^p(\mu)$ which fails the Schur property, by Lemma 2.1, $\|f\| = 1$. Assume $f \notin \operatorname{ext} B_{L^p(\mu, X)}$. There is $g \in L^p(\mu, X)$ with $\|g\| > 0$ and $\|f \pm g\| = 1$. Since $\|f\| = 1 = \|f \pm g\|$ and $f = \frac{1}{2}[(f + g) + (f - g)]$, and since $L^p(\mu)$ is strictly convex, we conclude that $\|f(t) \pm g(t)\| = \|f(t)\|$ for almost all $t \in \Omega$. Without loss of generality we may assume that $\|f(t) \pm g(t)\| = \|f(t)\|$ for all $t \in \Omega$ and that both $f(\Omega)$ and $g(\Omega)$ are separable.

Since $\|g\| > 0$, there is $M > 0$ and E in Σ such that $\mu(E) > 0$ and $1/M \leq \|g(t)\| \leq M$ for all t in E. Then $\mu(E) < \infty$. By Proposition 1.6, there exists a Rademacher tree of measurable sets $\{E_\alpha\}_{\alpha \in T}$ in Ω such that for $k > 0$, and $\alpha \in T_k$, we have

$$E_\phi = E, \qquad \mu(E_\alpha) > 0, \qquad \operatorname{diam} f(E_\alpha) < 2^{-k} \quad \text{and} \quad \operatorname{diam} g(E_\alpha) < 2^{-k}.$$

For each $\alpha \in T$, pick an element $t_\alpha \in E_\alpha$ and define, for $k \geq 0$,

$$g_k = \sum_{\alpha \in T_k} g(t_\alpha) \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}.$$

By Theorem 1.3, $\{g_k\}$ converges weakly to 0.

Claim. $\lim_k \|f + g_k\| = 1$.

If $t \in \Omega \setminus E$, then $(f \pm g_k)(t) = f(t)$, so $\|(f \pm g_k)(t)\| = \|f(t)\|$. If $t \in E$, then for $k > 1$, there is $\alpha \in T_k$ and $n \geq 1$ such that $t \in E_{\alpha n}$. Thus $g_k(t) = (-1)^n g(t_\alpha)$. Since

$$\operatorname{diam} f(E_\alpha) < 2^{-k}, \quad t \in E_\alpha, \quad t_\alpha \in E_\alpha, \quad \text{and} \quad \|f(t_\alpha) \pm g(t_\alpha)\| = \|f(t_\alpha)\|,$$

we have

$$\begin{aligned} \|(f \pm g_k)(t)\| &= \|f(t) \pm (-1)^n g(t_\alpha)\| \\ &\leq \|f(t) - f(t_\alpha)\| + \|f(t_\alpha) \pm (-1)^n g(t_\alpha)\| \\ &= \|f(t) - f(t_\alpha)\| + \|f(t_\alpha)\| \\ &\leq \|f(t)\| + 2\|f(t) - f(t_\alpha)\| < \|f(t)\| + 2^{-k+1}. \end{aligned}$$

Therefore $\|f \pm g_k\| < \|f\| + 2^{-k+1}\mu(E)^{1/p}$. It follows that $\lim_k \|f \pm g_k\| = \|f\| = 1$.

Since $\lim_k \|f + g_k\| = 1$ and weak-$\lim_k (f + g_k) = f$, we have

$$\lim_k (f + g_k) = f,$$

i.e., $\lim_k \|g_k\| = 0$. On the other hand, since $\|g(t)\| \geq 1/M$ for $t \in E$, we have $\|g_k\| \geq (1/M)\mu(E)^{1/p} > 0$, which is impossible. Therefore $f \in \operatorname{ext} B_{L^p(\mu, X)}$. QED

We say that (Ω, Σ, μ) is not purely atomic if there is E in Σ such that $0 < \mu(E) < \infty$, and E contains no atoms, that is, (E, Σ_E, μ_E) is atom-free, where μ_E be the restriction of μ to $\Sigma_E = \{F\colon F \in \Sigma \text{ and } F \subset E\}$.

COROLLARY 2.3 [ST]. *Suppose that (Ω, Σ, μ) is not purely atomic. If $L^p(\mu, X)$ has the Kadec-Klee property, then X is strictly convex.*

Proof. Since (Ω, Σ, μ) is not purely atomic, there is E in Σ such that $0 < \mu(E) < \infty$ and (E, Σ_E, μ_E) is atom-free. Since $L^p(\mu_E, X)$ is isometrically isomorphic to a subspace of $L^p(\mu, X)$, the space $L^p(\mu_E, X)$ has the Kadec-Klee property. By Theorem 2.2, every unit vector in $L^p(\mu_E, X)$ is an extreme point of the unit ball, thus $L^p(\mu_E, X)$ is strictly convex. Therefore X is also strictly convex. QED

If $K \subset X$, the slice of K determined by the functional x^* in X^* and $\delta > 0$ is the subset of K given by

$$S(x^*, K, \delta) = \{x \in K\colon x^*(x) > \sup x^*(K) - \delta\}.$$

Let $x \in K$. Then x is called a denting point of K if the family of all slices of K containing x is a neighborhood base of x with respect to the relative norm topology on K. And x is said to be a point of continuity of K if the relative weak and norm topologies on K coincide at x. If $K \subset X^*$, $K \neq \emptyset$, then weak* slices, weak* denting points, and weak* points of continuity of K are defined similarly. We use dent K (resp. pc K, w*dent, w*-pc K) to denote the set of denting points (resp. points of continuity, weak* denting points, weak* points of continuity) of K.

By definition, a denting point is a point of continuity, and a point of continuity is a point of sequential continuity. It is known that $x \in \operatorname{dent} K$ if and only if $x \in \operatorname{pc} K$ and $x \in \operatorname{ext} K$ [LLT]. Thus by Theorem 2.2, the following assertion follows.

COROLLARY 2.4. *Suppose that (Ω, Σ, μ) is atom-free and f in $L^p(\mu, X)$. Then f is a point of continuity of $B_{L^p}(\mu, X)$ if and only if f is a denting point of $B_{L^p}(\mu, X)$.*

A Banach space X has the RNP if every non-empty bounded closed set K in X has a denting point. X has the CPCP (resp. PCP) if for every non-empty bounded closed convex (resp. bounded closed) set K in X, pc $K \neq \phi$. It is obvious that the RNP implies the PCP, and the PCP implies the CPCP, but these three properties are distinct. The dual version of PCP, in which one considers weak* point of continuity, is the same as the corresponding dual version of RNP, which in turn is the same as RNP itself [St]. However, the dual version of CPCP, denoted by C*PCP, is distinct from RNP [GMS2]. It is clear that C*PCP implies CPCP, though the converse is not true [DGHZ].

COROLLARY 2.5. *Suppose (Ω, Σ, μ) is not purely atomic. Then the RNP and the CPCP are equivalent in both $L^p(\mu, X)$ and $L^p(\mu, X)^*$.*

Proof. Suppose that $L^p(\mu, X)$ has the CPCP. Let $|\ \ |$ be an equivalent norm on X. Choose E in Σ such that $0 < \mu(E) < \infty$ and (E, Σ_E, μ_E) is atom-free. Since $L^p(\mu, X)$ has the CPCP, the space $L^p(\mu_E, (X, |\ \ |))$ which is isomorphic to a subspace of $L^p(\mu, X)$ also has the CPCP. Hence there exists f in pc $B_{L^p(\mu_E, (X, |\ \ |))}$. Then f must be a denting point of $B_{L^p(\mu_E, (X, |\ \ |))}$ following Corollary 2.4. By a result in [LL], it follows that $f(t)/|f(t)| \in$ dent $B_{(X, |\ |)}$ for almost all $t \in \operatorname{supp} f$. Thus dent $B_{(X, |\ |)}$ is not empty. Therefore X has the RNP (see e.g. p. 30 [Bi]), and hence $L^p(\mu, X)$ has the RNP [DU]. The converse is obvious.

Now suppose that $L^p(\mu, X)^*$ has the CPCP. The space $L^q(\mu, X^*)$, being a subspace of $L^p(\mu, X)^*$, also has the CPCP. As a consequence of the previous paragraph, the space $L^q(\mu, X^*)$ has the RNP. Thus X^* has the RNP, which implies that $L^p(\mu, X)^*$ has the RNP [DU]. The converse is also obvious. QED

Recall that a normed space Y is said to be finitely representable in a normed space E, if for each $\varepsilon > 0$ and finite dimensional subspace F of Y, there is a 1-1 linear operator

$$T: F \to T(F) \subset E \text{ with } \|T\|\,\|T^{-1}\| \leq 1 + \varepsilon.$$

If (P) is a property defined for Banach spaces, X is said to have the property "Super (P)" if every Banach space finitely representable in X has the property (P). It is known that X is super-reflexive if and only if it is super-Radon-Nikodym. It is an open problem whether super-RNP and super-PCP are equivalent.

PROPOSITION 2.6. *Suppose $X \oplus_p X$ is finitely representable in X for some p.*

Then X has the super-RNP *if and only if X has the super*-CPCP.

Proof. Suppose X has the super-CPCP. Let Y be a Banach space finitely representable in X. Then $l^p(Y_n)$, where $Y_n = Y$, $n \geq 1$, is finitely representable in $l^p(X_n)$, where $X_n = X$. Let μ be the Lebesgue measure on $[0, 1)$. Then μ is atom-free.

Claim. $L^p(\mu, Y)$ is finitely representable in X.

Let E be the linear span of simple functions in $L^p(\mu, Y)$. Since E is dense in $L^p(\mu, Y)$, the space $L^p(\mu, Y)$ is finitely representable in E. It is obvious that E is finitely representable in $l^p(Y_n)$, in fact, every finite dimensional subspace G of E is isometric to a subspace of $l^p(Y_n)$. Thus $L^p(\mu, Y)$ is finitely representable in $l^p(X_n)$. Since $X \oplus_p X$ is finitely representable in X, it follows that $l^p(X_n)$ is also finitely representable in X. Thus $L^p(\mu, Y)$ is finitely representable in X.

Since X has the super-CPCP, the space $L^p(\mu, Y)$ has the CPCP. By Corollary 2.5, $L^p(\mu, Y)$ has the RNP. Thus Y has the RNP. Therefore X has the super-RNP. The converse is obvious. QED

COROLLARY 2.7. *Suppose that (Ω, Σ, μ) is a measure space which is not purely atomic or which contains infinitely many atoms of finite positive measure. Then in both $L^p(\mu, X)$ and $L^p(\mu, X)^*$, super*-RNP *and super*-CPCP *are equivalent.*

Proof. In each case, $L^p(\mu, X) \oplus_p L^p(\mu, X)$ is finitely representable in $L^p(\mu, X)$. Thus $L^p(\mu, X)$ has the super-RNP if and only if it has the super-CPCP.

Now suppose that $L^p(\mu, X)^*$ has the super-CPCP, then $L^q(\mu, X^*)$, being a subspace of $L^p(\mu, X)^*$, also has the super-CPCP. Thus $L^q(\mu, X^*)$ has the super-RNP, and in particular X^* has the RNP. Therefore $L^p(\mu, X)^* = L^q(\mu, X^*)$ [DU], and so $L^p(\mu, X)^*$ has the super-RNP. The converse is obvious. QED

Suppose K is a subset of X and $x \in K$. For a given $\varepsilon > 0$, we say that x is an ε-strongly extreme point in K if there is a $\delta > 0$ such that for any y in X, the conditions $d(x + y, K) < \delta$ and $d(x - y, K) < \delta$ imply that $\|y\| < \varepsilon$, where $d(x, K)$ is the distance between x and K. Then x is called a strongly extreme point of K if x is an ε-strongly extreme point in K for all $\varepsilon > 0$. We use str-ext K to denote the set of the strongly extreme points of K. By definition, strongly extreme points are extreme points, but the converse is not true [M]. It is obvious that if K is convex and $d(x \pm y, K) < \delta$ then for any $0 \leq \lambda \leq 1$, we have $d(x \pm \lambda y, K) < \delta$. Thus if K is convex and x is not

ε-strongly extreme in K, then for any $\delta > 0$, there exists y in X such that $d(x \pm y, K) < \delta$ and $\|y\| = \varepsilon$.

THEOREM 2.8. *Suppose that* (Ω, Σ, μ) *is atom-free and* f *is a* $\sigma(L^p(\mu, X), L^q(\mu, X^*))$*-point of sequential continuity of* $B_{L^p}(\mu, X)$, *i.e.*,

$$\lim_k f_k = f \quad \text{if } \sigma(L^p(\mu, X), L^q(\mu, X^*)) - \lim_k f_k = f$$

and $\{f_k\}$ *is in* $B_{L^p}(\mu, X)$. *Then* $\|f\| = 1$ *and* $f(t)/\|f(t)\| \in \text{str-ext}\, B_X$ *for almost all* t *in* $\operatorname{supp} f$. *Thus* f *is a strongly extreme point of* $B_{L^p}(\mu, X)$.

Proof. By Theorem 2.2, the norm $\|f\| = 1$. Without loss of generality, we may assume that $f(\Omega)$ is separable. Define

$$D = \left\{t : t \in \operatorname{supp} f \text{ and } \frac{f(t)}{\|f(t)\|} \notin \text{str-ext}\, B_X\right\}$$

and define, for each $m \geq 1$, the set

$$D_m = \left\{t : t \in D,\ \|f(t)\| > 1/m, \text{ and } \frac{f(t)}{\|f(t)\|} \text{ is not } 1/m\text{-strongly extreme in } B_X\right\}.$$

Then D is the union of D_m. Assume that it is not true that $f(t)/\|f(t)\| \in$ str-ext B_X for almost all t in $\operatorname{supp} f$, that is, $\mu^*(D) > 0$, where μ^* is the outer measure associated to μ. Then there is m such that $\mu^*(D_m) > 0$. Choose a measurable set $E \subset \operatorname{supp} f$ with $\mu(E) = \mu^*(D_m)$ and $D_m \subset E$. It is obvious that $\mu(E) < \infty$. By Proposition 1.6, there is a Rademacher tree of measurable sets $\{E_\alpha\}_{\alpha \in T}$ in Ω such that for $1 \leq i \leq m$, $k > 0$, and $\alpha \in T_k$, we have

$$E_\phi = E, \qquad \mu(E_\alpha) > 0, \quad \text{and} \quad \operatorname{diam} f(E_\alpha) < 2^{-k}.$$

It is obvious that for each $\alpha \in T$, $\mu^*(A \cap E_\alpha) = \mu(E_\alpha)$. For each $\alpha \in T$, pick an element $t_\alpha \in A \cap E_\alpha$ and choose $x_\alpha \in X$ such that

$$\|x_\alpha\| = 1/m \quad \text{and} \quad \left\| \frac{f(t_\alpha)}{\|f(t_\alpha)\|} \pm x_\alpha \right\| \leq 1 + \frac{1}{2^{|\alpha|}}.$$

For each $k > 0$, define

$$g_k = \sum_{\alpha \in T_k} \|f(t_\alpha)\| x_\alpha \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}.$$

Claim. For $k \geq 1$, $\|g_k\| \leq 3\|f\| + 2^{-k+2}\mu(E)^{1/p}$ and $\lim_k \|f \pm g_k\| = \|f\| = 1$.

If $t \in \Omega \setminus E$, then $(f \pm g_k)(t) = f(t)$, so $\|(f \pm g_k)(t)\| = \|f(t)\|$. If $t \in E$, then for $k > 1$, there is $\alpha \in T_k$ and $n \geq 1$ such that $t \in E_{\alpha n}$. Thus $g_k(t) = (-1)^n \|f(t_\alpha)\| x_\alpha$ and so we have

$$\begin{aligned}
\|(f \pm g_k)(t)\| &\leq \|f(t) - f(t_\alpha)\| + \|f(t_\alpha) \pm (-1)^n \|f(t_\alpha)\| x_\alpha\| \\
&= \|f(t) - f(t_\alpha)\| + \|f(t_\alpha)\| \left\| \frac{f(t_\alpha)}{\|f(t_\alpha)\|} \pm (-1)^n x_\alpha \right\| \\
&\leq \|f(t) - f(t_\alpha)\| + \left(1 + \frac{1}{2^{|\alpha|}}\right) \|f(t_\alpha)\| \\
&\leq \left(1 + \frac{1}{2^{|\alpha|}}\right) \|f(t)\| + \left(2 + \frac{1}{2^{|\alpha|}}\right) \|f(t) - f(t_\alpha)\| \\
&\leq (1 + 2^{-k}) \|f(t)\| + 2^{-k+2}.
\end{aligned}$$

Therefore $\|f \pm g_k\| \leq (1 + 2^{-k})\|f\| + 2^{-k+2}\mu(E)^{1/p}$. It follows that

$$\|g_k\| \leq 3\|f\| + 2^{-k+2}\mu(E)^{1/p} \quad \text{and} \quad \lim_k \|f \pm g_k\| = \|f\| = 1.$$

Since $\{g_k\}$ is a bounded Rademacher sequence in $L^p(\mu, X)$, by Proposition 1.1, it is $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-null. Thus

$$\sigma(L^p(\mu, X), L^q(\mu, X^*)) - \lim_k f + g_k = f \quad \text{and} \quad \lim_k \|f + g_k\| = \|f\| = 1.$$

Since f is a $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-point of sequential continuity of $B_{L^p}(\mu, X)$, we conclude that $\lim_k f + g_k = f$. Thus $\lim_k \|g_k\| = 0$. On the other hand, since $\|g_k(t)\| \geq 1/m^2$ for $t \in E$, the norm

$$\|g_k\| \geq (1/m^2)\mu(E)^{1/p} > 0,$$

which is a impossible. Therefore

$$f(t)/\|f(t)\| \in \text{str-ext } B_X$$

for almost all t in $\operatorname{supp} f$. Hence f is a strongly extreme point of $B_{L^p}(\mu, X)$ [Sm2]. QED

In addition to its sequential generalization, the point of continuity has a "slice generalization", namely, the point of small combination of slices (SCS-point). Let K be a convex set of X, the point $x \in K$ is called a SCS-point of K [GGMS] if for each $\varepsilon > 0$, there exist slices S_i of K and $\lambda_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ such that $\operatorname{diam} \sum_{i=1}^n \lambda_i S_i < \varepsilon$ and $x \in \sum_{i=1}^n \lambda_i S_i$. Let SCS($K$) denote the set of all SCS-points of K. If K is in X^*, a w*-SCS-point of K is defined similarly except the slices S_i of K are weak* slices. It is clear that pc $K \subset$ SCS(K) (resp. w*-pc $K \subset$ w*-SCS(K)) for all convex sets K in X (resp. X^*).

It is known [GGMS], [R1] that X (resp. dual space X^*) is strongly (resp. w*-strongly) regular if and only if every non-empty bounded closed convex set K in X (resp. X^*) is contained in the norm-closure (resp. weak* closure) of SCS(K) (resp. w*-SCS(K)). Schachermayer [Sc] proved that a Banach space has the RNP if and only if it is strongly regular and it has the Krein-Milman Property. The "point-version" of this result is also true and it extends the result in [LLT].

PROPOSITION 2.9. *Let K be a closed convex set in X^* and let $\overline{K}^*$ be the weak* closure of K. Then*:

(1) w*-pc K = w*-pc $\overline{K}^*$.
(2) w*-SCS(K) = w*-SCS($\overline{K}^*$).
(3) w*-dent $\overline{K}^*$ = w*-dent K = (w*-pc K) $\cap$ ext K = w*-SCS(K) $\cap$ ext K.

Proof. (1) Let $x^* \in$ w*-pc $\overline{K}^*$. Since the weak* and norm topologies on $\overline{K}^*$ coincide at x^*, we have $x^* \in \overline{K} = K$. Thus $x^* \in$ w*-pc K.

Conversely, if $x^* \in$ w*-pc K, then for each $\varepsilon > 0$, there are $x_1, \ldots, x_n$ in X and $\delta > 0$ such that $\operatorname{diam} V < \varepsilon$, where

$$V = \{y^*: y^* \in K, (y^*, x) > (x^*, x) - \delta, i = 1, \ldots, n\}.$$

Let

$$U = \{y^*: y^* \in \overline{K}^*, (y^*, x) > (x^*, x) - \delta, i = 1, \ldots, n\}.$$

Then U is a w*-neighborhood of x^* in $\overline{K}^*$ and V is weak* dense in U. Thus $\operatorname{diam} U = \operatorname{diam} V < \varepsilon$. So $x^* \in$ w*-pc $\overline{K}^*$.

(2) Let $x^* \in$ w*-SCS($\overline{K}^*$). It is obvious that every weak* slice of $\overline{K}^*$ contains a point of K. Hence, by the definition of w*-SCS-points, $x^* \in \overline{K} = K$. Therefore $x^* \in$ w*-SCS(K).

Conversely, if $x^* \in$ w*-SCS(K), then for each $\varepsilon > 0$, there exist w*-slices S_j of K and $\lambda_i > 0, i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ such that $\operatorname{diam} \sum_{i=1}^n \lambda_i S_i < \varepsilon$. We assume $S_i = S(x_i, K, \delta_i)$ for some x_i in X and $\delta_i > 0$. Since

$\sum_{i=1}^{n}\lambda_i S(x_i, \overline{K}^*, \delta_i)$ is a subset of the weak* closure of $\sum_{i=1}^{n}\lambda_i S_i$, we have $\operatorname{diam}\sum_{i=1}^{n}\lambda_i S(x_i, \overline{K}^*, \delta_i) < \varepsilon$. Hence $x^* \in$ w*-SCS $\overline{K}^*$.

(3) It is obvious that

$$\text{w*-dent}\,\overline{K}^* \subset \text{w*-dent}\,K \subset (\text{w*-pc}\,K) \cap \text{ext}\,K \subset \text{w*-SCS}(K) \cap \text{ext}\,K.$$

To complete the proof we only need to show

$$(\text{w*-SCS}\,K) \cap \text{ext}\,K \subset \text{w*-dent}\,\overline{K}^*.$$

So let $x^* \in$ w*-SCS$(K) \cap$ ext K. For each $\varepsilon > 0$, there exist weak* slices S_i of K and $\lambda_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n}\lambda_i = 1$ such that $\operatorname{diam}\sum_{i=1}^{n}\lambda_i S_i < \varepsilon$ and $x^* \in \sum_{i=1}^{n}\lambda_i S_i$. Since $x^* \in$ ext K, x^* must belong to $\bigcap_{j=1}^{n} S_i$. Thus $\bigcap_{j=1}^{n} S_i$ is a weak* neighborhood of x^*. Note that $\operatorname{diam}\bigcap_{j=1}^{n} S_i \le \operatorname{diam}\sum_{i=1}^{n}\lambda_i S_i < \varepsilon$, so $x^* \in$ w*-pc K.

Next we show that $x^* \in$ ext $\overline{K}^*$. Assume $x^* = (y^* + z^*)/2$ for some y^*, z^* in $\overline{K}^*$. Since $x^* \in$ w*-pc $K =$ w*-pc $\overline{K}^*$, it follows that $y^*, z^* \in$ w*-pc $\overline{K}^*$ (see the proof of Lemma 2.1). By (1), $y^*, z^* \in K$. Thus $x^* = y^* = z^*$ because $x^* \in$ ext K. So $x^* \in$ ext $\overline{K}^*$. Since x^* is a weak* point of continuity and an extreme point of the weak* compact convex set $\overline{K}^* \cap B_{X^*}(x, 1)$, the weak* slices of $\overline{K}^* \cap B_{X^*}(x, 1)$ containing x^* is a norm neighborhood base at x^*. Therefore $x^* \in$ w*-dent $\overline{K}^* \cap B_{X^*}(x, 1)$. Hence $x^* \in$ w*-dent $\overline{K}^*$ [B]. QED

COROLLARY 2.10. *Let K be a closed convex set in X and let $\overline{K}^*$ be the weak* closure of K in X^{**}. Then*:

(1) pc $K =$ w*-pc $\overline{K}^*$.

(2) w*-dent $\overline{K}^* =$ dent $K =$ pc $K \cap$ ext $K =$ SCS$(K) \cap$ ext K.

Proof. This follows immediately from Proposition 2.9 and the facts that w*-dent $K =$ dent K, w*-pc $K =$ pc K, and w*-SCS$(K) =$ SCS(K), QED

Note that for any $f \in L^q(\mu, X^*)$ and $g \in L^p(\mu, X)$, the action of f on g is defined by

$$(f, g) = \int_\Omega (f(t), g(t))\, d\mu(t) \quad \text{[DU]}.$$

It is obvious that the space $L^q(\mu, X^*)$ is a subspace of $L^p(\mu, X)^*$, and that $L^q(\mu, X^*)$ norms $L^p(\mu, X)$. So if $K = B_{L^q(\mu, X^*)}$, then $\overline{K}^* = B_{L^p(\mu, X)^*}$. Hence the following result is a corollary of Proposition 2.9.

COROLLARY 2.11. *The following assertions are true*:
(1) w*-pc $B_{L^q(\mu, X^*)}$ = w*-pc $B_{L^p(\mu, X)^*}$.
(2) w*-SCS $B_{L^q(\mu, X^*)}$ = w*-SCS $B_{L^p(\mu, X)^*}$.
(3) w*-dent $B_{L^p(\mu, X)^*}$ = w*-dent $B_{L^q(\mu, X^*)}$ = w*-SCS$(B_{L^q(\mu, X^*)}) \cap$ ext $B_{L^q(\mu, X^*)}$.

If (Ω, Σ, μ) is atom-free, then every weak* point of continuity f of $B_{L^q(\mu, X^*)}$ is an extreme point of $B_{L^q(\mu, X^*)}$. (Corollary 2.4), by Corollary 2.11, it is a weak* denting point of $B_{L^q(\mu, X^*)}$. Thus we have the following result.

COROLLARY 2.12. *Suppose that (Ω, Σ, μ) is atom-free and f in $L^p(\mu, X)^*$. Then f is a weak* point of continuity of $B_{L^p(\mu, X)^*}$ if and only if f is a weak* denting point of $B_{L^p(\mu, X)^*}$.*

The next example shows that we can not replace the point of sequential continuity by SCS-point in Theorem 2.2.

Example 2.13. Let Y be a Banach space such that it contains no copies of l^1 but its dual Y^* does not have the RNP [GMS2]. Let $X = Y^*$ and let $K = B_{L^p(\mu, X)}$. By taking equivalent norms, we may assume that w*-dent $B_{Y^*} = \phi$ [Bi]. Let μ be the Lebesque measure on $[0, 1)$. Since Y contains no copy of l^1, the space $L^q(\mu, X)$ also contains no copy of l^1 [P]. By a result of J. Bourgain [Ba], $L^q(\mu, Y)^*$ is weak* strongly regular. Thus K is contained in the weak* closure of w*-SCS(K). So the weak* closure of the w*-SCS-points is $B_{L^q(\mu, Y)^*}$. Were a w*-SCS point f an extreme point, that point f would be a weak* denting point of $B_{L^p(\mu, Y^*)}$ by Corollary 2.11. But then by a result in [HL], for almost all t in the support of f, $f(t)/\|f(t)\|$ would be a weak* denting point of B_{Y^*}, which contradicts the fact that w*-dent $B_{Y^*} = \phi$. Therefore none of these w*-SCS-points is an extreme point of K. By definition, w*-SCS$(K) \subset$ SCS(K), so in Theorem 2.2 we can not replace the point of sequential continuity by the SCS-point.

If (Ω, Σ, μ) is purely atomic and finite, then there exists an at most countable partition π of Ω such that every element in π is an atom of positive measure. For each E in π, let X_E be the space X. Define mapping T from $L^p(\mu, X)$ to $l^p(X_E)_{E \in \pi}$ by

$$T(f)(E) = \mu(E)^{1/p} \int_E f(t)\, d\mu(t).$$

Thus $T(f)(E) = \mu(E)^{1/p} f(t)$ for almost all t in E. It is obvious that T is an isometric embedding. Partly because of this, in the rest of this section we will consider the space $l^p(X_i)$, instead of $L^p(\mu, X)$ with (Ω, Σ, μ) being purely atomic.

PROPOSITION 2.14. *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $f=(f(i))_{i\in I}$ be a unit vector in $l^p(X_i)$. Then $f\in$ psc $B_{l^p(X_i)}$ (resp. pc $B_{l^p(X_i)}$; ext $B_{l^p(X_i)}$; or dent $B_{l^p(X_i)}$) if and only if $f(i)/\|f(i)\|\in$ psc B_{X_i} (resp. pc B_{X_i}; ext B_{X_i}; or dent B_{X_i}) for $i\in$ supp f.*

Moreover, the weak version of this statement is also true.*

Proof. Suppose $f\in$ psc $B_{l^p(X_i)}$. Fix $i\in I$ with $f(i)\neq 0$. We use $B_X(x,r)$ to denote the ball in X with center x and radius r. Let $\{x_n\}$ be a sequence in $B_{X_i}(0,\|f(i)\|)$ such that w-$\lim_n x_n=f(i)$. For each n define

$$f_n(j)=\begin{cases} f(j) & \text{if } j\neq i\\ x_n & \text{if } j=i\end{cases}.$$

Then $f_n\in B_{l^p(X_i)}$ and weak-$\lim_n f_n=f$. Hence $\lim_n\|f_n-f\|=0$ and so $\lim_n\|x_n-f(i)\|=0$. Therefore $f(i)\in$ psc $B_{X_i}(0,\|f(i)\|)$ which is equivalent to $f(i)/\|f(i)\|\in$ psc B_{X_i}.

Conversely, suppose $f_n\in B_{l^p(X_i)}$ with weak-$\lim_n f_n=f$. Then weak-$\lim_n f_n(i)=f(i)$, $i\in I$ and w-$\lim_n\frac{1}{2}(f_n+f)=f$. Since $\|f\|=1$ we must have $\lim_n\|\frac{1}{2}(\|f_n(\cdot)\|+\|f(\cdot)\|)\|=1$ in $l^p(I)$. By the uniform convexity of $l^p(I)$, $\lim_n\|\,\|f_n(\cdot)\|-\|f(\cdot)\|\,\|=0$. So for each $i\in I$, $\lim_n\|f_n(i)\|=\|f(i)\|$. Using the fact that $f(i)\in$ psc $B_{X_i}(0,\|f(i)\|)$, we can conclude that $\lim_n\|f_n(i)-f(i)\|=0$. Hence $\lim_n\|f_n-f\|=0$, and so $f\in$ psc $B_{l^p(X_i)}$.

The proofs for pc, w*-psc and w*-pc points are similar while that for extreme points can be found in [Sm1]. The conclusion for denting (resp. w*-denting) points follows from Proposition 2.9. QED

As a corollary of Proposition 2.14, if (Ω,Σ,μ) is purely atomic and f is a unit vector in $L^p(\mu,X)$, then $f\in$ psc $B_{L^p(\mu,X)}$ (resp. pc $B_{L^p(\mu,X)}$; dent $B_{L^p(\mu,X)}$) if and only if $f(t)/\|f(t)\|\in$ psc B_X (resp. pc B_X; dent B_X) for almost all t in supp f.

For the proof of our next result, we need the following facts: X has the CPCP (resp. PCP) if and only if given $\varepsilon>0$ and any non-empty bounded convex (resp. bounded) set K in X, there is a relatively weakly open set V in K with diameter less than ε; X^* has the C*PCP if and only if given $\varepsilon>0$ and any non-empty bounded convex set K in X^*, there is a relatively weak* open set V in K with diameter less than ε (see [R2]).

THEOREM 2.15. *Let $\{X_i,\ i\in I\}$ be a family of Banach spaces. Then:*

(1) *$l^p(X_i)$ has the* CPCP *(resp.* PCP*) if and only if each X_i has the* CPCP *(resp.* PCP*).*

(2) *$l^p(X_i)^*$ which can be identified as $l^q(X_i^*)$ has the* C*PCP *if and only if each X_i^* has the* C*PCP.

Proof. Assume that each X_i has the CPCP and $I = \{1,2\}$. Since the CPCP is an isomorphic invariant, it suffices to show that the space

$$X = \{(x_1, x_2): x_i \in X_i, i = 1,2, \|(x_1, x_2)\| = \max(\|x_1\|, \|x_2\|)\}$$

has the CPCP.

Let A be a non-empty bounded convex set in X and let P_i: $X \to X_i$, $i = 1,2$, be the natural projection. Let $A_1 = P_1(A)$. Since X_1 has the CPCP, there exist x_j^*, $a_j > 0$, $j = 1, \dots, n$ such that $\operatorname{diam} \bigcap_{j=1}^n S(x_j^*, A_1, a_j) < \varepsilon$. Let

$$A_2 = P_2\left[A \cap \left(P_1^{-1}\left(\bigcap_{j=1}^n S(x_j^*, A_1, a_j)\right)\right)\right].$$

Then A_2 is a non-empty bounded convex set in X_2. Since X_2 has the CPCP there are y_k^*, $b_k > 0$, $k = 1, \dots, m$ such that $\operatorname{diam} \bigcap_{k=1}^m S(y_k^*, A_2, b_k) < \varepsilon$. Put

$$V = \{(x_1, x_2): (x_1, x_2) \in A, x_j^*(x_1) > \sup x_j^*(A_1) - a_j, y_k^*(x_2) > \sup y_k^*(A_2) - b_k, \quad j = 1, \dots, n, k = 1, \dots, m\}.$$

Then V is a weakly open set in A with diameter less than ε. Therefore X has the CPCP.

To prove the general case, let $E = l^p(X_i)$ and let A be a non-empty bounded closed convex set in E. Without loss of generality, assume that $\sup\{\|x\|, x \in A\} = 1$. Given $\varepsilon > 0$, we can choose $0 < \varepsilon_1 < 1 - [1 - (\varepsilon/3)^p]^{1/p}$ and $x = (x_i)_{i \in I}$ in A with $\|x\|^p > 1 - \varepsilon_1$. Then there exists $i_k \in I$, $k = 1, \dots, n$, such that $\sum_{k=1}^n \|x_{i_k}\|^p > 1 - \varepsilon_1$. For each $k = 1, \dots, n$, choose $x_{i_k}^*$ in $X_{i_k}^*$ such that $\|x_{i_k}^*\|^q = \|x_{i_k}\|^p$ and $(x_{i_k}^*, x_{i_k}) = \|x_{i_k}\|^p$. Let $x^* = (x_i^*)_{i \in I}$ where $x_i^* = 0$ for all $i \neq i_k$, $k = 1, \dots, n$. Then $x^* \in l^q(X_i^*)$, $\|x^*\| \le 1$ and $(x^*, x) = \sum_{k=1}^n \|x_{i_k}\|^p > 1 - \varepsilon_1$.

Let $E_1 = l^p(X_{i_1}, \dots, X_{i_n})$, $E_2 = l^p(X_i: i \in I, i \neq i_k, k = 1, \dots, n)$ and let P: $E \to E_1$ be the natural projection. Without loss of generality, we may regard E_1 and E_2 as subspaces of E. Let $\delta = \sup x^*(A) - 1 + \varepsilon_1$. Then for any $y = (y_i)_{i \in I}$ in $S(x^*, A, \delta)$ we have

$$\|Py\| \ge (x^*, Py) = (x^*, y) > 1 - \varepsilon_1 > \left[1 - (\varepsilon/3)^p\right]^{1/p}.$$

Hence

$$\|y - Py\| = \left(\|y\|^p - \|Py\|^p\right)^{1/p} < \varepsilon/3.$$

By the first part of the proof, E_1 has the CPCP. So there is a weakly open set

V_1 in E_1 with

$$\operatorname{diam}\{V_1 \cap P[S(x^*, A, \delta)]\} < \varepsilon/3.$$

Let $V = (V_1 \oplus E_2) \cap S(x^*, A, \delta)$. Then V is non-empty and weakly open in A and for any y and z in V, we have

$$\|y - z\| \leq \|y - Py\| + \|Py - Pz\| + \|Pz - z\| < \varepsilon.$$

Hence the diameter of V is less than or equal to ε and so E has the CPCP. The proofs of the remaining assertions are similar. QED

Remark 2.16. The PCP is a three-space property; i.e., if Y is a subspace of X such that both Y and X/Y have the PCP, then X also has the PCP [R2], and this fact implies that $l^p(X_i)_{i \in I}$ has the PCP if I is finite and X_i has the PCP for every $i \in I$. However it is unknown whether CPCP or C*PCP is a three-space property.

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Miami University
Oxford, Ohio
University of Iowa
Iowa City, Iowa