CONFORMALLY INVARIANT CLUSTER VALUE THEORY

BY

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1. Introduction

The purpose of this paper is to show how potential and probability theory can be used jointly to set up a conformally invariant cluster value theory of analytic functions. The analytic functions studied are the intrinsically natural ones, those whose domains and ranges are Riemann surfaces, and the methods used would not become any simpler if the functions were meromorphic functions defined on plane domains with smooth boundaries.

The present paper does not pretend to completeness, of course. What it contains is the substructure of a theory, and the proofs of a few key theorems. It is noteworthy that the Fatou boundary limit theorem for numerically-valued functions regular on a disc, and its generalization to functions of bounded type, are valid in our general context. This makes it possible to give a simple interpretation, in terms of boundary functions, to Heins's class **Bl** of analytic functions, a generalization of Seidel's class **U** and Storvick's class **L**. As an application to a classical situation, a disc covering theorem with a hypothesis of a different type from Bloch's is proved.

The probabilistic basis of the work is the theory of Brownian motion on a Riemann surface. This provides a suitable path system on any Riemann surface, replacing, for example, the set of radii to the perimeter of a disc. The key potential-theoretic tool is the Martin boundary R^M of a hyperbolic Riemann surface R, together with the Martin topology and the Cartan-Brelot-Naïm fine topology on $R \cup R^M$.

The reader will observe that sometimes the new theorems are not strictly generalizations of their classical versions, because the classical versions do not have an invariant form, so that the general theorems do not reduce to exactly the classical ones under the classical hypotheses. For example, the Fatou theorem that a function bounded and regular on a disc has an angular limit at almost every (Lebesgue measure) perimeter point becomes the theorem that an analytic function from a hyperbolic Riemann surface R_1 into a hyperbolic Riemann surface R_2 has a boundary limit (on $R_2 \cup R_2^M$) on approach in terms of the fine topology to almost every (harmonic measure) point of R_1^M. This general theorem is a perfect generalization of the classical theorem even though angular approach to a point of the perimeter of a disc is not the same as approach to the point in the fine topology. In fact if a bounded regular function on a disc has a fine limit at a perimeter point, it has this limit as a limit along certain continuous (conditional Brownian) paths, and hence has this limit also on angular approach, by a classical argument. Thus the gen-

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eral theorem when applied to the classical case yields a result which is at least as strong as the classical one. In other applications to classical situations, however, the situation is less clear. The appropriate invariant concept corresponding to a cluster value of a function on a disc, along a nontangential sequence to a perimeter point seems to be a fine cluster value of a function on a hyperbolic Riemann surface R at a minimal point of R^M (see Section 4). The relation between the two concepts if R is a disc has not yet been worked out.

Thus, as might be expected, a conformally invariant theory, an *intrinsic* theory, is more than a generalization. It is to some extent a recasting of the theory from a different point of view. We remark that the inappropriateness of sectorial approach to disc boundary points in potential theory and allied subjects can be seen from the well-known fact that the classical boundary limit theorems for positive harmonic and superharmonic functions on a disc differ (angular approach is admissible for the first class, only radial or similar approach is admissible for the second), whereas approach in the fine topology serves in both cases.

The methods used will be probabilistic, corresponding to the fact that in the present state of mathematics certain potential-theoretic results are easier to prove probabilistically than by purely potential-theoretic methods. A nonprobabilistic statement of each theorem will be given, however.

To clarify the historical background, references will be made to original papers, but most of the cluster value theorems referred to can be found, with proofs, in the books of Noshiro [2] and Tsuji [2].

2. Functions and paths on Riemann surfaces

A Riemann surface is, roughly, a connected Hausdorff space in which each point is in an open set, called a parametrized neighborhood below, which is the one-to-one conformal image of a plane disc. The nomenclature of the book of Ahlfors-Sario [1] will be used. "The Riemann surface R has a positive boundary, is hyperbolic, has a Green function, has a nonconstant positive superharmonic function" are equivalent assertions. In the contrary case, R "has a null boundary, is parabolic". An open connected set R_0 on a Riemann surface R is itself a Riemann surface, under the obvious conventions. If R is hyperbolic, so is R_0. If R is parabolic, R_0 is parabolic if and only if $R - R_0$ has capacity zero.

If f is a continuous function from one Riemann surface R_1 into a second, R_2, and is regular in terms of the local parameters, we shall call f an analytic function from R_1 to R_2, or an R_2-valued analytic function on R_1. It is trivial that R_2 must then be parabolic if R_1 is.

If a path on a noncompact Riemann surface leaves every compact set, we shall say it "goes to ∞". If a Riemann surface R is hyperbolic, a more interesting compactification than that implied by the above is obtained by the adjunction of the Martin boundary R^M, whose properties are fundamental in the work of this paper.

Brownian motion paths on abstract Euclidean spaces have already been discussed by Kakutani [1] and by the author [3]. These discussions yielded rather awkward and unnatural definitions of Brownian motion on a Riemann surface (considered as a covering surface of the plane). We shall need a more appropriate definition, a conformally invariant one, which we now give.

Let R be a Riemann surface, and let p be a function of the triple (t, ξ, A), where t is a strictly positive number, $\xi \in R$, and A is a Borel subset of R. If p satisfies the following conditions (a)–(e), p will be called a Brownian motion transition function on R.

(a) $p(t, \xi, \cdot)$ is a measure of Borel subsets of R, with $p(t, \xi, R) \leqq 1$.

(b) $p(t, \cdot, A)$ is a Baire function, for each pair (t, A).

(c) If $0 < s, t$,

$$p(s+t, \xi, A) = \int_R p(t, \eta, A) p(s, \xi, d\eta). \tag{2.1}$$

(d) If $\xi \in R$, there is a strong Markov process with state space R, initial point ξ, transition function p, and continuous sample functions.

(e) Let ξ be a point of R. There is then a parametrized neighborhood A of ξ with the following properties. (e1) Almost every path of the process described in (d) meets $R - A$. (e2) Let R_0 be an open subset of A whose closure is a compact subset of A. Let u be a function defined and superharmonic on a neighborhood of this closure. Let $\{z(t), 0 \leqq t < \infty\}$ be a stochastic process as in (d), with initial point in R_0, and let T be the time that a path from ξ first meets the boundary of R_0. Define $z_1(t) = z[\min(t, T)]$, and let $\mathfrak{F}(t)$ be the least Borel field of sets with respect to which $z_1(s)$ is measurable, for every $s \leqq t$. Then the stochastic process $\{z_1(t), \mathfrak{F}(t), 0 \leqq t < \infty\}$ is a supermartingale (omitting $t = 0$ if $u(\xi) = \infty$). Any stochastic process as described in (d) will be called a Brownian motion process from ξ. If $p(t, \xi, R) = 1$, when $0 < t < \infty$, almost all Brownian paths from ξ are well-defined on the parameter interval $[0, \infty)$, and are said to have infinite lifetimes. Otherwise they may have finite lifetimes, and in fact $p(t, \xi, R)$ is the probability of a life $\geqq t$. If T is a path lifetime, and if a path has some asymptotic property as the parameter increases to T, we shall say that the path "end" has the property. We shall see below that every Riemann surface has many quite different Brownian motion transition functions. Even the fact of having infinite path lifetimes is not an invariant of the surface.

Our hypotheses have given us the basis for work already done elsewhere (Kakutani [1], Doob [3]) in a less flexible way, and we only outline the basic results on the interrelations between Brownian motion processes and potential theory as they will be needed in what follows.

If a subset of a Riemann surface R has capacity zero, almost no Brownian path from a point of the surface meets the set (aside perhaps from the initial path point). If R is parabolic, almost every Brownian path from a point has everywhere dense ends, and in fact the path meets any preassigned set of strictly positive capacity at points arbitrarily near the path lifetime. On the other hand, if R is hyperbolic, almost every Brownian path from a point of R converges to ∞.

If R_1 is an open connected subset of the Riemann surface R, and if p is a Brownian motion transition function on R, let $p_1(t, \xi, A)$ be the probability that a Brownian motion path on R from ξ in R_1 lies in $A \subset R_1$ at time t, without having first reached the boundary of R_1. Then p is a Brownian motion process transition function whose paths are paths of the Brownian motion process on R considered only until the time they first reach $R - R_1$. We shall call p_1 and the Brownian motion process on R_1 just described the abbreviated versions of p and the process on R, adapted to R_1.

If two Riemann surfaces are conformally equivalent, the map establishing the equivalence takes a Brownian motion transition function [process] on one surface into one on the other. In particular, any one-to-one conformal map of a Riemann surface onto itself takes a Brownian motion transition function and process into a second such pair.

If u is a superharmonic function on a Riemann surface R, u is finite-valued and continuous, as a function of the parameter, on almost every Brownian path from a point ξ of R (except that of course u is infinite-valued at the parameter value 0 if $u(\xi) = \infty$). Moreover hypothesis (e2) remains true if R_0 is, more generally, any open subset of R, containing ξ, whose closure is a compact subset of R. (This assertion can be considerably generalized—see Doob [3].)

In general, all results in the preceding reference on Brownian motion on a Green space, dimension 2, remain valid except those tied to the heat equation. In fact in that reference the canonical local transformation was a rigid motion, whereas here it is a directly conformal map. Thus, for example, it remains true that the distribution of the first point in which a Brownian path from ξ in an open connected subset R_0 of the given Riemann surface meets the boundary of R_0 is harmonic measure on this boundary, relative to R_0, with reference point ξ. More generally, suppose that R is a hyperbolic Riemann surface, and adjoin its Martin boundary R^M. Then almost every Brownian motion path on R from a point of R converges to some minimal point of R^M. Let $\mu(\xi, A)$ be the probability that a path from ξ converges to a point of $A \subset R^M$. Then $\mu(\xi, \cdot)$ is harmonic measure (in the sense now to be made clear). If u is positive and superharmonic on R, there is a function ϕ on the Martin boundary such that, for almost all Brownian motion paths from a point ξ of R, u has the limit $\phi(\eta)$ if the path converges to η, and

$$\mu(\xi) \geqq \int_{R^M} \phi(\eta)\mu(\xi, d\eta). \tag{2.2}$$

The nonprobabilistic interpretation of this boundary limit theorem is the following. The class of subsets of the Martin boundary of harmonic measure 0 is independent of the reference point, and a member class will be described as having harmonic measure 0. Moreover the class of functions on the Martin boundary which are summable using harmonic measure is independent of the reference point, and will be described as the class of functions summable with respect to harmonic measure. In these terms, the above function u has the fine limit $\phi(\eta)$ at η on the Martin boundary (that is, the limit $\phi(\eta)$ on approach to η in the Cartan-Brelot-Naïm topology) except possibly for an η set of harmonic measure 0, ϕ is summable with respect to harmonic measure, and (2.2) holds. Finally, from the point of view of the Dirichlet problem: the Perron-Wiener-Brelot method applied to find a harmonic function on R with assigned arbitrary boundary function ϕ on the Martin boundary, yields a "solution" u if and only if ϕ is summable with respect to harmonic measure. The solution is defined by (2.2) with equality, and u has the function ϕ as a fine limit at almost every Martin boundary point (harmonic measure).

It is important to observe that, although there are always many quite different Brownian motion transition functions on a Riemann surface, all the above assertions are valid for any choice of the transition function. Roughly speaking, the different transition functions correspond to different speeds of traversing the same paths.

Example 1. Let R be the Riemann sphere. In this case, Yosida [1] has obtained an explicit formula for a transition probability function, invariant under sphere rotations, $p(t, \xi, A) = p(t, \gamma\xi, \gamma A)$ for every rotation γ, and this transition function is easily seen to satisfy our conditions for a Brownian motion transition function. Under stereographic projection we thus have a Brownian motion transition function for the extended plane, invariant under certain linear transformations.

Example 2. If R is any open subset of the extended plane, the transition function of Example 1, abbreviated to match R, is a Brownian motion transition function for R. If R is an open subset of the finite plane, a second example is the ordinary plane Brownian motion transition function abbreviated to match R. Note that, if R is a disc, every linear transformation of the disc onto itself takes a Brownian motion transition function into a new (in general different) one.

In the following work and later we shall consider derivable functions on a Riemann surface. It is important that derivability, and the nonvanishing of both partial derivatives of a function has an invariant meaning in terms of local parameters.

We shall now show that there are many Brownian motion transition functions on an arbitrary Riemann surface R. Let u be a function on R with continuous second partial derivatives in terms of local parameters, whose first partial derivatives do not both vanish at any point, and such that there is a covering of R by parameter neighborhoods, mapped conformally on unit discs, in terms of which parametrizations the first partial derivatives are uniformly bounded. If (ξ, η) is a local parameter, define $\sigma^2 = u_\xi^2 + u_\eta^2$. Then define a diffusion process on R with local displacement coefficient 0 and dispersion coefficient σ^2. This pair of coefficients transforms as it should under parameter transformations, and Itô [2] has shown that there is a corresponding Markov transition function. The corresponding processes are known from Itô's previous work [1] to have continuous sample functions, if separable as we shall always suppose. It is easy to deduce that, if $C(R)$ is the Banach space of continuous functions on R, vanishing at ∞ if R is not compact, with uniform norm, and if $U_t f$ is defined by

$$(2.3) \qquad (U_t f)(\xi) = \int_R f(\eta) p(t, \xi, d\eta),$$

then U_t takes $C(R)$ into itself and defines a strongly continuous semigroup. Hence the process has the strong Markov property (Loève [1]). Itô proves that if f has continuous second derivatives,

$$(2.4) \qquad \lim_{t \to 0} [(U_t f)(\xi) - f(\xi)]/t = \sigma^2 \Delta f/2,$$

where Δ is the Laplace operator. This indicates (unrigorously) that if $\Delta f \leqq 0$, so that f is superharmonic, $U_t f$ decreases as t increases, that is, f defines a supermartingale on the sample functions of an Itô process. To obtain this result rigorously we proceed somewhat differently. In order to check the property (e) of a Brownian motion process we first consider a special case. In the following, the process $\{z(t), 0 \leqq t < \infty\}$ is a Markov process as constructed by Itô. Let R be the finite plane, and let p be a transition function on R as found by Itô, satisfying all the conditions we have verified so far, using the trivial identity parametrization. Moreover, contradicting for the moment one of our conditions on the variance function σ, suppose that σ has compact support. In view of the latter condition, we cannot hope to verify property (e1) but we shall prove instead that *almost every path from a point at which σ is strictly positive must reach the region of vanishing of σ.* Since Itô [2] proved that almost no path reaches interior points of this set, it is already known that the paths are confined to the support of σ. In Itô's development the process $\{z(t), 0 \leqq t < \infty\}$ satisfies the equation

$$(2.5) \qquad z(t) = \int_0^t \sigma[z(s)]\, dy(s),$$

where ξ is the initial point of paths and the $y(t)$ process is ordinary plane Brownian motion (unit variance parameter for each component). Since any stochastic integral like that on the right defines a martingale (Doob [2]), the $z(t)$ process is a bounded martingale. Hence there is necessarily convergence when $t \to \infty$, so we need only show that the convergence must be to the set of zeros of σ. On a path converging to a nonzero of σ, the integrand $\sigma[z(s)]$ would be asymptotically a strictly positive constant, whereas the integral would converge, and this is absurd. Thus a limited version of (e1) is satisfied. Furthermore, still in our special case when R is the finite plane, Itô proved that if u is a function on the plane, with continuous second partial derivatives, then $u[z(t)]$ can be written in the form

$$(2.6)\qquad u[z(t)] = u(\xi) + \int_0^t (\sigma^2 \Delta u/2)\, ds + \int_0^t \sigma[u_1\, dy^{(1)}(s) + u_2\, dy^{(2)}(s)],$$

where the functions in the integrands have argument $z(s)$, u_1, u_2 are the first partial derivatives of u, and $y^{(1)}(s)$, $y^{(2)}(s)$ are the components of $y(s)$. In particular, if $\Delta u = 0$, the first integral on the right drops out, and the $u[z(t)]$ process is a martingale, since the stochastic integral on the right always determines a martingale. More generally, if u is a superharmonic function, so that $\Delta u \leqq 0$, $u[z(t)]$ determines a process which is the sum of a martingale and a process with monotone decreasing sample functions, and is therefore a supermartingale. Hence the stopped $u[z(t)]$ process described in (e2) is a supermartingale, since stopping preserves the supermartingale property. Now suppose more generally that u is specified as in (e2), defined and superharmonic only in a neighborhood of the closure of R_0. Then in a smaller neighborhood u is the limit of a monotone increasing sequence of functions, defined on the whole plane, with continuous second derivatives, superharmonic in a neighborhood of the closure of R_0. The result already obtained, applied to the members of this sequence, yields (e2).

Since the requirement (e2) is an in-the-small requirement, it is clear that our treatment of the plane special case shows that the constructed process in the general case satisfies the set conditions for Brownian motion processes. We have thus shown that every Riemann surface has these processes.

We observe that, if u is a function defined on the direct product of a neighborhood of the closure of R_0 (defined as above) with the interval $[0, t_1]$, and if the $z_1(t)$ process is defined as in (e), then a slight extension of the reasoning just used would prove that $u[t_0 - t, z_1(t)]$ defines a supermartingale, for $t < t_0 < t_1$, excluding $t = 0$ if $u[t_0, z(0)] = \infty$, if u is superparabolic in the sense that

$$(2.7)\qquad \partial u/\partial t \geqq \sigma^2 \Delta u/2.$$

(There is a corresponding generalization if u does not have the appropriate derivatives.) We shall go no farther in this particular direction, nor shall we investigate here how near the general case the Brownian motion processes we have obtained on an arbitrary Riemann surface are, nor what is a proper definition of superparabolic functions in the general case.

3. Conditional Brownian motion

Let h be a strictly positive superharmonic function on a Riemann surface R. Then, in Hunt's terminology, h is excessive in the sense that if p is a Brownian motion transition function on R,

$$(3.1)\qquad h(\xi) \geqq \int_R h(\eta) p(t, \xi, d\eta),$$

and in the limit when $t \to 0$ there is equality. (See Doob [4] for a proof of this well-known fact correct in the present context.) Hence if p^h is defined by

$$p^h(t, \xi, A) = \int_R h(\eta)p(t, \xi, d\eta)/h(\xi), \tag{3.2}$$

p^h is a Markov transition function with state space R, corresponding to "h-path processes". Suppose first that h is harmonic. Then just as in an earlier more special discussion (Doob [4]) the paths of a Markov process with transition function p^h can be identified with those of a conditional Brownian motion process. If h is constant, the process is of course simply a Brownian motion process. If h is not constant, the surface must be hyperbolic, and then almost all paths of the h-path process are continuous, converge to points of the Martin boundary, and the distribution of endpoints is h-harmonic measure on this boundary. The h-harmonic functions corresponding to h-harmonic measure are the quotients u/h, where u is harmonic. If h is minimal, almost all h-paths converge to the point of the Martin boundary corresponding to h.

If h is superharmonic, h-paths are again almost all continuous, but may simply stop in regions in which h is not harmonic. Almost none reach the Martin boundary if h is a potential. In particular, if h is the Green function with pole α, h-paths almost all converge to α.

4. Cluster values

We shall need an analysis of various types of cluster values. If R is a Riemann surface, limit concepts on R involving the Cartan-Brelot-Naïm fine topology will be qualified by "fine". A connection with probability theory that we shall use frequently is the following. Let α be a minimal point of the Martin boundary of R^M, corresponding to the minimal harmonic function h. An analytic subset A of R has fine limit point α if and only if almost every h-path from a point of R meets A arbitrarily near α. In the contrary case $(R - A) \cup \{\alpha\}$ is a fine neighborhood of α. A function f on R has the [fine] cluster value b at α if b is the limit of f on a subset of R with [fine] limit point α (Doob [5]), the fine limit b at α if b is the limit of f on a subset of R which is a fine deleted neighborhood of α (Naïm [1]).

The "fine boundary function" of a function f from the hyperbolic Riemann surface R to a topological space S is the function defined at each point of R^M at which f has a fine limit, with value this limit. In particular, suppose that S is a Riemann surface, and let f be an S-valued analytic function on R. Here if S is a noncompact parabolic surface, we allow f to have the fine limit ∞, and if S is hyperbolic, we allow f to have fine limits in S^M. Let B be the domain of the fine boundary function f' of f, let B_0 be a subset of B of harmonic measure 0, and consider the closure on S (to which ∞ or S^M has been adjoined) of $f'(B - B_0)$. The intersection of all these closures (equal to the intersection of countably many, and therefore attained by a proper choice of B_0) will be called the essential closed range of the fine boundary function. The essential closed range may be empty, because the domain of the fine boundary function may be empty. If we make the same definition except that we replace B by the part of B in an arbitrary neighborhood of a point α of R^M, and take the intersection of the corresponding essential range closures for all these neighborhoods, this intersection (which is again attained for a suitable choice of B_0) will be called the essential cluster set at α of the fine boundary function. If in this discussion the domain of the fine boundary function is replaced by a subset, the corresponding concepts become those relative to that subset.

Let R be a disc. Then ξ is called an angular limit of a function f on R at a boundary point α if f has limit ξ at α in every angle with vertex α opening into R (that is with rays meeting R); ξ is called an angular cluster value of f at α if f has limit ξ on a sequence of points converging to α in some angle as above.

If R is any hyperbolic Riemann surface, the role played when R is a disc by an angular limit [angular cluster value] is played by a fine limit [fine cluster value] at a minimal point of R^M. The latter concepts do not reduce to the former ones if R is a disc, however. Some of the relations between these concepts will now be discussed.

If R is a disc, let f be a meromorphic function on R, and suppose that f is normal in the Lehto-Virtanen [1] sense that the family of transforms of f by the linear transformations taking R onto itself is a normal family. Lehto and Virtanen proved that if f has the limit α on a continuous path to a boundary point of R, then f has angular limit α at the point. Hence f has angular limit α at a boundary point whenever f has fine limit α there. We can go further in the same direction. Let f be meromorphic and normal on the disc R, and suppose that f has angular cluster value ξ at $\alpha, f(\alpha_n) \to \xi$, where the sequence $\{\alpha_n\}$ lies in some angle with vertex α as above. Then a normal family argument (applied to the sequence of functions obtained by mapping R linearly onto itself so that α_n goes into the center of R and α goes into itself) shows that, for every $\varepsilon > 0, f$ is within ε of ξ on discs converging to α and meeting the rays of some angle (depending on ε) with vertex α. Since it is easy to see that the union of such a class of discs has α as fine limit point, we have obtained the following theorem.

THEOREM 4.1. *Let f be a normal meromorphic function on a disc. Then every angular cluster value at a boundary point is also a fine cluster value at the point.*

It is not known whether or not the converse of this theorem is true. There is a global theorem in the same direction, with fewer hypotheses on f, as follows.

THEOREM 4.2. *Let f be a function from the disc R to a metric space, and suppose that at every point α of a perimeter set A there is an angle opening into R*

in which f has a limit $\phi(\alpha)$. Then f has $\phi(\alpha)$ as fine limit at almost every point α of A.

To see this it is sufficient to make the only apparently more restrictive hypothesis that the angles are all obtained by rotations of R from a single angle, and we shall do so. The perimeter set for which there is convergence in the angles is measurable, so we can assume that A is measurable, and even, applying the theorems of Egoroff and Lusin, and decreasing A slightly, that A is closed, that there is uniform convergence to ϕ in the angles, and that the restriction of ϕ to A is continuous. Applying a standard technique used in cluster value theory (see for example Tsuji [2], p. 338) we then find a simply connected domain R_0 in R, bounded by a rectifiable curve, containing all points in a large closed subdisc D concentric with R and all points in the given angles with vertices on A outside D. This is done in such a way that f outside D and in one of the angles is within ε of its limit. Now according to a theorem of Naïm [1] (Theorem 25) R_0 is a deleted neighborhood in the fine topology of almost every (R_0 harmonic measure) point of A, and the classes of subsets of A of harmonic measure 0 for R and R_0 are the same. Moreover in our application $f(\xi)$ is within 2ε of $\phi(\alpha)$ if ξ is in R_0 and is near enough to α in A. Since ε is arbitrary, f must have the fine limit ϕ almost everywhere on A, as was to be proved.

The converse of this theorem is false. In fact if f is defined as 1 on the disc R except at the points of a countable dense set, where f is defined as 0, f has fine limit 1 at every perimeter point but does not have a limit at the boundary in any angle with vertex on the boundary.

Let f be a function from the disc R to a compact metric space, and consider the angular cluster set C_α of f at a boundary point α. Then for almost all α, C_α is the same as the cluster set of f at α in any angle with vertex α, opening into R. In fact the usual proof of the Plessner theorem (see Noshiro [2] or Tsuji [2]) gives this result immediately. The following theorem links angular with fine cluster values.

THEOREM 4.3. *Let f be a function from the disc R to a compact metric space. Then at almost every boundary point of R the fine cluster set of f is a subset of the angular cluster set of f.*

To prove the theorem we simply apply Theorem 4.2 to $u_n(f)$, $n \geqq 1$, where $u_n(\eta)$ is the distance on the range space from η to the complement of the n^{th} set of a countable open neighborhood basis of the range space.

Example 1. Consider the function h on the unit disc R: $\{|z| < 1\}$,

$$h(z) = (1 - |z|^2)/|1 - z|^2. \tag{4.1}$$

This function is minimal harmonic, corresponding to the perimeter point 1. Let $u = -\log[|z - 1|/2]$ on R. Then u is positive and harmonic, and according to a theorem of Naïm [1], u/h has $\inf(u/h)$ as fine limit at 1.

Since u/h has limit 0 on the radius to 1, the fine limit must also be 0 (because the radius has the point 1 as fine limit point). Similarly $1/h$ has fine limit 0 at 1. Now consider the function f on R defined by

$$f(z) = (z-1)^{-1} \exp\left(\frac{z+1}{z-1}\right). \tag{4.2}$$

This function is the quotient of two bounded regular functions. Moreover

$$\log |f| = -2h + u - \log 2. \tag{4.3}$$

Hence $-\log |f|$ has fine limit ∞ at 1, that is, f has fine limit 0 at 1, even though f obviously has ∞ as limit along a tangential (circular) path to 1. Lehto [2] noted that this function also has 0 as an angular limit at 1.

If α is a Martin boundary point of the hyperbolic Riemann surface R, it is natural to call α a regular boundary point if the harmonic measure of the boundary set contained in each neighborhood of α on $R \cup R^M$, at ξ in R, has limit 1 when $\xi \to \alpha$. If the same condition is satisfied even though R is replaced by the part of R in an arbitrary open neighborhood of α, α will be called a locally regular boundary point.

Let A be a subset of R^M, measurable with respect to harmonic measure. Then η will be called a cluster value at α relative to A of a function f on R if there is a sequence $\{\xi_n\}$ on R for which $\xi_n \to \alpha \in R^M$, $f(\xi_n) \to \eta$, $u_n \to 1$, where u_n is the harmonic measure at ξ_n of the part of A in any preassigned neighborhood of α. If A contains almost all (harmonic measure) points of R^M near α, and if α is a locally regular point of R^M, then any cluster value at α is a cluster value relative to A, and in fact in this case R can even be replaced by its part in any open neighborhood of α.

5. Brownian paths on covering surfaces

Let R_1 and R_2 be Riemann surfaces, and let R_1 be a regular covering surface of R_2. If p_2 is a Brownian motion transition function on R_2, it determines one on R_1 as follows. Let ξ_1 be any point of R_1, and let ξ_2 be the point of R_2 under ξ_1. Brownian motion paths from ξ_2 are lifted in the usual way into continuous curves on R_1 with initial point ξ_1. If A_1 is a Borel subset of R_1, let $p_1(\xi_1, t, A_1)$ be the probability of the set of image paths on R_1 which lie in A_1 at time t. It is easily verified that p_1 is then a Brownian motion transition function on R_1. Clearly, if γ is a cover transformation, $p_1(\gamma\xi_1, t, \gamma A_1) = p_1(\xi_1, t, A_1)$. Conversely any Brownian motion transition function p_1 on R_1 which is invariant under the group of cover transformations defines a Brownian motion transition function p_2 on R_2 by

$$p_2(\xi_2, t, A_2) = p_1(\xi_1, t, A_1), \tag{5.1}$$

where A_1 is the class of all points of R_1 over a point of A_2, and ξ_1 is any point over ξ_2.

Example 1. Let R_1 be the conformal universal covering surface of the torus R_2. Then R_1 is the finite plane, and R_2 is obtained from R_1 by identification of congruent points in a rectangular lattice. One example of a Brownian motion process on R_1 is an ordinary plane Brownian motion process (with any variance parameter). A corresponding Brownian motion process on R_2 is defined by (5.1).

Example 2. Let R_1 be the conformal universal covering surface over R_2, the finite complex plane less two points. Take as Brownian motion transition function on R_2 the ordinary plane Brownian motion transition function, with any variance parameter. Note that the lifetime of the paths is ∞, that is, the transition function has maximum value 1. This Brownian motion transition function lifts into one with the same property on R_1. Now R_1 is conformally equivalent to the interior of a disc by way of the elliptic modular function. Hence another possibility for a Brownian motion transition function on R_1 is the image of the ordinary plane Brownian motion transition function, abbreviated to the disc. This transition function has maximum value less than 1, for each initial point, corresponding to finite lifetimes for the Brownian paths.

We have supposed heretofore that R_1 is a regular covering surface of R_2. Now suppose more generally that R_1 is a smooth (no branch points), but not necessarily regular, covering surface of R_2. Then it remains true that a Brownian transition function on R_2 determines one on R_1, just as in the regular case. This correspondence has the following property. If ξ_2 is a point of R_2, ξ_1' and ξ_1'' are points of R_1 over ξ_2, and if A_2 is a sufficiently small open neighborhood of ξ_2, there are open neighborhoods A_1' of ξ_1' and A_1'' of ξ_1'', the local images of A_2, such that the Brownian process from ξ_1', stopped when the paths meet the boundary of A_1', has the same joint distributions of its random variables (under the map from A_1' to A_1'' by way of A_2) as the process from ξ_1'' stopped when its paths meet the boundary of A_1''; in fact the joint distributions are the same as those for the Brownian process from ξ_2 stopped when its paths meet the boundary of A_2 (under the map just referred to). Conversely any Brownian transition function on R_1 with this property yields a Brownian transition function on R_2 in the obvious way—by building up the corresponding process locally.

Finally, if R_1 is a covering surface of R_2, with no further restrictions, its set B_1 of branch points is countable, projecting down into a countable subset B_2 of R_2. Since B_2 has capacity zero, almost no Brownian path from a point of $R_2 - B_2$ ever passes through a point of B_2, so that, just as above, any Brownian motion transition function on R_2 defines one on $R_1 - B_1$.

The treatment we have given of Brownian motion processes on covering surfaces requires few changes to be applicable to conditional Brownian processes. Let h_2 be a strictly positive harmonic function on R_2, and let f be the map taking a point of R_1 into the covered point of R_2. Then an h_2-path process from a point ξ_2 of R_2 generates an $h_2(f)$-path process from any point ξ_1 of R_1 over ξ_2. Let τ_2 be the lifetime of a path γ_2 on R_2, and let τ_1 be the lifetime of the lifted path γ_1. If $\tau_1 < \tau_2$, R_1 cannot be compact, and γ_1

approaches ∞. If R_1 is parabolic, $h_2(f)$ must be a constant function, so h_2 is also a constant function, and the paths considered are Brownian paths for which the probability of going to ∞ is 0. Thus the only interesting case when $\tau_1 < \tau_2$ occurs when R_1 is hyperbolic. Then γ_1 not only approaches the point ∞ but even approaches a specific point of R_1^M (neglecting a set of paths of probability 0) because the path γ_1 is an $h_2(f)$-path, and conditional Brownian paths almost all have this character. If $\tau_1 = \tau_2$, γ_1 is a typical $h_2(f)$-path, and has the properties discussed in Section 3.

Going in the other direction, suppose that R_1 is hyperbolic, and that η_1 is a minimal Martin boundary point of R_1. Define Brownian paths on R_1 from ξ_1 by lifting paths on R_2 with initial point ξ_2. Suppose that h_1 is the minimal harmonic function on R_1 corresponding to η_1. The probability that the projections of h_1-paths from ξ_1 have a prescribed limiting conduct is a bounded h_1-harmonic function of ξ_1, and, since h_1 is minimal, this function must be a constant function. Moreover, by a standard argument (Doob [3]) the constant must be 0 or 1. Hence almost all these paths have the same asymptotic conduct on R_2. If they converge, they must almost all converge to the same point. Now if h_1 is derived from a harmonic function h_2 on R_2, by way of $h_1 = h_2(f)$, and if R_2 is also hyperbolic, the projected paths are h_2-paths, or initial segments of h_2-paths, which we know almost all converge to points of $R_2 \cup R_2^M$, and so almost all converge to the same point, according to the above argument. If h_1 is not so derived, we can obtain an only slightly weaker conclusion as follows. Suppose again that R_2 is hyperbolic, and make 1-paths (that is Brownian paths) correspond on the two Riemann surfaces. As η_1 varies on R_1^M we obtain almost all 1-paths on R_1 from ξ_1, projecting into 1-paths on R_2 from ξ_2 or initial segments of such paths, which we know almost all converge to points of $R_2 \cup R_2^M$. Hence, we conclude: if h_1 is the minimal function corresponding to almost any point of R_1^M, almost every h_1-path from a point ξ_1 of R_1 projects into a path from $f(\xi_1)$, converging to a point of $R_2 \cup R_2^M$ independent of the path.

6. Review of the Fatou and related theorems

Let f be a function defined and meromorphic on a disc, R_1. Then under various conditions on f it is known that f has a finite angular limit at almost every (harmonic measure or equivalently Lebesgue measure) boundary point. The result is true for example, according to Fatou [1] if f is bounded. Moreover F. and M. Riesz [1] proved that if the Fatou boundary function of a bounded f is constant on a set of strictly positive measure, f is identically constant. By combining these results it follows that a function "of bounded type", that is, one which is the quotient of two bounded regular functions, also has a finite angular limit at almost every boundary point. R. Nevanlinna [1] proved that a regular function f on a disc is such a quotient if and only if the supremum of the average of $\log^+ |f|$ over concentric perimeters is finite.

According to a theorem of Plessner [1] if f is a function defined and meromorphic on a disc, then at almost every point of the boundary either f has an angular limit, or the cluster set of f in any angle with vertex at the point is the extended plane.

If R_2 is a Riemann surface, and if f is an analytic function from the disc R_1 to R_2, the natural generalization of the condition that f be of bounded type is that f be "Lindelöfian". The definition of this concept will be recalled below. Under this hypothesis Heins [3] proved that f has an angular limit, either a point of R_2 or ∞, at almost every boundary point of R_1. Note that the limit ∞ may occur at every perimeter point. In fact this is true for $R_1 = R_2$, both being the unit disc with the origin as center, and $f(z) = z$. This example illustrates the obvious fact that compactification of R_2 by adjoining only a single point at infinity cannot be expected to yield fine structure results involving ∞.

Let f be a superharmonic function, defined and bounded below on a hyperbolic Riemann surface R. Then it has been proved (Doob [4]) that f has a finite fine limit at almost every (harmonic measure) point of R^M. In particular, the result is of course also true for f a bounded analytic function and is easily generalized to larger classes of regular or even meromorphic functions.

If an analytic function on a disc has a boundary function which is nearly constant on a large set, the analytic function is a constant function. We have already noted the Riesz theorem that the Fatou boundary function of a bounded nonconstant regular function on a disc is not constant on a boundary set of strictly positive measure. A very general theorem in this direction is due to Tsuji [1]. He proved that if f is a function defined and meromorphic on a disc, and if there is a set A of capacity zero such that the angular cluster set at each point of a perimeter set of strictly positive measure lies in A, then f is a constant function.

We now proceed to generalize the theorems stated in this section, together with related theorems, to the case of analytic functions from one Riemann surface to a second.

7. Generalizations of the Fatou and related theorems

Our analysis of the relation between probability paths on covering surfaces and on the covered surfaces leads quite trivially to generalizations of the theorems of Section 6. Let R_1 and R_2 be arbitrary Riemann surfaces, and let f be an R_2-valued analytic function on R_1. The surface R_1 thereby becomes a covering surface of R_2: ξ_1 covers ξ_2 if $f(\xi_1) = \xi_2$. The results we have obtained concerning Brownian paths, which we shall need in this section, can be outlined loosely as follows.

(a) Our path systems map into path systems under the transformation determined by an analytic function.

(b) Our paths on a hyperbolic surface converge to individual boundary points.

(c) Our 1-paths on a parabolic surface are everywhere dense, even if initial segments of the paths are deleted.

(d) Our paths miss sets of zero capacity, but have strictly positive probability of meeting sets of strictly positive capacity.

We first apply these facts when R_1 and R_2 are hyperbolic. In this case the Brownian paths on R_1 can be derived from Brownian paths on R_2 (which converge to points of R_2^M) or to pieces of such paths, which converge to the piece endpoints. In other words we have the following Theorem 7.1. (Strictly speaking we have ignored the possible existence of branch points, but these can in fact be ignored because they form a set of capacity zero.) Theorems like the following can be stated in terms of the properties of a function on Brownian paths (p), on conditional Brownian paths (cp), or in terms of fine limit concepts (f), but only our first theorem will be stated in all three ways, to exhibit the principle involved. Note that the results are independent of the particular Brownian motion processes used, as shown for example by the (f) version.

THEOREM 7.1p. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the hyperbolic Riemann surface R_2. Then f has a limit (on $R_2 \cup R_2^M$) on almost every Brownian path from a point of R_1 to R_1^M.*

In terms of conditional paths this theorem can be phrased as follows (see the corresponding discussion by the author in [4]). We recall that almost every (harmonic measure) point of R_1^M is minimal.

THEOREM 7.1cp. *If f is as in Theorem 7.1p, let η_1 be almost any (harmonic measure) minimal point of R_1^M, corresponding to the minimal harmonic function h_1. Then f has a limit (on $R_2 \cup R_2^M$) on almost every h_1-path from a point of R_1 to η_1. The limit is independent of the initial point of R_1 and of the path.*

Finally, we rephrase the result in terms of the fine topology. The rephrasing incidentally shows that the results are not dependent on how the Brownian processes on R_1 and R_2 discussed in Section 5 are related. We also add a further result, (b) below, which could have been stated in Theorem 7.1cp, and which is in fact trivial although important, but which was omitted to avoid superfluous repetition. The relations between the fine topology and conditional probability paths are discussed in Doob [4].

THEOREM 7.1f. *Let f be as in Theorem 7.1p. If η_1 is almost any (harmonic measure) point of R_1^M, then*

(a) *f has a fine limit at η_1, say η_2 (in $R_2 \cup R_2^M$), and*

(b) *if A_1 is an analytic subset of R_1 for which $A_1 \cup \{\eta_1\}$ is a fine neighborhood of η_1, $f(A_1) \cup \{\eta_2\}$ is a fine neighborhood of η_2.*

Part (b) is essential to the understanding of why, if R_1, R_2, and R_3 are hyperbolic Riemann surfaces, if f_1 is analytic from R_1 into R_2, and if f_2 is analytic from R_2 into R_3, then $f_2(f_1)$ has a fine limit at almost every point of R_1^M. In view of (b) this limit assertion can be explained by following the successive maps as well as by simply applying the theorem to the combined map $f_2(f_1)$. (Theorem 7.3 is also needed to make the reasoning complete.)

We now apply the general principles stated at the beginning of this section to an analytic function from a hyperbolic Riemann surface R_1 to a parabolic Riemann surface R_2. We determine the Brownian paths on R_1 by lifting those on R_2, as usual, but this coupling of path systems on the two surfaces is irrelevant to the final result, whose validity is independent of the Brownian processes used to obtain it. Under our hypotheses on the Riemann surfaces, the images of Brownian paths on R_1 are either full Brownian paths on R_2 (which are everywhere dense and even have everywhere dense ends) or are initial pieces of these paths, which of course converge to their own endpoints. Thus we have proved the following theorem, corresponding to Plessner's (see Section 6) in the classical case. (If R_2 is hyperbolic, the Plessner theorem becomes even simpler, in view of Theorem 7.1.)

THEOREM 7.2p. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the parabolic Riemann surface R_2. Then for almost every Brownian path from a point of R_1 to R_1^M, either* (a) *f has a limit (a point of R_2) on the path, or* (b) *f has R_2 as a cluster set on the path. If Case* (b) *occurs with strictly positive probability, $R_2 - f(R_1)$ has capacity zero.*

The assertion (b) means that the set of limiting values of f on the path, as the parameter approaches the path lifetime, is R_2. The last assertion of the theorem follows from the fact that if $R_2 - f(R_1)$ has strictly positive capacity, $f(R_1)$ is a hyperbolic Riemann surface, so that, according to Theorem 7.1p, (b) cannot occur with strictly positive probability.

We omit the conditional-path version of Theorem 7.2, but state the fine-topology version.

THEOREM 7.2f. *If R_1, R_2, f are as in Theorem* 7.2p, *then at almost every (harmonic measure) point of R_1^M, either* (a) *f has a fine limit (a point of R_2), or* (b) *f has every point of R_2 as a fine cluster value. If* (b) *occurs on a subset of R_1^M of strictly positive harmonic measure, $R_2 - f(R_1)$ has capacity zero.*

Before leaving this analogue of Plessner's theorem, we remark that the last assertion can be strengthened to give local results. In fact if (b) occurs on a subset A_1 of R_1^M, of strictly positive harmonic measure, we can replace R_1 by an open subset which includes a neighborhood of some point of R_1^M at which A_1 is metrically dense. If we then apply the theorem to the restriction of f to this neighborhood, we find that there is a subset B_1 of A_1 of harmonic measure zero, and a subset C_2 of R_2 of capacity zero such that $R_2 - f(R_1 \cap N) \subset C_2$ for every neighborhood N of any point of $A_1 - B_1$.

If f is a meromorphic function on a disc in Theorem 7.2, we can combine this theorem with Plessner's classical one and obtain, using Theorem 4.2, the following result. *If f is a meromorphic function on a disc R, then at almost every* (*Lebesgue measure*) *point of the perimeter of R either* (a) *f has both an angular limit and an equal fine limit, or* (b) *f has every point of the plane as cluster value in every angle opening into R with vertex at the point, and f has a fine limit at the point, or* (c) *f has every point of the plane both as cluster value in every angle opening into R with vertex at the point and as fine cluster value.*

It would be interesting to find an example to show that Case (b) can really occur on a perimeter set of strictly positive measure, or a proof that it cannot. It follows readily from theorems of Meier [1] that for every point ξ of the extended plane f assumes the value ξ infinitely often in every angle with vertex α on the perimeter, opening into R, for almost every α for which (b) is true. Thus, for example, (b) can be excluded if the function is regular.

We now apply the general principles stated at the beginning of this section to a function f from a hyperbolic Riemann surface R_1 to a Riemann surface R_2 using this time the fact that (roughly) Brownian paths on R_2 neither meet a preassigned set of zero capacity (excluding the path initial point) nor, if R_2 is hyperbolic, converge to a preassigned subset of R_2^M of harmonic measure zero. If R_2 is parabolic, the paths do not approach ∞. Then the Brownian paths on R_1 leading to paths on R_2 with forbidden conduct must have probability zero, and we thus obtain the following theorem, a generalization of Tsuji's theorem stated in Section 6. (From now on we shall state only the fine-topology version of a theorem, and shall omit the qualifying "f" from its number.)

THEOREM 7.3. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the Riemann surface R_2. Let A_1 be a subset of R_1^M of strictly positive outer harmonic measure. Let A_2 be a set with the following properties. If R_2 is compact, A_2 is a subset of R_2 of capacity zero. If R_2 is parabolic but not compact, A_2 is a subset of $R_2 \cup \{\infty\}$, with $A_2 \cap R_2$ having capacity zero. If R_2 is hyperbolic, A_2 is a subset of $R_2 \cup R_2^M$ for which $A_2 \cap R_2$ has capacity zero and $A_2 \cap R_2^M$ has harmonic measure zero. Then if all the fine cluster values of f at the points of A_1 lie in A_2, f is identically constant.*

Note that, in view of Theorem 7.2, the hypothesis of the present theorem is not really more general than that f has a fine limit at every point of A_1, which limit is a point of A_2.

We have not applied the general principles stated at the beginning of this section to analytic functions from one parabolic Riemann surface R_1 to a second parabolic surface R_2. In this case the full Brownian paths on the two surfaces can be made to correspond to each other. One way of saying this is simply that if ξ_2 is a point of R_2, the image of ξ_1 on R_1, and if the latter is not a branch point of R_1 considered as a covering surface, then the corresponding branch of the inverse of f at ξ_2 can be continued analytically along almost every Brownian path from ξ_2. This result is reminiscent of the Gross star theorem on the continuation of the inverse of a function meromorphic on the finite plane along rays to ∞.

The theorems of this section have all had rather trivial qualitative proofs, once our foundations were laid. The generalization of the theorem that a function of bounded type on a disc has an angular limit at almost every perimeter point (see Section 6) needs more quantitative considerations, which we now give.

Let R_1 and R_2 be hyperbolic Riemann surfaces, and let f be an analytic function from R_1 to R_2. Then, if g_i is the Green function of R_i,

$$g_2[f(\xi_1), \xi_2] = \textstyle\sum_{f(\eta_1)=\xi_2} g_1(\xi_1, \eta_1) + u_{\xi_2}(\xi_1), \qquad \xi_i \in R_i, f(\xi_1) \neq \xi_2, \tag{7.1}$$

where in the sum each term is counted as many times as f has the value ξ_2 at η_1, and u_{ξ_2} is a positive harmonic function on R_1. Note that the sum on the right makes sense even if R_2 is not hyperbolic. Heins [2] calls a function f analytic from a hyperbolic Riemann surface R_1 with Green function g_1 to a Riemann surface R_2 "Lindelöfian" if the sum on the right converges whenever $f(\xi_1) \neq \xi_2$. Then f is necessarily Lindelöfian if R_2 is hyperbolic. If R_1 is a disc and if R_2 is the extended plane, so that f is a meromorphic function on the disc, f is Lindelöfian if and only if it is of bounded type. For this and other reasons the class of Lindelöfian functions seems to be a natural generalization of the class of functions of bounded type.

We shall need the following lemma, due to Heins ([2], p. 433), which we restate in probability language.

LEMMA 7.4. *Let f be a Lindelöfian analytic function from the hyperbolic Riemann surface R_1 to the parabolic Riemann surface R_2. Then if $\xi_1 \in R_1$, the infimum over compact proper subsets C_2 of R_2 of the probability that a Brownian path on R_1 from ξ_1 meets $f^{-1}(R_2 - C_2)$ arbitrarily near ∞ is 0.*

This lemma has the following trivial corollary.

COROLLARY. *If R_1, R_2, f are as in the lemma, the supremum over compact proper subsets C_2 of R_2 of the probability that a Brownian path starting from a point ξ_1 of R_1 will lie entirely in $f^{-1}(C_2)$ is one.*

We shall now prove our desired generalization of Theorem 7.1, which we state only in its fine-topology form.

THEOREM 7.5. *Let f be a Lindelöfian analytic function from the hyperbolic Riemann surface R_1 to the Riemann surface R_2. Then if η_1 is almost any (harmonic measure) point of R_1^M,*

(a) *f has a fine limit at η_1, say η_2 (in R_2 if R_2 is parabolic, in $R_1 \cup R_2^M$ if R_2 is hyperbolic), and*

(b) *if A_1 is an analytic subset of R_1 for which $A_1 \cup \{\eta_1\}$ is a fine neighborhood of η_1, $f(A_1) \cup \{\eta_2\}$ is a fine neighborhood of η_2.*

The proof is now nearly trivial. We can suppose that R_2 is parabolic since otherwise Theorem 7.1f would be applicable. If C_2 is an open subset of R_2, with compact closure $\bar{C}_2 \neq R_2$, C_2 is a hyperbolic Riemann surface, aside from possible disconnectedness, which does not affect the argument. Let ξ_1 be any point of R_1 which is not a branch point of the surface as a covering surface of R_2. We apply Theorem 7.1p, considering paths from ξ_1, and the restriction of f to the open component C_1 of $f^{-1}(C_2)$ containing ξ_1. We take as the Brownian paths on C_1 those on R_1 cut off when they first reach $R_1 - C_1$, if ever. Then f has a limit on almost all Brownian paths in C_1 from ξ_1, which means, in view of the Corollary, that f has a limit on almost every Brownian path in R_1 from ξ_1. If R_2 is noncompact (and parabolic), the limit is almost never ∞ by Theorem 7.3. Thus part (a) of the present theorem is true in its form as a theorem on limits along Brownian paths, and therefore in the fine-limit form in which it is actually stated. The overall picture is now like that in Theorem 7.1, so (b) is also true.

We observe that this theorem answers a natural question about Theorem 7.1. In that theorem, suppose that R_2 is immersed in a larger Riemann surface R_3. Then f can be considered a function from R_1 to R_3. Will this function still have a fine limit at almost every (harmonic measure) point of R_1^M ? If R_3 is hyperbolic, Theorem 7.1f itself gives an affirmative answer; if R_3 is parabolic, Theorem 7.5 gives an affirmative answer, since f is Lindelöfian. (A direct proof can also be given by analyzing the relation between Martin boundary points of R_2 and ordinary boundary points of R_2 relative to R_3.)

8. Functions of type Bl

Let f be a function defined and regular on the unit disc $|z| < 1$. Suppose that $|f| \leqq 1$ and that the modulus of the Fatou boundary function is 1 almost everywhere on the perimeter. Seidel [1] called the class of such functions the class **U**, and this class has been studied extensively by him and other authors. Storvick [1] generalized the class **U** into the class **L** defined as follows. A function f, meromorphic on the unit disc, is in the class **L** if its range R has complement of strictly positive capacity (which implies that f is of bounded type) and if the Fatou boundary function of f, almost everywhere on $|z| = 1$, has its values on the boundary of R.

Now let R_1 and R_2 be hyperbolic Riemann surfaces, and let f be an analytic function from R_1 to R_2. Then (7.1) holds, and Heins [1] calls the class of functions f for which u_{ξ_2} (for all ξ_2) dominates no bounded strictly positive harmonic function the class **Bl**. This is his generalization of the classes **U** and **L**, and it is clear from his work that the class **Bl** is indeed an appropriate generalization of these classes. We shall now show that the class **Bl** can be characterized by the exact analogue of Seidel's characterization of the class **U** and Storvick's of **L**. To see this we note first that a positive harmonic function u on R_1 which dominates no strictly positive bounded harmonic function, is in our language simply one which has fine boundary function vanishing almost everywhere on R_1^M. (In fact if there is a bounded strictly positive harmonic function h dominated by u, the fine boundary function of h does not vanish almost everywhere on R_1^M and is dominated by the fine boundary function of u. Conversely if u has fine boundary function ϕ not vanishing almost everywhere, $\phi_a = \min [\phi, a]$ is bounded and does not vanish almost everywhere, for sufficiently large a, whereas ϕ_a is the fine boundary function of its corresponding Dirichlet solution, which is a strictly positive bounded harmonic function dominated by u.) We now apply this remark to the definition of the class **Bl**.

The sum in (7.1), as a function of ξ_1, is a potential, and so has fine boundary function zero almost everywhere on R_1^M, according to a theorem of Naïm [1] (or see Doob [4]). Thus if ξ_2 is fixed, the function of ξ_1 on the left, and the function u_{ξ_2} on the right, have the same fine boundary function almost everywhere on R_1^M. It follows that f is of type **Bl** if and only if $g_2(f, \xi_2)$ has fine boundary function zero almost everywhere on R_1^M. If $g_2(f, \xi_2)$ has this fine boundary function, it is clear that f must have its fine boundary function almost everywhere confined to R_2^M. Conversely, if f has this property, and in view (i) of the fact that $g_2(\cdot, \xi_2)$ has fine boundary function zero almost everywhere on R_2^M and (ii) of Theorems 7.1f and 7.3, it follows that f is of type **Bl**. We have thus proved the following theorem.

THEOREM 8.1. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the hyperbolic Riemann surface R_2. Then f is of type* **Bl** *if and only if the fine boundary function of f on R_1^M has as value a point of R_2^M at almost every (harmonic measure) point of R_1^M.*

The criterion of Theorem 8.1 suggests that one can generalize the **Bl** definition as follows: Let f be an analytic function from the hyperbolic Riemann surface R_1 to the Riemann surface R_2, and suppose that f has the following properties. Let A_1 be the subset of R_1^M at each point of which f has a fine limit. The limit is allowed to be ∞ if R_2 is parabolic and not compact, or a point of R_2^M if R_2 is hyperbolic. It is supposed that A_1 has strictly positive harmonic measure and that the fine limit at a point of A_1 lies almost always outside $f(R_1)$. We now show that this situation is not essentially more general than the **Bl** one. In discussing this situation we may as well assume that $R_2 = f(R_1)$, and we shall do so. Then R_2 cannot be compact. If R_2 is parabolic, f must have fine limit ∞ almost everywhere on A_1, impossible by Theorem 7.3. Hence R_2 is parabolic, and A_1 includes almost all points of R_1^M, by Theorem 7.1f. We are thus back in the conditions of the Heins **Bl** definition.

Heins proved that if f is of type **Bl** from R_1 to R_2, if A_2 is an open connected subset of R_2, and if A_1 is a component of the inverse image of A_2 under f, then the restriction of f to A_1 is of type **Bl**, it being understood that R_2 is now replaced by A_2. This fact has a simple intuitive proof with the present background. In fact let Brownian paths on A_1 be taken as the inverse images of those on R_2 from a point of A_2, up to the time the latter first leave A_2, if ever. (Branch points and their images are to be avoided as initial points.) Now the **Bl** hypothesis on f is that the paths of a Brownian process on R_2 almost all go to R_2^M as their images on R_1 go to R_1^M. It is then clear that the corresponding property holds for the restricted f, as was to be proved.

If f is of type **Bl** from R_1 to R_2, $R_2 - f(R_1)$ must have capacity zero. In fact otherwise Brownian paths from a point of $f(R_2)$ would have strictly positive probability of hitting $R_2 - f(R_1)$, whereas they actually almost all go to R_2^M as their images on R_1 go to R_1^M.

The following theorem is the analogue in the present case of theorems of Seidel [1] and Storvick [1]. Note that, in their cases, the domain R_1 is a disc, and a fine limit at the boundary is necessarily also an angular limit (see Section 4) because the functions in the classes **U** and **L** are normal. See also Noshiro's book [2] for related theorems.

THEOREM 8.2. *If f is of type* **Bl** *from R_1 to R_2, and if α_2 is either a point of $R_2 - f(R_1)$ or a minimal point of R_2^M, then f has α_2 as a fine limit at some point of R_1^M (and therefore as limit along some continuous path to the point).*

At this level of generality Heins [1] proved that if $\alpha_2 \in R_2 - f(R_1)$, then α_2 is a limit along a path to ∞ on R_1.

In proving Theorem 8.2 it is sufficient to consider only the case when $\alpha_2 \in R_2^M$ since the other case can be reduced to that one by replacing R_2 by $f(R_1)$. Suppose then that α_2 is a minimal point of R_2^M, corresponding to the minimal harmonic function h_2 on R_2. The h_2-paths from a point of R_2 (we exclude an image of a branch point as initial path point) correspond to $h_2(f)$-paths, or initial pieces of these paths, from a point of R_1, and because of the **Bl** property the full paths correspond to each other. But then f has the limit α_2 on almost every $h_2(f)$-path, that is, f has the fine limit α_2 at a set of points of R_1^M of $h_2(f)$-harmonic measure 1.

We now consider the local approach to the **Bl** property. Let R_1 and R_2 be any two Riemann surfaces, and let f be an analytic function from R_1 to R_2. Let A_2 be any open connected set on R_2 which, considered as a Riemann surface, is hyperbolic, and let A_1 be one of the open components of $f^{-1}(A_2)$. Then A_1 is also hyperbolic, and, following Heins, we say that f has the **Bl** property at a point if the point is in the closure of $f(R_1)$ and if it is a point of some A_2 for which the restriction of f to A_1, for each choice of the component A_1, has the **Bl** property as a function from A_1 to A_2. Note that then A_2 can contain only a set of capacity zero outside $f(R_1)$, or even outside $f(A_1)$. If f has the property **Bl** at every point of R_2, f is said to be locally of type **Bl**.

THEOREM 8.3. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the Riemann surface R_2. Then f is of type* **Bl** *at ξ_2 on R_2 if and only if ξ_2 is in the closure of $f(R_1)$ but is not in the essential closed range of the fine boundary function of f.*

(See Section 4 for the definition of the essential closed range of the fine boundary function, which may be empty.) Heins [2] proved a corresponding result, generalizing a theorem of Lehto [1], under the stronger hypothesis that f is Lindelöfian, and using the angular instead of the fine boundary function because he supposed that R_1 was a disc. Under his hypotheses the fine boundary function is defined at almost every point of R_1^M (the ordinary disc boundary) according to Theorem 7.1f. The latter theorem involved the Martin compactification of R_2^M if R_2 is hyperbolic. Hence all the more, if R_2 is compactified in the noncompact case merely by the adjunction of ∞, f has a fine boundary function defined almost everywhere on R_1^M. Heins [3] proved that there is also an angular limit at almost every point of R_1^M, using the same conpactification of R_2. Since these two types of boundary functions obviously agree everywhere where both are defined, Theorem 8.3 reduces to Heins's result in his special case.

To prove Theorem 8.3 let ξ_2 be a point of the closure of $f(R_1)$, and let A_2 be an open neighborhood of ξ_2. Suppose that f has a fine limit which is a point of A_2 at each point of a subset of R_1^M of strictly positive outer harmonic measure. Then there must be an initial distribution of Brownian paths on R_2, giving a set of Brownian paths on R_2 not of zero probability, corresponding to Brownian paths on R_1 starting from an open component A_1 of $f^{-1}(A_2)$ and going to R_1^M while the paths on R_2 go to points of A_2. But then f cannot have the property **Bl** from A_1 to A_2. It follows that if f has the **Bl** property at ξ_2, this point is not in the essential range C_2 of the fine boundary function. By definition of the **Bl** property, ξ_2 must be a point of the closure of $f(R_1)$ if f is to have the property **Bl** at ξ_2. Conversely suppose that ξ_2 is in the closure of $f(R_1)$ but is not in C_2. Let A_2 be any open neighborhood of ξ_2 whose closure is compact and contains no point of C_2. Then tracing Brownian paths from a point of A_2 in $f(R_1)$, excluding images of branch points as initial points, corresponds to tracing image paths from a point in an open component A_1 of $f^{-1}(A_2)$. In order to prove that f has the property **Bl** at ξ_2 we show that the probability is zero that some of the latter paths reach R_1^M while the former paths go to points of A_2 (without having left this set). But this is just the case when f has a limit, in A_2, on each of a set of paths to R_1^M not having zero probability, and this contradicts the fact that C_2 contains no points of the closure of A_2. The proof of the theorem is complete.

The following theorem is a trivial consequence of the preceding one, and we therefore can omit the proof. Part (a) is due to Heins [1].

THEOREM 8.4. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the Riemann surface R_2.*

(a) *If R_2 is hyperbolic, f is locally of type* **Bl** *if and only if f is of type* **Bl.**

(b) *If R_2 is parabolic, f is locally of type* **Bl** *if and only if f has a fine limit at almost no point of R_1^M.*

Note that in (b) if f is of type **Bl**, $R_2 - f(R_1)$ must have zero capacity or

$f(R_1)$ would be hyperbolic, f would be Lindelöfian, and f would therefore have a fine limit at almost every point of R_1^M.

In proving the next theorem we shall need the fact that if f is a function from a hyperbolic Riemann surface R to a separable metric space, the fine boundary function is defined on a Borel subset of R^M and is a Borel measurable function. To see this we use the fact proved by Naïm [1] that the set of those points in R^M which are not limit points in the fine topology of a specified subset of R is an F_σ set. Let $\{\eta_n, n \geqq 1\}$ be dense in the range space, and let B_{jn} be the open ball with center η_n and radius $1/j$. Let A_{jn} be R^M less the set of points of R^M which are limit points of $R - f^{-1}(B_{jn})$ in the fine topology. Then A_{jn} is an F_σ set, and the domain of the fine boundary function is $L = \bigcap_j \bigcup_n A_{jn}$. The proof of measurability of the fine boundary function is routine.

We now show how the principles outlined at the beginning of Section 7 lead in a trivial way to a generalization of a theorem going back to Löwner in 1923 and since generalized by so many authors that we only refer to the books of Tsuji ([2], p. 322) and Noshiro ([2], p. 34) for further references. The general idea is that a conformal map increases the harmonic measure of boundary sets. In the following, $\mu(\xi, A, R)$ is the harmonic measure of the subset A of R^M relative to the point ξ on the hyperbolic Riemann surface R.

THEOREM 8.5. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the hyperbolic Riemann surface R_2. Let A_2 be a Borel subset of R_2^M, and let A_1 be the inverse image of A_2 on R_1^M under the fine boundary function of f. Then*

$$\mu(f(\xi_1), A_2, R_2) \geqq \mu(\xi_1, A_1, R_1), \qquad \xi_1 \in R_1. \tag{8.1}$$

There is equality at one point of R_1 (equivalently at every point of R_1) if and only if f is of type **Bl**, *whenever A_2 is chosen to make the left side of* (8.1) *strictly positive.*

Heins [1] (p. 468) has already given the necessary and sufficient conditions for equality stated here. Since both sides of (8.1) define harmonic functions of ξ_1, equality at a single point implies equality everywhere. We can assume in proving (8.1) that ξ_1 is not a branch point of R_1, regarded as a covering surface of R_2. Then Brownian paths from $\xi_2 = f(\xi_1)$ correspond to image Brownian paths from ξ_1 as usual. The left side of (8.1) is the probability of the set of paths from ξ_2 to A_2. Now neglecting a set of paths of zero probability, tracing a path from ξ_2 to a point of A_2 corresponds either to tracing a path from ξ_1 to a point of A_1 or tracing a path from ξ_1 to a point of $R_1^M - A_1$ which is reached before the path on R_2 reaches R_2^M. Since the first possibility accounts for almost all paths to A_1, the stated inequality is true, and there is equality if and only if f has a fine limit which is a point of R_2 at almost no point of R_1^M, that is, if and only if f is of type **Bl**, whenever A_2 is chosen to make the left side of (8.1) strictly positive.

We observe that it would be possible, but would not really increase the generality, to rephrase the theorem in terms of inner and outer harmonic measures, allowing A_2 to be an arbitrary subset of R_2^M.

9. Applications to the ranges of analytic functions

The first theorem we prove in this section generalizes a theorem of Lehto [1] who considered meromorphic functions of bounded type on a disc. Note that in our theorem the results are not much stronger under the hypothesis that the functions are Lindelöfian, corresponding to Lehto's hypothesis, than without this hypothesis. Lehto obtains a slightly larger class of points as asymptotic limits however.

THEOREM 9.1. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the Riemann surface R_2, and let A_2 be the essential closed range of the fine boundary function of f (see Section 4). Let B_2 be an open component of $R_2 - A_2$ containing a point of $f(R_1)$. Then $f(R_1)$ includes all of B_2 except for a set of capacity zero; each exceptional value is a fine limit of f at some point of R_1^M if f is Lindelöfian, a limit of f along a continuous path to ∞ if f is not Lindelöfian.*

Let C_2 be an open connected subset of B_2, chosen so that (a) C_2 contains a point of $f(R_1)$, (b) the closure of C_2 is compact, does not meet A_2, and is a proper subset of R_2. Then almost all Brownian paths from a point of $C_2 \cap f(R_1)$ to C_2^M have images on R_1 which are Brownian paths to the Martin boundaries of the open components of $f^{-1}(C_2)$. That is, the restriction of f to any one of these components, say C_1, as a map from C_1 to C_2 has the property **Bl**. Hence (see Section 8) $f(C_1)$ includes all of C_2 except possibly a set of zero capacity, and so $f(R_1)$ includes all of B_2 except possibly a set of zero capacity. If α_2 is a point of C_2 not in $f(C_1)$, Theorem 8.2 implies that α_2 is the limit of f along a path to some point of C_1^M, which means a path to ∞ on R_1. If R_2 is parabolic, α_2 need not be a fine limit at a point of R_1^M. (Take R_1 to be the disc $|z| < 1$, R_2 the extended plane, $f(z) = \exp(1/z)$, $\alpha_2 = 0$.) If R_2 is hyperbolic, however, let α_2 satisfy the conditions

$$\alpha_2 \notin f(R_2), \qquad \alpha_2 \in C_2 - f(C_1), \qquad f(C_1) \cap C_2 \neq \emptyset.$$

Then α_2 is a minimal boundary point of $R_2 - \{\alpha_2\}$ corresponding to the minimal harmonic function h, the Green function of R_2 with pole α_2. Almost all h-paths on R_2 from a point other than α_2 go to α_2. The probability that an h-path from a point of $C_2 \cap f(C_1)$ meets α_2 before reaching A_2 is strictly positive. But then the probability that $h(f)$-paths on R_1 from a point of C_1 meet a point of R_1^M at which the fine boundary function of f has value α_2 must also be strictly positive. Thus f has fine limit α_2 at at least one point of R_1^M. Finally, if R_2 is parabolic but if f is Lindelöfian, f must have α_2 as a fine limit at some point of R_1^M because this case can be reduced to the hyperbolic case just as in the proof of Theorem 7.5.

The first theorem of the following type, drawing conclusions about the range of a meromorphic function from the fact that its cluster set at a boundary point α contains a point not in the cluster set of the boundary function at α, seems to be the theorem of Iversen [1] and Gross [1]. For references to the enormous amount of later work in the same direction see Noshiro [2].

THEOREM 9.2. *Let f be an analytic function from the hyperbolic Riemann surface R_1 to the Riemann surface R_2. Let α_1 be a point of R_1^M, let A_1 be a Borel subset of R_1^M, and let A_2 be the essential fine boundary cluster set of f on A_1 at α_1. Suppose that α_2 is a point of R_2 which is a cluster value of f at α_1 relative to A_1 (see Section 4). Suppose that $\alpha_2 \notin A_2$, and let B_2 be the open component of $R_2 - A_2$ containing α_2. Then $B_2 - B_2 \cap f(R_1)$ has zero capacity, and every point in this set is the limit of f on a continuous path to ∞.*

Our hypothesis implies that, deleting from A_1 a set of harmonic measure zero and all points outside some neighborhood of α_1, there is a sequence $\{\xi_n\}$ in R_1 with the following properties: the harmonic measure of A_1 at ξ_n goes to 1 when $n \to \infty$; $\xi_n \to \alpha_1$; $f(\xi_n) \to \alpha_2$; the closure of the set of fine limits (if any exist) at the points of A_1 is a set $\bar{A}_2$ not containing α_2. We can suppose, changing ξ_n slightly if necessary, that ξ_n is not a branch point of R_1 considered as a covering surface of R_2. Let D_2 be an arbitrary compact subset of B_2 of strictly positive capacity. We suppose from now on that A_1 has been shrunk enough so that D_2 is in an open component C_2 of $R_2 - \bar{A}_2$, in fact in the same one as α_2. We also suppose below that n is so large that $f(\xi_n)$ is in this same component. To prove the first assertion of the theorem it is sufficient to prove that $f(R_1) \cap D_2$ is not empty. Suppose that this intersection is empty, and consider the probability that a Brownian path on R_2 from $f(\xi_n)$ meets D_2 before $\bar{A}_2$. This probability is strictly positive and has a strictly positive infimum as n varies. Now (neglecting path sets of zero probability) a Brownian path from $f(\xi_n)$ to D_2 which does not meet $\bar{A}_2$ corresponds to a Brownian path from ξ_n to $R_1^M - A_1$. By hypothesis the probability of the set of paths of the latter type, which is at most the harmonic measure of $R_1^M - A_1$, goes to 0 with $1/n$, contradicting our description of path probabilities on R_2. We have now proved that $B_2 - B_2 \cap f(R_1)$ has zero capacity. A standard examination of the inverse images of a decreasing sequence of neighborhoods of any point of this difference, shrinking to the point, shows that the point is the limit of f on some path to ∞, and the proof of the theorem is now complete. The theorem is easily strengthened by adding hypotheses to ensure that R_1 can be replaced by its intersection with any open neighborhood of α_1.

10. Application to the range of an analytic function on a disc

In this section we shall consider only functions on the unit disc $R: |z| < 1$. If f is a regular function on this disc, with $|f'(0)| \geqq 1$, Bloch [1] obtained his classical result that $f(R)$ contains a disc of radius independent of f, univalently covered by f. A related result (Doob [1]) whose exact relationship with Bloch's theorem is not yet clear is the following, which we state formally, for ease of reference.

THEOREM 10.1. *Let f be a regular function on the disc R, and let A be an open arc of the perimeter. Suppose that $f(0) = 0$ and that*

$$\liminf_{\xi \to \alpha} |f(\xi)| \geqq 1 \tag{10.1}$$

if $\alpha \in A$. Then there is a number $\delta_1(|A|)$ depending on the length of A, $|A|$, but not on f, such that $f(R)$ includes a disc in R of radius $\delta_1(|A|)$.

In this section we shall extend Theorem 10.1 by relaxing the conditions on A and on the character of f near A. We first remark that Theorem 10.1 remains valid for meromorphic functions, even with a loosening of the restriction at A, according to the following theorem.

THEOREM 10.2. *Let f be a meromorphic function on the disc R, and let A be an open boundary arc. Suppose that $f(0) = 0$ and that (10.1) is true at all points α of A less a subset of zero capacity. Then there is a number $\delta_2(|A|)$, depending on $|A|$ but not on f, such that $f(R)$ includes a disc in R of radius $\delta_2(|A|)$.*

We first note that Theorem 10.1 is still true, with perhaps a smaller δ_1, if f is allowed to be meromorphic, according to the following argument. There is nothing to prove if $f(R) \supset R$, whereas if b is an excluded point of R, Theorem 10.1 can be applied to the function g,

$$g = (1 + |b|)f/(f - b), \tag{10.2}$$

to obtain the desired result. Furthermore this extension of Theorem 10.1 yields Theorem 10.2 unless the function f in question has a cluster value of modulus < 1 at some point α of A. But the theorem is surely true for such a function, because if this cluster value has modulus $c < 1$, and if A_1 is the closed subset of A (allowing endpoints of A if necessary) of zero capacity at each point of which $|f|$ has inferior limit $\leqq c'$, $c < c' < 1$, f has all its boundary limit values at α along $A - A_1$ of modulus $\geqq c'$, so according to a cluster value theorem of Noshiro [1], $f(R)$ includes every point of modulus $< c'$, save possibly two.

The next theorem shows what is still true if f need satisfy (10.1) only at almost every point of A, and in fact the actual hypothesis supposes far less than this. The hypothesis is satisfied, for example, if at almost every point α of A there is a linear path to α along which every cluster value of f has modulus $\geqq 1$.

THEOREM 10.3. *Let f be a meromorphic function on the disc R, and let A be an open boundary arc. Suppose that $f(0) = 0$, and suppose that at almost every point of A, f either has no fine limit or has a fine limit of modulus $\geqq 1$.*

Then there is a number $\delta_3(|A|)$, depending on $|A|$ but not on f, such that $f(R)$ includes some disc in R of radius $\delta_3(|A|)$ except for a set of zero capacity.

We observe that if f has no fine limit at points of a subset of A of strictly positive measure, then $f(R)$ includes the whole plane less a set of zero capacity, according to Theorem 7.2, so that the fewer points of A at which the fine limit exists the better! To prove the theorem we need only remark that if (10.1) is satisfied at every point of A, we can apply Theorem 10.1, whereas in the contrary case there is a cluster value of modulus < 1 at a point of A, and we can then apply Theorem 9.2 to deduce that $f(R)$ includes all of R save a possible exceptional set of zero capacity.

The following examples show that Theorems 10.2 and 10.3 cannot be strengthened in their present settings. Let B be a compact subset of R which does not contain the center and for which $R - B$ is connected. Let f be a regular function mapping R onto $R - B$, with $f(0) = 0$, derived from a univalent conformal map onto the conformal universal covering surface of $R - B$. Then f has a limit of modulus 1 at each point of a certain union A of boundary open arcs, and all fine and angular limits at other boundary points are in B. Then f is of type **Bl** from R to $R - B$. According to Theorem 8.5 the Lebesgue measure of A is equal to the harmonic measure of the outer boundary of $R - B$ relative to the center, multiplied by 2π.

In particular, let B be finite. Then (10.1) is satisfied almost everywhere on the boundary, but $f(R)$ need cover no disc in R of radius independent of f, because B can be chosen as dense as desired in R. Thus Theorem 10.2 becomes false if "less a set of zero capacity" is replaced by "less a set of measure zero" even if f is supposed regular and bounded.

If B is a finite union of disjunct closed discs, $f(R)$ need cover no disc in R, save for a set of capacity zero, of radius independent of f, even if the length of A is arbitrarily near 2π, because of the variety of possible choices of B. Thus Theorem 10.3 becomes false if the arc A is replaced by a measurable boundary set of measure $|A|$.

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Note added in proof. The author has belatedly seen a paper by C. CONSTANTINESCU AND A. CORNEA, *Über das Verhalten der analytischen Abbildungen Riemannscher Flächen auf dem idealen Rand von Martin*, Nagoya Math. J., vol. 17 (1960), pp. 1–87. Although these authors do not phrase their results in the language of the fine topology, their paper contains results which can be shown to be equivalent to those in Section 7 (including answers to questions raised in that section in connection with the Plessner theorem), as well as to Theorems 8.1, 8.3, and 8.5. The methods used are purely function-theoretic, with no use of or application to probability theory.

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