THE FUNDAMENTAL SOLUTION OF A LINEAR PARABOLIC EQUATION CONTAINING A SMALL PARAMETER[1]

BY

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1. Introduction

We consider the second order linear parabolic differential equation containing a parameter $\varepsilon > 0$

$$L_\varepsilon(u) \equiv \varepsilon \sum_{i,j=1}^{n} a_{ij}(x, y) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i(x, y) \frac{\partial u}{\partial x_i} + b(x, y)u - \frac{\partial u}{\partial y} = f(x, y), \tag{1.1}$$

where the matrix $(a_{ij}(x, y))$ is positive definite and symmetric for all $x = (x_1, \cdots, x_n)$ in Euclidian n-space E^n and y in some closed interval I. We are interested in the behavior as $\varepsilon \to 0+$ of the solution $u(x, y; \varepsilon)$ of the initial value problem (i.v.p.)

$$L_\varepsilon(u) = f \quad \text{for} \quad y > 0, \qquad u(x, 0; \varepsilon) = g(x), \tag{1.2}$$

where g is a given function; in particular, in the connection between $u(x, y; \varepsilon)$ and the solution $v(x, y)$ of the corresponding i.v.p.

$$L_0(v) = f \quad \text{for} \quad y > 0, \qquad v(x, 0) = g(x), \tag{1.3}$$

for the *reduced equation*

$$L_0(v) \equiv \sum_{i=1}^{n} a_i(x, y) \frac{\partial v}{\partial x_i} + b(x, y)v - \frac{\partial v}{\partial y} = f(x, y). \tag{1.4}$$

For example, in the special case $L_\varepsilon(u) \equiv \varepsilon \Delta u - \partial u/\partial y$, where Δ is the n-dimensional Laplace operator, it is well known that for suitable f and g the solution of the i.v.p. (1.2) is given by[2]

$$u(x, y; \varepsilon) = \int \Gamma(x, y; \xi, 0; \varepsilon) g(\xi) \, d\xi - \int_0^y d\eta \int \Gamma(x, y; \xi, \eta; \varepsilon) f(\xi, \eta) \, d\xi, \tag{1.5}$$

where

$$\Gamma(x, y; \xi, \eta; \varepsilon) = 2^{-n} \{\pi^n \varepsilon^n (y - \eta)^n\}^{-1/2} \cdot \exp\left\{ - \frac{1}{4\varepsilon(y - \eta)} \sum_{i=1}^{n} (x_i - \xi_i)^2 \right\}. \tag{1.6}$$

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[2] If no domain of integration is indicated, the integral is to be taken over the whole of E^n.

The solution of the corresponding reduced i.v.p. (1.3) is given by

$$v(x, y) = g(x) - \int_0^y f(x, \nu)\, d\nu.$$

It can be easily shown that

$$u(x, y; \varepsilon) = \pi^{-n/2}\left\{\int g(x + 2\alpha\varepsilon^{1/2}y^{1/2})e^{-|\alpha|^2}\, d\alpha - \int_0^y d\eta \int f(x + 2\alpha\varepsilon^{1/2}(y-\eta)^{1/2}, \eta)e^{-|\alpha|^2}\, d\alpha\right\},$$

where for any $\tau < y$

$\alpha_j = (\xi_j - x_j)/2\varepsilon^{1/2}(y-\tau)^{1/2}$ $(j = 1, \cdots, n)$ and $|\alpha| = (\sum_{j=1}^n \alpha_j^2)^{1/2}$.

Thus for suitable f and g, $\lim_{\varepsilon\to 0+} u(x, y; \varepsilon)$ exists, and

$$\lim_{\varepsilon\to 0+} u(x, y; \varepsilon) = v(x, y). \tag{1.7}$$

In the general case we show under fairly weak hypothesis that (1.7) holds, where $v(x, y)$ is understood to be the *weak* solution of (1.3). Moreover, we obtain an estimate of the rate of convergence of $u(x, y; \varepsilon)$ to $v(x, y)$ in terms of the moduli of continuity of the coefficients of L_ε and the data. Our method is based on the fundamental solution (f.s.) of the homogeneous equation

$$L_\varepsilon(u) = 0, \tag{1.8}$$

i.e., a function Γ analogous to (1.6). By means of the f.s. we obtain the explicit formula (1.5) for $u(x, y; \varepsilon)$. Once the behavior of the f.s. as $\varepsilon \to 0+$ is known, the limiting behavior of $u(x, y; \varepsilon)$ can be read off from (1.5). The major portion of this paper (§§2–5) is devoted to the construction of a representation of the f.s. which is valid for all $\varepsilon > 0$ and whose behavior as $\varepsilon \to 0+$ can be easily analyzed. The remainder of the paper is concerned with the application of these results to the i.v.p. (1.2).

In a recent paper O. A. Ladyjzenskaja [9] has dealt with the i.v.p. for a parabolic system of equations of the form (1.1), i.e., where the coefficients a_{ij}, a_i, and b are interpreted as $N \times N$ matrices, while the functions u, f, and g are N-vectors. She proves that if the coefficients and data are sufficiently regular, then $u(x, y; \varepsilon)$ converges in the L^2-sense as $\varepsilon \to 0+$ to the weak solution of (1.3). Our result on the i.v.p. can be viewed as an extension of Ladyjzenskaja's in the special case of a single equation since, under a similar hypothesis, we prove the pointwise convergence of u to v. A similar result is probably true for parabolic systems; work on this problem is now in progress.

2. The structure of the fundamental solution

A function $\Gamma(x, y; \xi, \eta; \varepsilon)$ defined for all x, ξ in E^n, y, η in I such that $y > \eta$, and $\varepsilon > 0$ is said to be a f.s. of (1.8) if, as a function of x, y, it is a solution of (1.8), and if for suitable functions $g(x)$,

$$\lim_{y \to \eta +} \int \Gamma(x, y; \xi, \eta; \varepsilon) g(\xi)\, d\xi = g(x). \tag{2.1}$$

If Γ is a f.s. of (1.8), then the solution of the i.v.p. (1.2) is given by (1.5) provided that the right-hand side exists and has the required derivatives. For fixed $\varepsilon > 0$, Dressel [4], Èĭdel'man [5] (for parabolic systems of equations), Feller [6] (for $n = 1$), and Pogorzelski [10] have proved the existence of the f.s. under various hypotheses and have given parametrix representations of the f.s. For second order equations the most general of these results is due to Pogorzelski. As a by-product of the present investigation, we obtain the following mild generalization of Pogorzelski's result. Let I be some closed y-interval and let $R = E^n \times I$. We assume (i) $A(x, y) = (a_{ij}(x, y))$ is symmetric and positive definite for all (x, y) in R, and (ii) the coefficients of L_ε are bounded, uniformly continuous, and satisfy a uniform Hölder condition with respect to x in R. Then for each $\varepsilon > 0$ the f.s. of (1.8) exists and can be written in the form

$$\begin{aligned} \Gamma(x, y; \xi, \eta; \varepsilon) &= G(x, y; \xi, \eta; \varepsilon) \\ &\quad + \int_\eta^y d\tau \int G(x, y; s, \tau; \varepsilon) \psi(s, \tau; \xi, \eta; \varepsilon)\, ds \end{aligned} \tag{2.2}$$

for any (x, y), (ξ, η) in R such that $y > \eta$, where

$$\begin{aligned} G(x, y; \xi, \eta; \varepsilon) &= 2^{-n} \left\{ \varepsilon^n \pi^n \det \int_\eta^y A(\xi, \nu)\, d\nu \right\}^{-1/2} \exp \left\{ \int_\eta^y b(\xi, \nu)\, d\nu \right. \\ &\quad \left. - \frac{1}{4\varepsilon} (x - \xi + \alpha)^T \left(\int_\eta^y A(\xi, \nu)\, d\nu \right)^{-1} (x - \xi + \alpha) \right\}, \\ (x - \xi + \alpha) &= (x_1 - \xi_1 + \alpha_1, \cdots, x_n - \xi_n + \alpha_n), \\ \alpha_j(y; \xi, \eta) &= \int_\eta^y a_j(\xi, \nu)\, d\nu, \end{aligned} \tag{2.3}$$

and ψ is the solution of the integral equation

$$\begin{aligned} \psi(x, y; \xi, \eta; \varepsilon) &= L_\varepsilon[G(x, y; \xi, \eta; \varepsilon)] \\ &\quad + \int_\eta^y d\tau \int L_\varepsilon[G(x, y; s, \tau; \varepsilon)] \psi(s, \tau; \xi, \eta; \varepsilon)\, ds. \end{aligned} \tag{2.4}$$

This result is discussed in more detail in the appendix of this paper.

The choice of the parametrix G is by no means fixed. Indeed, we will show that (2.3) is not well suited for use in studying the ε-dependence of the f.s.[3] Qualitatively, *the parametrix must be, in some sense, an approximate solution of* (1.8), *and it must have the same singularities as* Γ. For fixed

[3] This is also true for the parametrices employed in the papers cited above.

$\varepsilon > 0$, in view of (2.1), the latter means that

$$\lim_{y\to\eta+} \int G(x, y; \xi, \eta; \varepsilon)g(\xi)\, d\xi = g(x). \tag{2.5}$$

Thus the sense in which G must be an approximate solution of (1.8) is the following. $L_\varepsilon(G)$ must be such that if ψ is the solution of (2.4), then

$$\lim_{y\to\eta+} \int g(\xi)\, d\xi \int_\eta^y d\tau \int G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon)\, ds = 0 \tag{2.5$'$}$$

for fixed $\varepsilon > 0$. A sufficient condition for (2.5′) to hold is that there exist constants γ, K, $l > 0$ independent of x, y, ξ, η, ε such that

$$| L_\varepsilon(G) | \leqq K[\varepsilon(y - \eta)]^{-(n+2-\gamma)/2} \exp\{-l| x - \xi |^2/\varepsilon(y - \eta)\}. \tag{2.6}$$

Since (2.3) satisfies the equation

$$\varepsilon \sum_{i,j=1}^n a_{ij}(\xi, y) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^n a_i(\xi, y) \frac{\partial u}{\partial x_i} + b(\xi, y)u - \frac{\partial u}{\partial y} = 0$$

for $y > \eta$, one can verify, using (i) and (ii), that (2.6) holds in this case.

In the ε-dependent case Γ has an additional singularity which arises from the connection between the solutions of (1.8) and those of the homogeneous reduced equation. Let $\tilde{u}(x, y; \varepsilon)$ be the (regular) solution of the i.v.p.

$$L_\varepsilon(u) = 0 \quad \text{for} \quad y > \eta, \qquad u(x, \eta; \varepsilon) = g(x),$$

and let $\tilde{v}(x, y)$ be the solution of the i.v.p.

$$L_0(v) = 0 \quad \text{for} \quad y > \eta, \qquad v(x, \eta) = g(x).$$

We assume that g and the coefficients of L_ε are sufficiently regular so that $\tilde{u}$ and $\tilde{v}$ exist, are unique, and are twice continuously differentiable in R for $y > \eta$. Moreover, we assume that $\tilde{u}$ is uniformly bounded in every closed subregion of R as $\varepsilon \to 0+$. Then it follows from Theorem 2 of [2] that $\tilde{u}(x, y; \varepsilon) = \tilde{v}(x, y) + O(\varepsilon)$ uniformly as $\varepsilon \to 0+$ for (x, y) contained in suitably chosen subregions S of R.[4] Thus for (x, y) in S

$$\lim_{\varepsilon\to 0+} \tilde{u}(x, y; \varepsilon) = \lim_{\varepsilon\to 0+} \int \Gamma(x, y; \xi, \eta; \varepsilon)g(\xi)\, d\xi = \tilde{v}(x, y). \tag{2.7}$$

This implies that G must be such that

$$\lim_{\varepsilon\to 0+} \int G(x, y; \xi, \eta; \varepsilon)g(\xi)\, d\xi = \tilde{v}(x, y), \tag{2.8}$$

and

$$\lim_{\varepsilon\to 0+} \int g(\xi)\, d\xi \int_\eta^y d\tau \int G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon)\, ds = 0 \tag{2.8$'$}$$

[4] The case $n = 1$ is dealt with in [2]; however, the result can be readily generalized for any $n \geqq 1$.

for (x, y) in S. It can be shown that for G given by (2.3)

$$\lim_{\varepsilon \to 0+} \int G(x, y; \xi, \eta; \varepsilon) g(\xi) \, d\xi = g(x).$$

Hence (2.8) is not generally satisfied, and a fortiori neither is (2.8′).

To analyze the nature of the singularity implied by (2.7) we will have to investigate the structure of the solutions of the reduced equation. Corresponding to the i.v.p. (1.3) we have the characteristic i.v.p.

$$\begin{aligned} dx_j/dy &= -a_j(x, y), \qquad x_j(\eta) = \alpha_j \qquad (j = 1, \cdots, n), \\ dv/dy &= b(x, y)v - f(x, y), \qquad v(\eta) = g(\alpha). \end{aligned} \tag{2.9}$$

These problems are equivalent in the following sense. If $v = v(x, y; \eta)$ is a solution of (1.3), then equations (2.9) are satisfied on the surface $v = v(x, y; \eta)$ in the $(n + 2)$-dimensional (x, y, v)-space. Conversely, if (2.9) has a solution $x_j = x_j(y; \alpha, \eta)$, $v = v(y; \alpha, \eta)$, and if, after eliminating the parameter α, v is a continuously differentiable function of x and y, then v is a solution of (1.3). Suppose that the a_j are sufficiently regular so that the first n equations of (2.9) have a unique solution

$$x = (\phi_1(y; \alpha, \eta), \cdots, \phi_n(y; \alpha, \eta)) = \phi(y; \alpha, \eta)$$

for all α in E^n and y, η in I. The curve $x = \phi(y; \alpha, \eta)$ is called the characteristic of L_0 through (α, η). It is easy to show that

$$\begin{aligned} v(y; \alpha, \eta) = g(\alpha) \exp \int_\eta^y b(\phi(\nu; \alpha, \eta), \nu) \, d\nu \\ - \int_\eta^y f(\phi(\tau; \alpha, \eta), \tau) \exp \left\{ \int_\tau^y b(\phi(\nu; \alpha, \eta), \nu) \, d\nu \right\} d\tau. \end{aligned}$$

In view of the assumed uniqueness of the characteristics of L_0, if we put $\alpha = \phi(\eta; x, y)$, then $\phi(\nu; \alpha, \eta) = \phi(\nu; x, y)$ and

$$\begin{aligned} v^*(x, y; \eta) = g(\phi(\eta; x, y)) \exp \int_\eta^y b(\phi(\nu; x, y), \nu) \, d\nu \\ - \int_\eta^y f(\phi(\tau; x, y), \tau) \exp \left\{ \int_\tau^y b(\phi(\nu; x, y), \nu) \, d\nu \right\} d\tau, \end{aligned} \tag{2.10}$$

where $v^*(x, y; \eta) \equiv v(y; \phi(\eta; x, y), \eta)$. Note that v^* is uniquely determined in cases where it is not differentiable, e.g., if a_j, b, g, and f are bounded and continuous, and if the a_j are Lipschitz continuous with respect to x. For this reason we call v^* the *weak solution* of (1.3). Every classical solution of (1.3) is a weak solution, and every continuously differentiable weak solution of (1.3) is a classical solution.

In the notation of (2.7) we have

$$\tilde{v}(x, y) = g(\phi(\eta; x, y)) \exp \int_\eta^y b(\phi(\nu; x, y), \nu) \, d\nu.$$

Hence (2.7) implies that Γ has a singularity at $\xi = \phi(\eta; x, y)$, $\varepsilon = 0$ for $y > \eta$, and (2.1) implies that Γ has a singularity at $\xi = x = \phi(y; x, y)$, $y = \eta$ for $\varepsilon > 0$. Since we have assumed uniqueness of the characteristics of L_0, we can solve $\xi - \phi(\eta; x, y) = 0$ for x and obtain $x = \phi(y; \xi, \eta)$ for all (x, y), (ξ, η) in R. Thus in terms of x and y, Γ has a singularity at $x = \phi(y; \xi, \eta)$ for $\varepsilon = 0$, $y > \eta$ and for $\varepsilon > 0$, $y = \eta$. In order that G have the same singularities as Γ, G will have to be, at least, an approximate solution of (1.8) in the neighborhood of $(\phi(y; \xi, \eta), y)$ for every (ξ, η) in R and $y > \eta$, where the degree of approximation is determined by the conditions (2.5′) and (2.8′). Essentially this means that the characteristics $x = \phi(y; \xi, \eta)$ play the same role in our theory as do the lines $x = \xi$ in the ε-independent case. We will show that a sufficient condition for (2.5′) and (2.8′) to hold is that there exist constants γ, k, $l > 0$ independent of x, y, ξ, η, ε such that

$$(2.11) \quad | L_\varepsilon(G) | \leqq k\varepsilon^{-(n-\gamma)/2}(y - \eta)^{-(n+2-\gamma)/2} \exp\{-l| x - \phi |^2/\varepsilon(y - \eta)\},$$

where $\phi = \phi(y; \xi, \eta)$. Note that the right-hand side of (2.11) has a weaker singularity with respect to ε than it has with respect to $y - \eta$. This is to be expected since there is no integration with respect to ε.

A parametrix G which satisfies (2.11) can be obtained, as we will show, by choosing the appropriate solution of a partial differential equation which is derived from (1.8) by replacing the coefficients by one or two terms of their Taylor expansions about $x = \phi(y; \xi, \eta)$. This equation is solved by Fourier transform methods in §4, and the relevant properties of G are developed there. Our main result (Theorem I) on the f.s. is stated at the beginning of §5, and its proof is carried out in that section. In §6 we prove our main results (Theorems II and III) on the existence of the solution of the i.v.p. for (1.1) and on its limiting behavior as $\varepsilon \to 0+$. The f.s. in the ε-independent case is discussed in the appendix. We begin in §3 with a statement of our hypothesis and with certain preliminary results concerning the characteristics of the reduced operator L_0.

3. Hypothesis and preliminary lemmas

Let I be some closed y-interval $y' \leqq y \leqq y''$, where y' and y'' are fixed finite numbers. We will denote by R the product space $E^n \times I$ and by R' the product space $I \times E^n \times I$. If a function $g(x, y)$ defined in R is bounded, uniformly continuous, and satisfies a uniform Hölder condition with exponent γ with respect to x in R, we say that g belongs to class $H(\gamma; x; R)$. By a uniform Hölder condition we mean there exist constants $H > 0$ and $0 < \gamma \leqq 1$ independent of (x', y), (x, y) in R such that

$$| g(x', y) - g(x, y) | \leqq H| x' - x |^\gamma,$$

where

$$| x | = \left(\textstyle\sum_{i=1}^n x_i\right)^{1/2}.$$

For functions defined in R', we define the class $H(\gamma; x; R')$ in a similar manner.

Unless the contrary is explicitly stated, all of the results in this paper are obtained under the following hypothesis ($\mathcal{H}$):

($\mathcal{H}_1$) $a_{ij}(x, y) = a_{ji}(x, y)$, and there exist constants $l_1^*, l_2^* > 0$ such that

$$l_1^* \lambda^T \lambda \leqq \sum_{i,j=1}^n a_{ij}(x, y)\lambda_i \lambda_j \leqq l_2^* \lambda^T \lambda$$

uniformly in R for all real n-vectors λ.

($\mathcal{H}_2$) $a_{ij}(x, y)$, $a_i(x, y)$, and $b(x, y)$ belong to class $H(\gamma; x; R)$.

($\mathcal{H}_3$) The $(\partial/\partial x_j)a_i(x, y)$ exist in R and belong to class $H(\gamma; x; R)$.

($\mathcal{H}_1$) and ($\mathcal{H}_2$) are sufficient for the existence of the f.s. of (1.8) for each $\varepsilon > 0$. ($\mathcal{H}_3$) is, in effect, a regularity condition on the characteristics of L_0. Some condition of this nature is needed in order to obtain a parametrix which satisfies (2.11). Note that since we do not assume that b is differentiable with respect to x, ($\mathcal{H}_2$) and ($\mathcal{H}_3$) are not sufficient for the existence of a classical solution of the i.v.p. (1.3) regardless of how smooth f and g are assumed to be.

Throughout the remainder of this paper the constants H and γ will refer to the common Hölder condition satisfied by all the quantities in ($\mathcal{H}_2$) and ($\mathcal{H}_3$), and the constant N to their common bound. In general, all constants which appear will be independent of all variables, parameters, and indices unless a dependence is explicitly indicated either by a superscript or in the usual function notation. Occasionally, to simplify the writing, we will use the summation convention and the notation $u_{,ij} = \partial^2 u/\partial x_i \partial x_j$ and $u_{,y} = \partial u/\partial y$.

The characteristics of L_0 are the solutions of the vector ordinary differential equation

$$dx/dy = -a(x, y), \tag{3.1}$$

where $x = (x_1, \cdots, x_n)^T$ and $a = (a_1, \cdots, a_n)^T$. As we indicated in §2, these characteristics play an important role in our theory. For convenience, we enumerate here the various properties of the characteristics which will be needed in the sequel.

LEMMA 3.1. *Let ($\mathcal{H}_2$) and ($\mathcal{H}_3$) hold. For every (ξ, η) in R there exists for all y in I a unique solution of (3.1), $x = \phi(y; \xi, \eta)$, such that $\phi(\eta; \xi, \eta) = \xi$. The matrices*

$$\phi_\xi(y; \xi, \eta) = \left(\frac{\partial}{\partial \xi_j} \phi_i(y; \xi, \eta)\right), \qquad \Phi_\xi(y; \xi, \eta) = \phi_\xi^{-1}(y; \xi, \eta)$$

and their first derivatives with respect to y exist and belong to $H(\gamma; \xi; R')$.[5] *If*

$$B(y; \xi, \eta) = \left(\frac{\partial}{\partial x_j} a_i(x, y)\right)_{x=\phi(y;\xi,\eta)}$$

[5] Statements about boundedness and continuity of matrices are with respect to the norm $|A| = \sum_{i,j} |a_{ij}|$, where $A = (a_{ij})$.

and E is the $n \times n$ identity matrix, then

$$\begin{aligned}\frac{\partial}{\partial y}\phi_\xi(y;\xi,\eta) &= -B(y;\xi,\eta)\phi_\xi(y;\xi,\eta),\\ \frac{\partial}{\partial y}\Phi_\xi(y;\xi,\eta) &= \Phi_\xi(y;\xi,\eta)B(y;\xi,\eta),\\ \phi_\xi(\eta;\xi,\eta) &= \Phi_\xi(\eta;\xi,\eta) = E,\end{aligned} \tag{3.2}$$

and

$$\det \phi_\xi(y;\xi,\eta) = [\det \Phi_\xi(y;\xi,\eta)]^{-1} = \exp\left\{-\int_\eta^y \operatorname{tr} B(\nu;\xi,\eta)\,d\nu\right\} \tag{3.3}$$

in R'.

Lemma 3.1 follows easily from standard results in the theory of ordinary differential equations (cf. Chapter 1 of [3], particularly Theorem 7.2). We omit the details. From the boundedness of ϕ_ξ and Φ_ξ together with (3.3) we conclude immediately that there exist constants $0 < d_1 < d_2$ such that

$$d_1\lambda^T\lambda \leqq \lambda^T\phi_\xi\phi_\xi^T\lambda \leqq d_2\lambda^T\lambda \quad\text{and}\quad \lambda^T\lambda/d_2 \leqq \lambda^T\Phi_\xi\Phi_\xi^T\lambda \leqq \lambda^T\lambda/d_1 \tag{3.4}$$

uniformly in R' for all real n-vectors λ.

LEMMA 3.2. *Let $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold. If (ξ, η), (σ, τ) are any two points of R and y_1, y_2 any two points of I, then*

$$e^{-P|y_2-y_1|} \leqq \frac{|\phi(y_1;\xi,\eta) - \phi(y_1;\sigma,\tau)|}{|\phi(y_2;\xi,\eta) - \phi(y_2;\sigma,\tau)|} \leqq e^{P|y_2-y_1|}, \tag{3.5}$$

where $P = n^{1/2}N$.

Proof. It follows from $(\mathcal{H}_3)$ that

$$|a(x', y) - a(x, y)| \leqq P|x' - x|$$

uniformly in R. Suppose $y_1 < y_2$. For any $y \geqq y_1$ we have

$$\begin{aligned}\phi(y;\xi,\eta) - \phi(y;\sigma,\tau) &= \phi(y_1;\xi,\eta) - \phi(y_1;\sigma,\tau)\\ &\quad + \int_{y_1}^y \{a(\phi(\nu;\sigma,\tau),\nu) - a(\phi(\nu;\xi,\eta),\nu)\}\,d\nu,\end{aligned} \tag{3.6}$$

and hence

$$\begin{aligned}|\phi(y;\xi,\eta) - \phi(y;\sigma,\tau)| \leqq\, &|\phi(y_1;\xi,\eta) - \phi(y_1;\sigma,\tau)|\\ &+ P\int_{y_1}^y |\phi(\nu;\xi,\eta) - \phi(\nu;\sigma,\tau)|\,d\nu.\end{aligned}$$

Thus by Theorem 2.1 of [3, Chapter 1]

$$|\phi(y_2;\xi,\eta) - \phi(y_2;\sigma,\tau)| \leqq |\phi(y_1;\xi,\eta) - \phi(y_1;\sigma,\tau)|e^{P(y_2-y_1)}.$$

Since (3.6) holds with y replaced by y_2, and y_1 replaced by $y \leqq y_2$, we also have

$$|\,\phi(y_1\,;\xi,\eta) - \phi(y_1\,;\sigma,\tau)\,| \leqq |\,\phi(y_2\,;\xi,\eta) - \phi(y_2\,;\sigma,\tau)\,|e^{P(y_2-y_1)},$$

which completes the proof.

An immediate consequence of Lemma 3.2 is the following. Let $c(x, y)$ be of class $H(\gamma; x, R)$ and let $c(y; \xi, \eta) \equiv c(\phi(y; \xi, \eta), y)$. Then

$$\begin{aligned} |\,c(y;\sigma,\eta) - c(y;\xi,\eta)\,| &\leqq H|\,\phi(y;\sigma,\eta) - \phi(y;\xi,\eta)\,|^{\gamma} \\ &\leqq He^{\gamma P(y''-y')}|\,\sigma - \xi\,|^{\gamma} = \hat{H}|\,\sigma - \xi\,|^{\gamma}, \end{aligned} \tag{3.7}$$

i.e., $c(y; \xi, \eta)$ is in class $H(\gamma; \xi, R')$.

Finally, we state as a lemma a well known property of the exponential function which will be used frequently in what follows.

LEMMA 3.3. *If $\alpha \geqq 0$, and if $p > 0$, $0 < h < 1$ are given constants, then there exists a constant $k > 0$ independent of α such that*

$$\alpha^p e^{-\alpha} \leqq k e^{-h\alpha}.$$

4. The parametrix and its properties

We now turn to the determination of a parametrix which has the properties outlined in §2, i.e., a solution with appropriate singular behavior of an equation which approximates to (1.8) along a characteristic of L_0. The simplest equation which will serve our purpose is

$$\begin{aligned} \Lambda_\varepsilon(u) \equiv \varepsilon a_{jk}(y;\xi,\eta)u_{,jk} + \{a_j(y;\xi,\eta) + (x_k - \phi_k)a_{j,k}(y;\xi,\eta)\}u_{,j} \\ + b(y;\xi,\eta)u - u_{,y} = 0, \end{aligned} \tag{4.1}$$

where $a_{jk}(y; \xi, \eta) = a_{jk}(\phi(y; \xi, \eta), y)$ etc., $\phi_k = \phi_k(y; \xi, \eta)$, and (ξ, η) is an arbitrary point of R. The usual method of solving parabolic equations with coefficients depending on y alone is by Fourier transforms; see, e.g., [8]. We pursue this method in a purely formal manner even though some of the coefficients of (4.1) depend linearly on x.

Let s be a real n-vector, and let

$$A(y;\xi,\eta) = (a_{ij}(y;\xi,\eta)), \qquad B(y;\xi,\eta) = \left(\frac{\partial a_i}{\partial x_j}(y;\xi,\eta)\right)$$

(cf. Lemma 3.1). The Fourier transform of a solution $u(x, y; \xi, \eta; \varepsilon)$ of (4.1) is given by

$$v(s, y; \xi, \eta; \varepsilon) = \int e^{ix^Ts} u(x, y; \xi, \eta; \varepsilon)\, dx\,.$$

If u and its x-derivatives tend to zero as $|\,x\,| \to \infty$, and if the appropriate integrals converge, then v satisfies the first order linear partial differential equation

(4.2) $s^T B v_{,s} + v_{,y} = v[-is^T\{d\phi/dy + B\phi\} - \varepsilon s^T A s + b - \text{tr } B],$

where $v_{,s} = (\partial v/\partial s_1, \cdots, \partial v/\partial s_n)$, and we have made use of (3.1). In (4.2), A, B, b, and ϕ are evaluated at $(y; \xi, \eta)$.

In the $(n + 1)$-dimensional (s, y)-space let α_j, $j = 1, \cdots, n$, be the coordinates of a point on the hyperplane $y = \eta$, and let α be the vector $(\alpha_1, \cdots, \alpha_n)^T$. Consider the following i.v.p.: find a solution of (4.2) for $y > \eta$ which satisfies

(4.3) $$v = e^{-i\xi^T\alpha} \quad \text{on} \quad y = \eta.$$

I.e., we are looking for a v such that its inverse Fourier transform, if it exists, behaves like a delta-function for $y = \eta$. The characteristic equations corresponding to (4.2) can be written in the form

(4.4) $ds/dy = s^T B, \qquad dv/dy = v[-is^T\{d\phi/dy + B\phi\} - \varepsilon s^T A s + b - \text{tr } B].$

The characteristic of (4.2) which passes through $(\alpha_1, \cdots, \alpha_n, \eta)$ is given by

(4.5) $$s^T = \alpha^T \Phi_\xi(y; \xi, \eta)$$

since, in view of (3.2), Φ_ξ is the fundamental matrix of the first equation of (4.4). Thus for each α

$$s^T\{d\phi/dy + B\phi\} = \alpha^T\{\Phi_\xi \, d\phi/dy + \Phi_\xi \, B\phi\} = \alpha^T(d/dy)(\Phi_\xi \, \phi),$$

and along the characteristic (4.5) through $(\alpha_1, \cdots, \alpha_n, \eta)$, v satisfies

(4.6) $dv/dy = v[-i\alpha^T(d/dy)(\Phi_\xi \, \phi) - \varepsilon\alpha^T\Phi_\xi \, A\Phi_\xi^T \, \alpha + b - \text{tr } B].$

If we now integrate (4.6) from η to $y > \eta$ and apply (4.3), (3.2), and $\phi(\eta; \xi, \eta) = \xi$, we obtain

(4.7) $$v = \exp\left\{-i\alpha^T\Phi_\xi \, \phi - \varepsilon\alpha^T C\alpha + \int_\eta^y (b - \text{tr } B) \, d\nu\right\},$$

where C is the symmetric matrix

(4.8) $$C(y; \xi, \eta) = \int_\eta^y \Phi_\xi(\nu; \xi, \eta) A(\nu; \xi, \eta) \, \Phi_\xi^T \, (\nu; \xi, \eta) \, d\nu.$$

From (3.3) we have $\det \Phi_\xi > 0$ in R'. Hence we can invert (4.5) and eliminate α from (4.7) to obtain

(4.9) $$v = \exp\left\{-is^T\phi - \varepsilon s^T\phi_\xi \, C\phi_\xi^T \, s + \int_\eta^y (b - \text{tr } B) \, d\nu\right\}$$

as the solution of the i.v.p. (4.2), (4.3).

Although v as given by (4.9) is a rigorous solution of (4.2), (4.3), there is as yet only a formal connection between v and equation (4.1). We now show that the inverse transform of v is indeed a solution of (4.1) for $y > \eta$ and $\varepsilon > 0$ and that it is a suitable parametrix. The inverse Fourier transform of (4.9) is

$$G(x, y; \xi, \eta; \varepsilon) = (2\pi)^{-n} \exp\left[\int_\eta^y \{b(\nu; \xi, \eta) - \operatorname{tr} B(\nu; \xi, \eta)\}\, d\nu\right] \cdot \int \exp\{is^T(x - \phi) - \varepsilon s^T \phi_\xi C \phi_\xi^T s\}\, ds. \tag{4.10}$$

where ϕ, ϕ_ξ, and C are evaluated at $(y; \xi, \eta)$. It follows from $(\mathcal{H}_1)$ and (3.4) that

$$(l_1^*/d_2)\lambda^T\lambda \leqq \lambda^T \Phi_\xi A \Phi_\xi^T \lambda = \lambda^T(\partial C/\partial y)\lambda \leqq l_2^*/d_1$$

uniformly in R' for any real n-vector λ, or, by integrating on y from η to $y > \eta$

$$0 < (l_1^*/d_2)(y - \eta)\lambda^T\lambda \leqq \lambda^T C \lambda \leqq (l_2^*/d_1)(y - \eta)\lambda^T\lambda \tag{4.11}$$

uniformly in R' for $y > \eta$ and $\lambda^T\lambda \neq 0$. Thus for every real s

$$\varepsilon(d_1 l_1^*/d_2)(y - \eta)s^T s \leqq \varepsilon s^T \phi_\xi C \phi_\xi^T s \leqq \varepsilon(d_2 l_2^*/d_1)(y - \eta)s^T s,$$

and it follows that the integral on the right-hand side of (4.10) is absolutely convergent for all (x, y), (ξ, η) in R' provided that $y > \eta$ and $\varepsilon > 0$. Moreover, we can differentiate G arbitrarily often with respect to x and once with respect to y, since the resulting integrals will still be absolutely convergent for $y > \eta$ and $\varepsilon > 0$.

To verify that $\Lambda_\varepsilon(G) = 0$ and to derive the various estimates of G and its derivatives which we will need, we can proceed directly from (4.10) by means of a rather elegant technique due to Ladyjzenskaja [8]. However, since we are dealing with a second order equation, it is possible to carry out the integrations in (4.10) and, for our purposes, more convenient to do so. For $y > \eta$, C is a symmetric matrix, and, by (4.11), it is positive definite. Hence there exists a unique lower triangular matrix M with positive diagonal elements such that

$$C(y; \xi, \eta) = M(y; \xi, \eta)M^T(y; \xi, \eta). \tag{4.12}$$

In fact, it can be shown using the so-called Schweins expansion [1, pp. 107–109] that if $M = (m_{ij})$ and $C = (c_{ij})$, then

$$m_{ij} = \begin{vmatrix} c_{11} & \cdots & c_{1j-1} & c_{1i} \\ \vdots & & \vdots & \vdots \\ c_{j1} & \cdots & c_{jj-1} & c_{ji} \end{vmatrix} \Bigg/ (C_{j-1} C_j)^{1/2}, \tag{4.13}$$

where C_j $(j = 1, \cdots, n)$ is the j^{th} order principal minor of C. $(C_n = \det C.)$ It follows from (4.11) that for $y > \eta$

$$0 < (l_1^*/d_2)^j(y - \eta)^j \leqq C_j \leqq (l_2^*/d_1)^j(y - \eta)^j, \qquad j = 1, \cdots, n. \tag{4.14}$$

Thus it is clear from (4.13) that $M(y; \xi, \eta)$ is continuous in R' for $y > \eta$ and that there exists a constant $k > 0$ independent of y, ξ, η in R' such that

$$| M(y; \xi, \eta) | \leqq k(y - \eta)^{1/2}. \tag{4.15}$$

By the standard process of completing the squares, the exponent in the integrand of (4.10) can be put in the form

$$-\alpha^T\alpha - (1/4\varepsilon)(x - \phi)^T\Phi_\xi^T C^{-1}\Phi_\xi(x - \phi),$$

where

$$\alpha = \varepsilon^{1/2}M^T\Phi_\xi^T s - (i/2\varepsilon^{1/2})M^{-1}\Phi_\xi(x - \phi).$$

If we introduce $\alpha_1, \cdots, \alpha_n$ as new variables and carry out the integrations, (4.10) becomes

$$(4.16) \quad G(x, y; \xi, \eta; \varepsilon) = h(y; \xi, \eta; \varepsilon) \exp\{-(1/4\varepsilon)z^T F(y; \xi, \eta)z\},$$

where $z(x, y; \xi, \eta) = x - \phi(y; \xi, \eta)$, and

$$(4.16') \quad \begin{aligned} h(y; \xi, \eta; \varepsilon) &= 2^{-n}\{\pi^n\varepsilon^n \det C(y; \xi, \eta)\}^{-1/2} \exp \int_\eta^y b(\nu; \xi, \eta)\, d\nu, \\ F(y; \xi, \eta) &= \Phi_\xi^T(y; \xi, \eta)C^{-1}(y; \xi, \eta)\Phi_\xi(y; \xi, \eta). \end{aligned}$$

Note that F is a symmetric matrix.

As we indicated above, G has derivatives of all orders in x and of first order in y for $y > \eta$ and $\varepsilon > 0$. In particular

$$(4.17) \quad \begin{aligned} G_{,j} &= -\frac{1}{2\varepsilon} e_j^T FzG, \qquad G_{,jk} = \left(-\frac{1}{2\varepsilon} e_j^T Fe_k + \frac{1}{4\varepsilon^2} z^T Fe_j\, e_k^T Fz\right) G, \\ G_{,y} &= \left(-\frac{1}{2}\frac{\partial}{\partial y} \log \det C + b - \frac{1}{2\varepsilon} a^T Fz - \frac{1}{4\varepsilon} z^T \frac{\partial F}{\partial y} z\right) G, \end{aligned}$$

where $e_j = (\delta_{1j}, \cdots, \delta_{nj})^T$ and a, b, C, F are evaluated at $(y; \xi, \eta)$. By a straightforward computation, using the definitions of C and F as well as certain elementary facts from matrix calculus, it can be shown that

$$\varepsilon a_{jk}\, G_{,jk} + a_j\, G_{,j} + z_k\, a_{j,k}\, G_{,j} + bG = \left(-\frac{1}{2}\frac{\partial}{\partial y} \log \det C - \frac{1}{4\varepsilon} z^T\Phi_\xi^T \frac{\partial C^{-1}}{\partial y} \Phi_\xi z \right.$$
$$\left. - \frac{1}{2\varepsilon} a^T Fz - \frac{1}{2\varepsilon} z^T \frac{\partial \Phi_\xi^T}{\partial y} C^{-1}\Phi_\xi z + b\right) G = \frac{\partial G}{\partial y},$$

i.e., $\Lambda_\varepsilon(G) = 0$. Thus we have

LEMMA 4.1. *If* ($\mathcal{H}$) *holds, then* G, *given by* (4.16), *is a solution of* (4.1) *for* $y > \eta$ *and* $\varepsilon > 0$.

We now obtain certain estimates for G and its derivatives which will lead to the verification of (2.5), (2.8), and (2.11), and which will permit us to carry out the construction of the f.s. of (1.8). If $m \geqq 0$ is an integer, let D^m denote any m^{th} order partial derivative with respect to x. We prove

LEMMA 4.2. *For any integer* $m \geqq 0$, D^mG *and* $(\partial/\partial y)D^mG$ *exist and are uniformly continuous in* $R \times R$ *for* $y > \eta$ *and* $\varepsilon > 0$. *There exist constants,*

l, $K^{(m)}$, $\tilde{K}^{(m)} > 0$ such that

$$
(4.18)\quad
\begin{aligned}
&| G(x, y; \xi, \eta; \varepsilon) | \leqq K^{(0)}[\varepsilon(y-\eta)]^{-n/2} \exp\{-l|z|^2/4\varepsilon(y-\eta)\},\\
&| D^m G(x, y; \xi, \eta; \varepsilon) | \leqq K^{(m)}[\varepsilon(y-\eta)]^{-(n+m)/2} \exp\{-l|z|^2/8\varepsilon(y-\eta)\},\\
&| (\partial/\partial y) D^m G(x, y; \xi, \eta; \varepsilon) |\\
&\qquad\leqq \tilde{K}^{(m)} \varepsilon^{-(n+m+1)/2}(y-\eta)^{-(n+m+2)/2} \exp\{-l|z|^2/8\varepsilon(y-\eta)\}
\end{aligned}
$$

in $R \times R$ for $y > \eta$ and $\varepsilon > 0$.

Proof. It follows from (4.11) that

$$
(d_1/l_2^*)(y-\eta)^{-1}\lambda^T\lambda \leqq \lambda^T C^{-1}\lambda \leqq (d_2/l_1^*)(y-\eta)^{-1}\lambda^T\lambda
$$

uniformly in R' for $y > \eta$. Hence, in view of (4.16′) and (3.4), we have

$$
(4.19)\qquad l|z|^2/(y-\eta) \leqq z^T F z \leqq \hat{l}|z|^2/(y-\eta)
$$

in R' for $y > \eta$, where $l = d_1(d_2 l_2^*)^{-1}$ and $\hat{l} = d_2(d_1 l_1^*)^{-1}$. Since b is bounded, (4.19) and (4.14) imply the first inequality of (4.18). The continuity of G is clear from its structure and ($\mathfrak{H}$).

For any integer $m \geqq 0$, $D^m G$ exists for $y > \eta$ and $\varepsilon > 0$, and can be written in the form

$$
(4.20)\qquad D^m G = h e^{-(1/4\varepsilon) z^T F z} \sum_{j=0}^{[m/2]} (2\varepsilon)^{j-m} \sum f^{m-j} z^{m-2j},
$$

where $\sum f^{m-j} z^{m-2j}$ denotes a sum of products of $m - j$ elements $f_{\alpha\beta}$ of F and $m - 2j$ components z_α of z. If $C = (c_{\alpha\beta})$, then it is clear from (4.8) that there exists a constant $k_1 > 0$ such that

$$
(4.21)\qquad | c_{\alpha\beta} | \leqq \sum | c_{ij} | = | C | \leqq k_1(y-\eta)
$$

uniformly in R' for $y > \eta$. Thus, if $C^{-1} = (c_{\alpha\beta}^{-1})$, we have in view of (4.14)

$$
(4.22)\qquad | c_{\alpha\beta}^{-1} | \leqq | C^{-1} | \leqq k_2(y-\eta)^{-1}
$$

uniformly in R' for $y > \eta$. Since $F = \Phi_\xi^T C^{-1} \Phi_\xi$, it follows that there exists a constant $k_3 > 0$ such that

$$
(4.23)\qquad | f_{\alpha\beta} | \leqq | F | \leqq k_3(y-\eta)^{-1}
$$

uniformly in R' for $y > \eta$. From (4.20), using (4.14), (4.19) and (4.23), we conclude that there is a constant $k_1^{(m)} > 0$ such that

$$
| D^m G | \leqq k_1^{(m)} \sum_{j=0}^{[m/2]} | z |^{m-2j} [\varepsilon(y-\eta)]^{-n/2-m+j} \exp\{-l|z|^2/4\varepsilon(y-\eta)\}.
$$

An application of Lemma 3.3 yields the second inequality of (4.18).

Since F, h, and z have first partial derivatives with respect to y for $y > \eta$, we have

$$\frac{\partial}{\partial y} D^m G = \frac{\partial G}{\partial y} \sum_{j=0}^{[m/2]} (2\varepsilon)^{j-m} \sum f^{m-j} z^{m-2j}$$
$$+ G \sum_{j=0}^{[m/2]} (2\varepsilon)^{j-m} \sum (z_{,y} f^{m-1} z^{m-1-2j} + f_{,y} f^{m-1-j} z^{m-2j}),$$

where $z_{,y} = \partial z_\alpha / \partial y$ for some α, $f_{,y} = \partial f_{\alpha\beta}/\partial y$ for some α, β, and $\partial G/\partial y$ is given by (4.17). From (3.1) we have $z_{,y} = -\phi_{\alpha,y} = a_\alpha$, and hence $z_{,y}$ is bounded. In addition we have from (3.2) and (4.8) that

$$| f_{,y} | \leqq | \partial F/\partial y | = | B^T F - FAF + FB | \leqq k_4(y - \eta)^{-2}$$

for some constant $k_4 > 0$. If we use this together with (4.21), (4.8), (4.14), (4.23), and the first inequality of (4.18), it is easy to show that

$$| \partial G/\partial y | \leqq k_5 \{ [\varepsilon(y - \eta)]^{-n/2} + \varepsilon[\varepsilon(y - \eta)]^{-(n+2)/2} + | z | [\varepsilon(y - \eta)]^{-(n+2)/2}$$
$$+ \varepsilon | z |^2 [\varepsilon(y - \eta)]^{-(n+4)/2} \} \exp \{ -l | z |^2 / 4\varepsilon(y - \eta) \},$$

for some constant $k_5 > 0$. Moreover, there exists a constant $k_2^{(m)} > 0$ such that

$$\left| \frac{\partial}{\partial y} D^m G \right| \leqq k_2^{(m)} \left\{ \left| \frac{\partial G}{\partial y} \right| \sum_{j=0}^{[m/2]} | z |^{m-2j} [\varepsilon(y - \eta)]^{-m+j} \right.$$
$$+ \sum_{j=0}^{[m/2]} \big(| z |^{m-1-2j} [\varepsilon(y - \eta)]^{-n/2-m+j}$$
$$\left. + \varepsilon | z |^{m-2j} [\varepsilon(y - \eta)]^{-n/2-m-1+j} \big) \exp \left(-l | z |^2 / 4\varepsilon(y - \eta) \right) \right\},$$

and by an application of Lemma 3.3 this completes the proof.

In a similar manner we show that G satisfies (2.11).

LEMMA 4.3. *There exists a constant* $K > 0$ *such that*

$$| L_\varepsilon[G(x, y; \xi, \eta; \varepsilon)] | \leqq K \varepsilon^{-(n-\gamma)/2} (y - \eta)^{-(n+2-\gamma)/2} \exp \{ -l | z |^2 / 8\varepsilon(y - \eta) \}$$

for $y > \eta$ *and* $\varepsilon > 0$, *where* γ *is the Hölder exponent of the coefficients of* L_ε.

Proof. Since the a_j are continuously differentiable with respect to x, it follows from Lemma 4.1, (4.17), and the Theorem of the Mean that

$$L_\varepsilon(G) = [\varepsilon\{a_{jk}(x, y) - a_{jk}(y; \xi, \eta)\}\{-(1/2\varepsilon) e_j^T F e_k + (1/4\varepsilon^2) e_j^T F z e_k^T F z\}$$
$$+ z_k \{a_{j,k}(\tilde{x}, y) - a_{j,k}(y; \xi, \eta)\}\{-(1/2\varepsilon) e_j^T F z\} + \{b(x, y) - b(y; \xi, \eta)\}] G,$$

where $\tilde{x} = \theta x + (1 - \theta)\phi(y; \xi, \eta)$ for some θ, $0 < \theta < 1$. Since $a_{jk}(y; \xi, \eta) = a_{jk}(\phi(y; \xi, \eta), y)$, we have

$$| a_{jk}(x, y) - a_{jk}(y; \xi, \eta) | \leqq H | x - \phi(y; \xi, \eta) |^\gamma = H | z |^\gamma.$$

Similar inequalities hold for b and for $a_{j,k}$ in view of the fact that $| \tilde{x} - \phi | =$

$\theta|z| < |z|$. It follows from (4.18) and (4.22) that there exists a constant $k > 0$ such that for $y > \eta$ and $\varepsilon > 0$

$$
\begin{aligned}
|L_\varepsilon(G)| \leqq kHK_0\{&|z|^\gamma(y-\eta)^{-1} + |z|^{2+\gamma}\varepsilon^{-1}(y-\eta)^{-2} \\
&+ |z|^{2+\gamma}\varepsilon^{-1}(y-\eta)^{-1} + |z|^\gamma\} \\
&\cdot [\varepsilon(y-\eta)]^{-n/2}\exp\{-l|z|^2/4\varepsilon(y-\eta)\},
\end{aligned}
\tag{4.24}
$$

and, by Lemma 3.3, the assertion follows.

In carrying out the construction of the f.s. it is useful to distinguish between the dependence of G on ξ through z, and through h and F. Thus for any σ in E^n we define

$$
\begin{aligned}
&G^\sigma(x, y; \xi, \eta; \varepsilon) \\
&\quad = h(y; \sigma, \eta; \varepsilon)\exp\{-(1/4\varepsilon)z^T(x, y; \xi, \eta)F(y; \sigma, \eta)z(x, y; \xi, \eta)\},
\end{aligned}
$$

with the convention that if $\sigma = \xi$ we will omit the superscript. Note that Lemma 4.2 holds when G is replaced by G^σ. Moreover, in view of Lemma 3.1, G^σ is differentiable with respect to ξ. We will now prove that D^mG^σ is Hölder continuous in σ. This result is essentially due to Pogorzelski [10], and it is crucial for the construction of the f.s. from a parametrix which is not differentiable with respect to ξ.

LEMMA 4.4. *Let σ and $\bar\sigma$ be any two points of E^n. There exists a constant $\hat{K}^{(m)} > 0$ such that*

$$|D^m(G^{\bar\sigma} - G^\sigma)| \leqq \hat{K}^{(m)}[\varepsilon(y-\eta)]^{-(n+m)/2}|\bar\sigma - \sigma|^\gamma \exp\{-l|z|^2/8\varepsilon(y-\eta)\}$$

uniformly in $R \times R$ for $y > \eta$ and $\varepsilon > 0$, where $G^{\bar\sigma}$ and G^σ are evaluated at $(x, y; \xi, \eta; \varepsilon)$.

Proof. In the notation of (4.20) we can write

$$
\begin{aligned}
D^m(G^{\bar\sigma} - G^\sigma) = {} & [h(\bar\sigma) - h(\sigma)]e^{-(1/4\varepsilon)z^TF(\bar\sigma)z}\sum_{j=0}^{[m/2]}(2\varepsilon)^{j-m}\sum f^{m-j}(\bar\sigma)z^{m-2j} \\
&+ h(\sigma)[e^{-(1/4\varepsilon)z^TF(\bar\sigma)z} - e^{-(1/4\varepsilon)z^TF(\sigma)z}]\sum_{j=0}^{[m/2]}(2\varepsilon)^{j-m}\sum f^{m-j}(\bar\sigma)z^{m-2j} \\
&+ G^\sigma\sum_{j=0}^{[m/2]}(2\varepsilon)^{j-m}\sum[f^{m-j}(\bar\sigma) - f^{m-j}(\sigma)]z^{m-2j},
\end{aligned}
$$

where $h(\bar\sigma) = h(y; \bar\sigma, \eta; \varepsilon)$, etc. From (4.8), Lemma 3.1, and (3.7) there exists a constant $k_1 > 0$ such that

$$|c_{\alpha\beta}(\bar\sigma) - c_{\alpha\beta}(\sigma)| \leqq |C(\bar\sigma) - C(\sigma)| \leqq k_1(y-\eta)|\bar\sigma - \sigma|^\gamma, \tag{4.25}$$

and from this together with (4.22) we have

$$
\begin{aligned}
|\overset{-1}{c}_{\alpha\beta}(\bar\sigma) - \overset{-1}{c}_{\alpha\beta}(\sigma)| &\leqq |C^{-1}(\bar\sigma) - C^{-1}(\sigma)| \\
&\leqq |C^{-1}(\bar\sigma)||C(\sigma) - C(\bar\sigma)||C^{-1}(\sigma)| \leqq k_2(y-\eta)^{-1}|\bar\sigma - \sigma|^\gamma.
\end{aligned}
$$

In view of (4.16′) and Lemma 3.1 we have for some constant $k_3 > 0$

$$|f_{\alpha\beta}(\bar\sigma) - f_{\alpha\beta}(\sigma)| \leqq |F(\bar\sigma) - F(\sigma)| \leqq k_3(y-\eta)^{-1}|\bar\sigma - \sigma|^\gamma, \tag{4.26}$$

which implies that there is a constant $k_1^{(m)}$ such that

$$\left| \sum_{j=0}^{[m/2]} (2\varepsilon)^{j-m} \sum [f^{m-j}(\bar{\sigma}) - f^{m-j}(\sigma)] z^{m-2j} \right| \leqq k_1^{(m)} | \bar{\sigma} - \sigma |^{\gamma} \sum_{j=0}^{[m/2]} | z |^{m-2j} [\varepsilon(y - \eta)]^{-m+j}.$$

By the Theorem of the Mean, (4.26), and (4.19) we obtain

$$| e^{-(1/4\varepsilon) z^T F(\bar{\sigma}) z} - e^{-(1/4\varepsilon) z^T F(\sigma) z} | \leqq k_4 | z |^2 | \bar{\sigma} - \sigma |^{\gamma} [\varepsilon(y - \eta)]^{-1} \exp \{ -l | z |^2 / 4\varepsilon(y - \eta) \}.$$

Again, by the Theorem of the Mean, taking into account the boundedness of b, (4.14), and (4.25), we have

$$| h(\bar{\sigma}) - h(\sigma) | \leqq k_5 [\varepsilon(y - \eta)]^{-n/2} \int_{\eta}^{y} | b(\nu; \bar{\sigma}, \eta) - b(\nu; \sigma, \eta) | \, d\nu$$

$$+ k_6 \, \varepsilon^{-n/2} (y - \eta)^{-3n/2} | \det C(\bar{\sigma}) - \det C(\sigma) | \leqq k_7 [\varepsilon(y - \eta)]^{-n/2} | \bar{\sigma} - \sigma |^{\gamma},$$

where k_5, k_6, $k_7 > 0$ are constants. By using these results together with some of the estimates obtained in the proof of Lemma 4.2, it is easy to show that

$$| D^m (G^{\bar{\sigma}} - G^{\sigma}) | \leqq k_2^{(m)} | \bar{\sigma} - \sigma |^{\gamma} \sum_{j=0}^{[m/2]} \{ | z |^{m-2j} [\varepsilon(y - \eta)]^{-n/2-m+j} + | z |^{m+2-2j} [\varepsilon(y - \eta)]^{-n/2-m-1+j} \} \exp \{ -l | z |^2 / 4\varepsilon(y - \eta) \},$$

and the result follows from Lemma 3.3.

We conclude this section by considering briefly integrals of the form

$$\mathcal{G}(x, y; \eta; \varepsilon) = \int G(x, y; \xi, \eta; \varepsilon) g(\xi) \, d\xi .$$

In particular, we show that (2.5) and (2.8) hold for bounded Hölder continuous g. More general results, for unbounded continuous g, will be given in §6. The essential property of G is the following.

LEMMA 4.5. *Let*

$$\mathcal{G}^*(x, y; \eta; \varepsilon) = \int G(x, y; \xi, \eta; \varepsilon) \, d\xi ;$$

then $\mathcal{G}^*$ *converges uniformly in* $R \times I$ *for* $y > \eta$, $\varepsilon > 0$, *and*

$$\mathcal{G}^*(x, y; \eta; \varepsilon) = \exp \int_{\eta}^{y} b(\nu; x, y) \, d\nu + O\{ [\varepsilon(y - \eta)]^{\gamma/2} \}$$

uniformly in $R \times I$.

Proof. Let $\sigma = \phi(\eta; x, y)$, and write $\mathcal{G}^*$ in the form

$$\mathcal{G}^* = \int G^{\sigma} \, d\xi + \int (G - G^{\sigma}) \, d\xi = \mathcal{G}_1^* + \mathcal{G}_2^* .$$

From (4.12) and (4.16′) we have that for $y > \eta$

$$F(\sigma) = \Phi_\xi^T(\sigma)M^{-T}(\sigma)M^{-1}(\sigma)\Phi_\xi(\sigma),$$

where $F(\sigma) = F(y; \sigma, \eta)$, etc. Let

$$\alpha = -(y-\eta)^{1/2}M^{-1}(\sigma)\Phi_\xi(\sigma)z(x, y; \xi, \eta); \tag{4.27}$$

then $|\alpha|^2 = (y-\eta)z^TFz$, and, by Lemma 3.1,

$$\det\left(\frac{\partial\alpha_i}{\partial\xi_j}\right) = (y-\eta)^{n/2}[\det C(\sigma)]^{-1/2} \cdot \exp\int_\eta^y \{\operatorname{tr} B(\nu; \sigma, \eta) - \operatorname{tr} B(\nu; \xi, \eta)\}\, d\nu .$$

Thus, for $y > \eta$ and $\varepsilon > 0$,

$$\mathcal{I}_1^* = 2^{-n}\pi^{-n/2}\exp\left\{\int_\eta^y b(\nu; \sigma, \eta)\, d\nu\right\}\left[\int [\varepsilon(y-\eta)]^{-n/2}\exp\left\{-\frac{|\alpha|^2}{4\varepsilon(y-\eta)}\right\}d\alpha + \int T[\varepsilon(y-\eta)]^{-n/2}\exp\left\{-\frac{|\alpha|^2}{4\varepsilon(y-\eta)}\right\}d\alpha\right] = \mathcal{I}_{11}^* + \mathcal{I}_{12}^*,$$

where

$$T = \exp\left\{-\int_\eta^y \operatorname{tr} B(\nu; \sigma, \eta)\, d\nu\right\} \cdot \left[\exp\int_\eta^y \operatorname{tr} B(\nu; \hat{\xi}, \eta)\, d\nu - \exp\int_\eta^y \operatorname{tr} B(\nu; \sigma, \eta)\, d\nu\right],$$

and from (4.27) by virtue of the uniqueness of the characteristics of L_0

$$\hat{\xi} = \phi(\eta; x + \phi_\xi(\sigma)M(\sigma)\alpha(y-\eta)^{-1/2}, y).$$

It follows from Lemmas 3.1, 3.2 and from (4.15) that there exists a constant $k_1 > 0$ such that

$$|\hat{\xi} - \sigma| \leqq n^{1/2}e^{P(y''-y')}|\phi_\xi(\sigma)|\,|M(\sigma)|(y-\eta)^{-1/2}|\alpha| \leqq k_1|\alpha|.$$

Hence by (3.7) and the boundedness of B, there exists a constant $k_2 > 0$ such that

$$|T| \leqq k_2|\alpha|^\gamma \exp\left\{-\int_\eta^y \operatorname{tr} B(\nu; \sigma, \eta)\, d\nu\right\}$$

for all α. Thus by the change of variables

$$\beta = \tfrac{1}{2}[\varepsilon(y-\eta)]^{-1/2}\alpha$$

we obtain

$$\mathcal{I}_{11}^* = \exp\int_\eta^y b(\nu; \sigma, \eta)\, d\nu \quad\text{and}\quad |\mathcal{I}_{12}^*| \leqq 2^\gamma k_2[\varepsilon(y-\eta)]^{\gamma/2}. \tag{4.28}$$

In view of Lemmas 3.2 and 4.4 there exists a constant $k_3 > 0$ such that

$$| G - G^{\sigma} | \leqq k_3[\varepsilon(y - \eta)]^{-n/2} | z |^{\gamma} \exp \{ -l | z |^2/8\varepsilon(y - \eta)\}$$

uniformly in $R \times R$ for $y > \eta$ and $\varepsilon > 0$. Hence

$$| \mathcal{G}_2^* | \leqq k_3 \int [\varepsilon(y - \eta)]^{-n/2} | z |^{\gamma} \exp \{ - l | z |^2/8\varepsilon(y - \eta) \} \, d\xi.$$

Let

$$\alpha = [l/8\varepsilon(y - \eta)]^{1/2} z;$$

then, by Lemma 3.1,

$$| \mathcal{G}_2^* | \leqq k_3 \left(\frac{8}{l}\right)^{(n+\gamma)/2} [\varepsilon(y - \eta)]^{\gamma/2} \int \det \Phi_{\xi}(y; \xi, \eta) e^{-|\alpha|^2} \, d\alpha \leqq k_4[\varepsilon(y - \eta)]^{\gamma/2}.$$

Since $b(\nu; \sigma, \eta) \equiv b(\nu; x, y)$, this estimate together with (4.28) completes the proof of Lemma 4.5.

From Lemma 4.5 we have immediately

COROLLARY 4.5. *If $g(x)$ belongs to class $H(\gamma'; x; E^n)$, then*

$$\mathcal{G}(x, y; \eta; \varepsilon) = g(\phi(\eta; x, y)) \exp \int_{\eta}^{y} b(\nu; x, y) \, d\nu + O\{[\varepsilon(y - \eta)]^{\hat{\gamma}/2}\}$$

uniformly in $R \times I$ for $y > \eta$ and $\varepsilon > 0$, where $\hat{\gamma} = \min(\gamma, \gamma')$.

Proof. Let $\sigma = \phi(\eta; x, y)$, and write

$$\mathcal{G} = g(\sigma)\mathcal{G}^* + \int [g(\xi) - g(\sigma)] G \, d\xi = g\mathcal{G}^* + \mathcal{G}_1 .$$

Since g belongs to class $H(\gamma'; x; E^n)$, we have, by Lemma 3.2,

$$| g(\xi) - g(\sigma) | \leqq H' | \xi - \sigma |^{\gamma'} \leqq k_5 | z |^{\gamma'}$$

for some constant $k_5 > 0$. Therefore, $\mathcal{G}_1$ can be estimated in the same way as $\mathcal{G}_2^*$ in the proof of the lemma, and the corollary follows easily.

5. Construction of the fundamental solution

Let

$$\mathcal{A}(\alpha, \beta, r) = \varepsilon^{-n/2+\alpha}(y - \eta)^{-(n+2)/2+\beta} \exp \{ -r\Delta^2(x, y; \xi, \eta)/\varepsilon(y - \eta)\},$$

where α, β, r are constants and

$$\Delta(x, y; \xi, \eta) = | \phi(y'; x, y) - \phi(y'; \xi, \eta) |.$$

Our main result concerning the f.s. of (1.8) is

THEOREM I. *If ($\mathcal{H}$) holds, then for all (x, y), (ξ, η) in R such that $y > \eta$, and for all $\varepsilon > 0$, (1.8) has a f.s. which can be written in the form*

$$\Gamma(x, y; \xi, \eta; \varepsilon) = G(x, y; \xi, \eta; \varepsilon) + \int_\eta^y d\tau \int G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon)\, ds, \tag{5.1}$$

where G is given by (4.16) *and* ψ *is the solution of the integral equation*

$$\psi(x, y; \xi, \eta; \varepsilon) = L_\varepsilon[G(x, y; \xi, \eta; \varepsilon)] + \int_\eta^y d\tau \int L_\varepsilon[G(x, y; s, \tau; \varepsilon)]\psi(s, \tau; \xi, \eta; \varepsilon)\, ds. \tag{5.2}$$

For $m = 0, 1, 2$, $D^m\Gamma$ *and* $\partial\Gamma/\partial y$ *are uniformly continuous in* $R \times R$ *for* $y > \eta$ *and* $\varepsilon > 0$. *Moreover, there exist constants* $Q^{(m)}$, $\tilde{Q} > 0$ *such that for* $m = 0$, 1, 2

$$| D^m\Gamma(x, y; \xi, \eta; \varepsilon) | \leqq Q^{(m)}\mathfrak{A}(-m/2, (2 - m)/2, \hat{r}/8) \quad \text{and} \quad | (\partial/\partial y)\Gamma(x, y; \xi, \eta; \varepsilon) | \leqq \tilde{Q}\mathfrak{A}(-\tfrac{1}{2}, 0, \hat{r}/8), \tag{5.3}$$

where $\hat{r} = (l/8) \exp\{-2P(y'' - y')\}$.

The proof of Theorem I is contained in the following lemmas. We begin by briefly sketching the sequence of ideas. If ψ is such that L_ε can be applied to Γ, (5.1), then the requirement that Γ be a solution of (1.8) leads to the equation (5.2) for ψ. Conversely, if ψ is a solution of (5.2), and if Γ admits the required derivatives, then Γ satisfies (1.8). Since (5.2) is of Volterra type, it has at most one solution. We show, by the method of successive approximations, that (5.2) does have a solution. If ψ were differentiable, it would be quite simple to prove that $L_\varepsilon(\Gamma)$ exists. Under our hypothesis, it is not generally the case that ψ has any derivatives. However, ψ does satisfy a Hölder condition with respect to x. Following Pogorzelski [10], we prove that the Hölder continuity of ψ is sufficient for the existence of $L_\varepsilon(\Gamma)$. It then follows that $L_\varepsilon(\Gamma) = 0$. Finally, we show that (2.1) holds for suitable initial functions g, and hence Γ is a f.s. of (1.8).

In order to carry out this program we will need the following preliminary result.

LEMMA 5.1. *Let* $w_1(x, y; \xi, \eta; \varepsilon)$ *be a continuous function of* ξ, η, *and let* $w_2(x, y; \xi, \eta; \varepsilon)$ *be a continuous function of* x, y *in* $R \times R$ *for* $y > \eta$ *and* $\varepsilon > 0$. *If for* $i = 1, 2$ *there are constants* α_i, β_i, $k_i > 0$, *and* $r > 0$ *such that*

$$| w_i | \leqq k_i\, \mathfrak{A}(\alpha_i, \beta_i, r),$$

then

$$W^*(x, y; \xi, \eta; \tau; \varepsilon) = \int w_1(x, y; s, \tau; \varepsilon)w_2(s, \tau; \xi, \eta; \varepsilon)\, ds$$

converges uniformly in $R \times R \times I$ *for* $\eta < \tau < y$ *and* $\varepsilon > 0$, *and there exists a constant* $k_3 > 0$ *such that*

$$| W^*(x, y; \xi, \eta; \tau; \varepsilon) | \leqq k_3(y - \tau)^{-1+\beta_1}(\tau - \eta)^{-1+\beta_2}\mathfrak{A}(\alpha_1 + \alpha_2, 1, r). \tag{5.4}$$

If, in addition, $\beta_i > 0$ *for* $i = 1, 2$, *then*

$$W(x, y; \xi, \eta; \varepsilon) = \int_\eta^y W^*(x, y; \xi, \eta; \tau; \varepsilon)\, d\tau$$

converges uniformly in $R \times R$ *for* $y > \eta$ *and* $\varepsilon > 0$, *and*

$$(5.5)\quad \begin{aligned} | W(x, y; \xi, \eta; \varepsilon) | \leqq k_3\, \Gamma(\beta_1)\Gamma(\beta_2)\{\Gamma(\beta_1 + \beta_2)\}^{-1} \\ \cdot\, \mathfrak{A}(\alpha_1 + \alpha_2 , \beta_1 + \beta_2 , r).\end{aligned}$$[6]

Proof. Let $\sigma = \phi(y'; s, \tau)$; then in view of Lemma 3.1

$$\det\,(\partial\sigma_i/\partial s_j) = \det \phi_\xi(y'; s, \tau) \geqq e^{-nN(y''-y')} = k_4^{-1} > 0.$$

Thus, using the bounds for the w_i and introducing σ as variable of integration, we obtain

$$| W^* | \leqq k_1\, k_2\, k_4\, \varepsilon^{-n+\alpha_1+\alpha_2}(y - \tau)^{-(n+2)/2+\beta_1}(\tau - \eta)^{-(n+2)/2+\beta_2} \cdot \int \exp\left\{ - \frac{r}{\varepsilon}\, Z(x, y; \xi, \eta; \sigma, \tau)\right\} d\sigma,$$

where for $\eta < \tau < y$

$$Z(x, y; \xi, \eta; \sigma, \tau) = \frac{| \phi(y'; x, y) - \sigma |^2}{y - \tau} + \frac{| \phi(y'; \xi, \eta) - \sigma |^2}{\tau - \eta} \geqq \frac{\Delta^2(x, y; \xi, \eta)}{y - \eta},$$

the bound being achieved if and only if

$$\sigma = \left(\frac{\tau - \eta}{y - \eta}\right) \phi(y'; x, y) + \left(\frac{y - \tau}{y - \eta}\right) \phi(y'; \xi, \eta).$$

Let

$$\mu = \left(\frac{r}{\varepsilon}\right)^{1/2} \left\{\left[\frac{y - \eta}{(y - \tau)(\tau - \eta)}\right]^{1/2} [\sigma - \phi(y'; \xi, \eta)] + \left[\frac{\tau - \eta}{(y - \tau)(y - \eta)}\right]^{1/2} [\phi(y'; \xi, \eta) - \phi(y'; x, y)]\right\};$$

then

$$| \mu |^2 = \frac{r}{\varepsilon}\left\{ Z(x, y; \xi, \eta; \sigma, \tau) - \frac{\Delta^2(x, y; \xi, \eta)}{y - \eta}\right\}$$

$$\text{and}\quad \det\left(\frac{\partial\mu_i}{\partial\sigma_j}\right) = \left[\frac{r(y - \eta)}{\varepsilon(y - \tau)(\tau - \eta)}\right]^{n/2}.$$

With μ as variable of integration in the above bound for W^* we obtain (5.4), where

$$k_3 = k_1\, k_2\, k_4\, r^{-n/2} \int e^{-|\mu|^2}\, d\mu = k_1\, k_2\, k_4 \left(\frac{\pi}{r}\right)^{n/2}.$$

[6] The symbol Γ with a single argument refers to the Euler Gamma Function.

If $\beta_i > 0$ for $i = 1, 2$, then it is clear that W converges uniformly for $y > \eta$, $\varepsilon > 0$, and (5.5) follows by integrating (5.4).

The existence of the solution of (5.2) is an easy consequence of Lemma 5.1. We have

LEMMA 5.2. *The solution* $\psi(x, y; \xi, \eta; \varepsilon)$ of (5.2) *exists and is uniformly continuous in* $R \times R$ *for* $y > \eta$ *and* $\varepsilon > 0$. *Moreover, there exists a constant* $Q > 0$ *such that* $|\psi| \leqq Q\mathfrak{A}(\gamma/2, \gamma/2, \hat{r})$, *where* $\hat{r} = (l/8) \exp\{-2P(y'' - y')\}$.

Proof. It is well known that if a solution of (5.2) exists, then it is unique and is given by

$$\psi(x, y; \xi, \eta; \varepsilon) = \sum_{m=0}^{\infty} \chi_m(x, y; \xi, \eta; \varepsilon), \tag{5.6}$$

where

$$\chi_0(x, y; \xi, \eta; \varepsilon) = L_\varepsilon[G(x, y; \xi, \eta; \varepsilon)],$$

$$\chi_m(x, y; \xi, \eta; \varepsilon) = \int_\eta^y d\tau \int \chi_0(x, y; s, \tau; \varepsilon)\chi_{m-1}(s, \tau; \xi, \eta; \varepsilon)\, ds \qquad (m \geqq 1), \tag{5.6$'$}$$

wherever the series (5.6) is uniformly and absolutely convergent. For any (x, y), (ξ, η) in R, we have from (3.5) that

$$|x - \phi(y; \xi, \eta)|^2 \geqq e^{-2P(y''-y')}\Delta^2(x, y; \xi, \eta),$$

and hence, by Lemma 4.3, $|\chi_0| \leqq K\mathfrak{A}(\gamma/2, \gamma/2, \hat{r})$. It follows from ($\mathfrak{H}$) and Lemma 4.2 that $\chi_0 = L_\varepsilon[G]$ is uniformly continuous in $R \times R$ for $y > \eta$ and $\varepsilon > 0$. It is easy to show by induction, using (5.6′) and Lemma 5.1, that χ_m is uniformly continuous in $R \times R$ for $y > \eta$ and $\varepsilon > 0$, and that

$$|\chi_m(x, y; \xi, \eta; \varepsilon)| \leqq K\Gamma(\gamma/2)q^m\{\Gamma[(m+1)\gamma/2]\}^{-1} \cdot \mathfrak{A}((m+1)\gamma/2, (m+1)\gamma/2, \hat{r})$$

for $m \geqq 0$, where $q = k_4 K\Gamma(\gamma/2)(\pi/\hat{r})^{n/2}$. Thus for $y > \eta$ and $\varepsilon > 0$, the series (5.6) converges uniformly and absolutely. The asserted properties follow immediately from those of the χ_m in view of the nature of the convergence.

We remarked above that in order to prove the existence of $L_\varepsilon(\Gamma)$ we will need the Hölder continuity of ψ with respect to x. We now prove that ψ is Hölder continuous.

LEMMA 5.3. *If* ψ *is the solution of* (5.2), *then there exists a constant* $\hat{Q} > 0$ *such that*

$$|\psi(\bar{x}, y; \xi, \eta; \varepsilon) - \psi(x, y; \xi, \eta; \varepsilon)| \leqq \hat{Q}|\bar{x} - x|^{\gamma/2}\varepsilon^{-n/2+\gamma/4} \cdot [\exp\{-\hat{r}\Delta^2(\bar{x}, y; \xi, \eta)/8\varepsilon(y-\eta)\} + \exp\{-\hat{r}\Delta^2(x, y; \xi, \eta)/8\varepsilon(y-\eta)\}]$$

in $R \times R$ *for* $y > \eta$ *and* $\varepsilon > 0$.

Proof. Consider

$$\psi(\bar{x}, y; \xi, \eta; \varepsilon) - \psi(x, y; \xi, \eta; \varepsilon) = \chi_0(\bar{x}, y; \xi, \eta; \varepsilon) - \chi_0(x, y; \xi, \eta; \varepsilon)$$
$$+ \int_\eta^y d\tau \int \{\chi_0(\bar{x}, y; s, \tau; \varepsilon) - \chi_0(x, y; s, \tau; \varepsilon)\}\psi(s, \tau; \xi, \eta; \varepsilon)\, ds$$
$$= \delta(\bar{x}, x, y; \xi, \eta; \varepsilon) + \int_\eta^y d\tau \int \delta(\bar{x}, x, y; s, \tau; \varepsilon)\, \psi(s,\tau;\xi,\eta;\varepsilon)\, ds$$

for fixed $y > \eta$ and $\varepsilon > 0$. Suppose that

$$(5.7) \qquad |x - \phi(y; \xi, \eta)| \leqq |\bar{x} - \phi(y; \xi, \eta)| \quad \text{and} \quad 2|\bar{x} - x| \leqq |\bar{x} - \phi(y; \xi, \eta)|,$$

and write $\delta = \delta_1 + \delta_2$, where

$$\delta_1(\bar{x}, x, y; \xi, \eta; \varepsilon) = \varepsilon\{a_{jk}(\bar{x}, y) - a_{jk}(x, y)\}G_{,jk}(\bar{x})$$
$$+ \{a_j(\bar{x}, y) - a_j(x, y) - (\bar{x}_k - x_k)a_{j,k}(y; \xi, \eta)\}G_{,j}(\bar{x})$$
$$+ \{b(\bar{x}, y) - b(x, y)\}G(\bar{x}),$$

$$\delta_2(\bar{x}, x, y; \xi, \eta; \varepsilon) = \varepsilon\{a_{jk}(x, y) - a_{jk}(y; \xi, \eta)\}\{G_{,jk}(\bar{x}) - G_{,jk}(x)\}$$
$$+ \{a_j(x, y) - a_j(y; \xi, \eta) - z_k\, a_{j,k}(y; \xi, \eta)\}\{G_{,j}(\bar{x}) - G_{,j}(x)\}$$
$$+ \{b(x, y) - b(y; \xi, \eta)\}\{G(\bar{x}) - G(x)\},$$

and $G(\bar{x}) = G(\bar{x}, y; \xi, \eta; \varepsilon)$. It follows from $(\mathcal{H})$ and Lemma 4.2 that

$$|\delta_1| \leqq \left\{\frac{\varepsilon H K^{(2)} |\bar{x} - x|^\gamma}{[\varepsilon(y - \eta)]^{(n+2)/2}} + \frac{K^{(1)}}{[\varepsilon(y - \eta)]^{(n+1)/2}} \sum_k |\bar{x}_k - x_k|\, |a_{j,k}(\tilde{x}, y) - a_{j,k}(y; \xi, \eta)| + \frac{H K^{(0)} |\bar{x} - x|^\gamma}{[\varepsilon(y - \eta)]^{n/2}}\right\} \exp\left\{-\frac{l|\bar{z}|^2}{8\varepsilon(y - \eta)}\right\},$$

where $\bar{z} = \bar{x} - \phi(y; \xi, \eta)$ and $\tilde{x} = \theta x + (1 - \theta)\bar{x}$ for some $0 < \theta < 1$. Since $|\tilde{x} - \phi(y; \xi, \eta)| = |\bar{x} - \phi + \theta(x - \bar{x})| \leqq |\bar{x} - \phi| + |x - \bar{x}| \leqq 2|\bar{x} - \phi|$, we have, in view of $(\mathcal{H}_3)$,

$$|a_{j,k}(\tilde{x}, y) - a_{j,k}(y; \xi, \eta)| \leqq H|\tilde{x} - \phi|^\gamma \leqq 2^\gamma H |\bar{x} - \phi|^\gamma.$$

Using this together with (5.7) and Lemma 3.3, we obtain

$$|\delta_1| \leqq k_1 |\bar{x} - x|^{\gamma/2}\left\{\frac{\varepsilon|\bar{z}|^{\gamma/2}}{[\varepsilon(y - \eta)]^{(n+2)/2}} + \frac{|\bar{z}|^{1+\gamma/2}}{[\varepsilon(y - \eta)]^{(n+1)/2}} + \frac{|\bar{z}|^{\gamma/2}}{[\varepsilon(y - \eta)]^{n/2}}\right\} \exp\left\{-\frac{l|\bar{z}|^2}{8\varepsilon(y - \eta)}\right\}$$
$$\leqq k_2 |\bar{x} - x|^{\gamma/2} \varepsilon^{-n/2+\gamma/4}(y - \eta)^{-(n+2)/2+\gamma/4} \exp\left\{-\frac{l|\bar{z}|^2}{16\varepsilon(y - \eta)}\right\}$$

for some constants $k_1, k_2 > 0$.

For any $m \geqq 0$ we have from the Theorem of the Mean

$$D^m G(\bar{x}) - D^m G(x) = \textstyle\sum_{i=1}^n (\bar{x}_i - x_i)(\partial/\partial x_i) D^m G(x^{(m)}),$$

where $x^{(m)} = \theta_m x + (1 - \theta_m)\bar{x}$ for some $0 < \theta_m < 1$. Thus, in view of ($\mathcal{H}$) and Lemma 4.2, there exists a constant $k_3 > 0$ such that

$$\begin{aligned} | \delta_2 | \leqq k_3 | \bar{x} - x | \Bigg[& \frac{\varepsilon | z |^\gamma}{[\varepsilon(y - \eta)]^{(n+3)/2}} \sum_{j,k} \exp\left\{ -\frac{l | x^{(jk)} - \phi |^2}{8\varepsilon(y-\eta)} \right\} \\ & + \frac{| z |^{1+\gamma}}{[\varepsilon(y - \eta)]^{(n+2)/2}} \sum_j \exp\left\{ -\frac{l | x^{(j)} - \phi |^2}{8\varepsilon(y - \eta)} \right\} \\ & + \frac{| z |^\gamma}{[\varepsilon(y - \eta)]^{(n+1)/2}} \exp\left\{ -\frac{l | x^{(0)} - \phi |^2}{8\varepsilon(y - \eta)} \right\} \Bigg]. \end{aligned}$$

It follows from (5.7) that

$$\begin{aligned} | x^{(m)} - \phi(y; \xi, \eta) | &= | \bar{x} - \phi + \theta_m(x - \bar{x}) | \\ &\geqq | \bar{x} - \phi | - \theta_m | \bar{x} - x | \geqq \tfrac{1}{2} | \bar{z} | \geqq | \bar{x} - x |. \end{aligned}$$

Together with $| z | \leqq | \bar{z} |$ and Lemma 3.3 this implies

$$\begin{aligned} | \delta_2 | &\leqq k_4 | \bar{x} - x |^{\gamma/2} \left\{ \frac{\varepsilon | \bar{z} |^{1+\gamma/2}}{[\varepsilon(y - \eta)]^{(n+3)/2}} + \frac{| \bar{z} |^{2+\gamma/2}}{[\varepsilon(y - \eta)]^{(n+2)/2}} \right. \\ &\qquad \left. + \frac{| \bar{z} |^{1+\gamma/2}}{[\varepsilon(y - \eta)]^{(n+1)/2}} \right\} \exp\left\{ -\frac{l | \bar{z} |^2}{32\, \varepsilon(y - \eta)} \right\} \\ &\leqq k_5 | \bar{x} - x |^{\gamma/2} \varepsilon^{-n/2+\gamma/4} (y - \eta)^{-(n+2)/2+\gamma/4} \exp\left\{ -\frac{l | \bar{z} |^2}{64\, \varepsilon(y - \eta)} \right\}, \end{aligned}$$

where k_4, $k_5 > 0$ are constants.

On the other hand, suppose that instead of (5.7) we have

$$(5.7') \qquad | x - \phi(y; \xi, \eta) | \leqq | \bar{x} - \phi(y; \xi, \eta) | \leqq 2 | \bar{x} - x |.$$

Then it follows from (4.24) that for some constant $k_6 > 0$

$$\begin{aligned} | \chi_0(\bar{x}) | &\leqq k_6 | \bar{x} - x |^{\gamma/2} \\ &\cdot \left\{ \frac{\varepsilon | \bar{z} |^{\gamma/2}}{[\varepsilon(y - \eta)]^{(n+2)/2}} + \frac{\varepsilon | \bar{z} |^{2+\gamma/2}}{[\varepsilon(y - \eta)]^{(n+4)/2}} + \frac{| \bar{z} |^{2+\gamma/2}}{[\varepsilon(y - \eta)]^{(n+2)/2}} + \frac{| \bar{z} |^{\gamma/2}}{[\varepsilon(y - \eta)]^{n/2}} \right\} \\ &\qquad \cdot \exp\left\{ -\frac{l | \bar{z} |^2}{4\varepsilon(y - \eta)} \right\}. \end{aligned}$$

Since $| z | \leqq 2 | \bar{x} - x |$, the same estimate holds for $\chi_0(x)$ with $\bar{z}$ replaced by z. Thus, by Lemma 3.3, we obtain

$$\begin{aligned} | \delta(\bar{x}, x, y; \xi, \eta; \varepsilon) | &\leqq k_7 | \bar{x} - x |^{\gamma/2} \varepsilon^{-n/2+\gamma/4} (y - \eta)^{-(n+2)/2+\gamma/4} \\ &\qquad \cdot [\exp\{ -l | \bar{z} |^2/8\varepsilon(y - \eta) \} + \exp\{ -l | z |^2/8\varepsilon(y - \eta) \}] \end{aligned}$$

whenever (5.7′) holds. This, together with the estimates for δ_1 and δ_2 when (5.7) holds, yields the complete estimate for δ in the half-space $|z| \leqq |\bar{z}|$. It is clear that, by interchanging the roles of $\bar{x}$ and x, the above argument can be carried through for $|\bar{z}| \leqq |z|$ with analogous results. Hence we have shown that for all $(\bar{x}, y)$, (x, y), (ξ, η) in R with $y > \eta$ and all $\varepsilon > 0$ there exists a constant $k_8 > 0$ such that

$$|\delta(\bar{x}, x, y; \xi, \eta; \varepsilon)| \leqq k_8 |\bar{x} - x|^{\gamma/2} \varepsilon^{-n/2+\gamma/4} (y - \eta)^{-(n+2)/2+\gamma/4} \cdot [\exp\{-l|\bar{z}|^2/64\varepsilon(y - \eta)\} + \exp\{-l|z|^2/64\varepsilon(y - \eta)\}]$$

In view of the fact that $|\psi| \leqq Q\mathfrak{A}(\gamma/2, \gamma/2, \hat{r})$, it follows from Lemma 5.1 that this estimate for δ also dominates

$$\int_\eta^y d\tau \int \delta\psi \, ds .$$

Thus the proof is complete.

We turn now to the question of the differentiability of Γ, i.e., of

$$V(x, y; \xi, \eta; \varepsilon) = \int_\eta^y d\tau \int G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon) \, ds.$$

For this purpose, we write V in the form

$$V(x, y; \xi, \eta; \varepsilon) = \int_\eta^y J(x, y; \xi, \eta; \tau; \varepsilon) \, d\tau ,$$

where

$$J(x, y; \xi, \eta; \tau; \varepsilon) = \int G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon) \, ds ,$$

and begin by examining the properties of J.

LEMMA 5.4. *The integrals*

$$J_m = \int D^m G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon) \, ds \qquad (m \geqq 0)$$

and

$$J_y = \int \frac{\partial}{\partial y} G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon) \, ds$$

converge uniformly, the functions J_m and J_y are uniformly continuous, and there exist constants $q^{(m)} > 0$, $\tilde{q} > 0$ such that

$$|J_m| \leqq q^{(m)} (y - \tau)^{-1+(2-m)/2} (\tau - \eta)^{-1+\gamma/2} \mathfrak{A}(\gamma/2 - m/2, 1, \hat{r})$$

and

$$|J_y| \leqq \tilde{q}(y - \tau)^{-1} (\tau - \eta)^{-1+\gamma/2} \mathfrak{A}(\gamma/2 - 1/2, 1, \hat{r})$$

in $R \times R \times I$ for $\eta < \tau < y$ and $\varepsilon > 0$. Moreover, $J_m = D^m J$ and $J_y = \partial J/\partial y$ for $\eta < \tau < y$ and $\varepsilon > 0$.

Proof. In view of Lemmas 4.2 and 5.2, D^mG, $\partial G/\partial y$, and ψ are uniformly continuous in $R \times R$ for $y > \eta$ and $\varepsilon > 0$, and satisfy

$$| D^mG | \leqq K^{(m)}\mathfrak{A}(-m/2, (2-m)/2, \hat{r}),$$

$$| \partial G/\partial y | \leqq \tilde{K}^{(0)}\mathfrak{A}(-\tfrac{1}{2}, 0, \hat{r}), \qquad | \psi | \leqq Q\mathfrak{A}(\gamma/2, \gamma/2, \hat{r}).$$

Thus the convergence, continuity, and bounds for J_m and J_y follow immediately from Lemma 5.1. Since J_m is uniformly convergent and uniformly continuous, and since J_{m-1} is convergent, it follows by a standard argument that $J_m = D^mJ$ for $m \geqq 1$, and similarly for $J_y = \partial J/\partial y$.

An immediate consequence of Lemmas 5.1 and 5.4 is the fact that for $m = 0, 1$ the integrals

$$V_m = \int_\eta^y D^mJ(x, y; \xi, \eta; \tau; \varepsilon)\, d\tau$$

converge uniformly to a uniformly continuous function in $R \times R$ for $y > \eta$ and $\varepsilon > 0$, where for some constants $\bar{q}^{(m)} > 0$ we have

$$| V_m | \leqq \bar{q}^{(m)}\mathfrak{A}(\gamma/2 - m/2, \gamma/2 + (2-m)/2, \hat{r}).$$

Moreover, it follows that $DV = V_1$. In order to extend this result to the case $m = 2$ (and for V_y) it is necessary to show that D^2J has a uniformly convergent integral on $\eta < \tau < y$, i.e., obtain a sharper estimate of D^2J than the one afforded by Lemma 5.4. To accomplish this we must use the Hölder continuity of ψ with respect to x and of D^2G^σ with respect to σ. The method we employ is due essentially to Pogorzelski [10].

LEMMA 5.5. *For $\eta < \tau < y$ and $\varepsilon > 0$ there exist constants $q_1^{(2)}, q_2^{(2)} > 0$ such that*

$$| D^2J | \leqq \begin{cases} q_1^{(2)}(\tau - \eta)^{-1+\gamma/2}\mathfrak{A}(\gamma/2 - 1, 0, \hat{r}) & \text{for} \quad \eta < \tau \leqq (y+\eta)/2 \\ q_2^{(2)}(y - \tau)^{-1+\gamma/4}\mathfrak{A}(\gamma/2 - 1, \gamma/4, \hat{r}/8) & \text{for} \quad (y+\eta)/2 \leqq \tau < y. \end{cases}$$

Proof. It follows from Lemma 5.4 that

$$| D^2J | \leqq q^{(2)}(y - \tau)^{-1}(\tau - \eta)^{-1+\gamma/2}\mathfrak{A}(\gamma/2 - 1/2, 1, \hat{r})$$

for $\eta < \tau < y$ and $\varepsilon > 0$. If we restrict τ to $\eta < \tau \leqq (\eta + y)/2$, we have $(y - \tau)^{-1} \leqq 2(y - \eta)^{-1}$, and the corresponding bound follows.

Now consider $(\eta + y)/2 \leqq \tau < y$. For any (x, y) in R choose α in E^n such that $| \alpha - \phi(y', x, y) | \leqq \tfrac{1}{2}$, and let $S(\tau)$, for any τ in I, denote the set of points (s, τ) which satisfy $| \alpha - \phi(y'; s, \tau) | \leqq 1$. Let $(\sigma(\tau), \tau)$ be an arbitrary interior point of $S(\tau)$, and write DJ in the form

$$DJ = \psi(\sigma, \tau)\left[\int_{S(\tau)} DG^\sigma\, ds + \int_{S(\tau)} D(G - G^\sigma)\, ds + \int_{E^n - S(\tau)} DG\, ds\right]$$
$$+ \int [\psi(s, \tau) - \psi(\sigma, \tau)]\, DG\, ds,$$

where $\psi(\sigma, \tau) = \psi(\sigma, \tau; \xi, \eta; \varepsilon)$, etc. Note that

$$DG^\sigma(x, y; s, \tau; \varepsilon) = \frac{\partial}{\partial x_i} G^\sigma = -\frac{\partial}{\partial s_i} G^\sigma + \frac{1}{2\varepsilon}\left(\frac{\partial \phi}{\partial s_i} - e_i\right)^T F(\sigma) z G^\sigma,$$

where $F(\sigma) = F(y; \sigma, \tau)$. For each τ in I, $S(\tau)$ is the image of the unit sphere $S(y')$ under the mapping induced by the characteristics of L_0, and, in view of Lemma 3.1, this mapping is one-to-one and continuously differentiable. Thus if $\partial S(\tau)$ denotes the boundary of $S(\tau)$, we have by the Divergence Theorem

$$DJ = \psi(\sigma,\tau)\left[-\int_{\partial S(\tau)} G^\sigma \cos(n^*, i)\, ds^* + \frac{1}{2\varepsilon}\int_{S(\tau)}\left(\frac{\partial\phi}{\partial s_i} - e_i\right)^T F(\sigma) z G^\sigma\, ds \right.$$
$$\left. + \int_{S(\tau)} D(G - G^\sigma)\, ds + \int_{E^n - S(\tau)} DG\, ds\right] + \int [\psi(s,\tau) - \psi(\sigma,\tau)]\, DG\, ds,$$

where $\cos(n^*, i)$ is the cosine of the angle between the exterior normal to $\partial S(\tau)$ at s^* and the s_i-axis. Since $(\eta + y)/2 \leqq \tau < y$ and $\varepsilon > 0$, we can differentiate DJ with respect to x and obtain

$$D^2 J = \psi(\sigma,\tau)\left[-\int_{\partial S(\tau)} DG^\sigma \cos(n^*, i)\, ds^* \right.$$
$$(5.8) \qquad + \frac{1}{2\varepsilon}\int_{S(\tau)}\left(\frac{\partial\phi}{\partial s_i} - e_i\right)^T F(e_j G^\sigma + z DG^\sigma)\, ds + \int_{S(\tau)} D^2(G - G^\sigma)\, ds$$
$$\left. + \int_{E^n - S(\tau)} D^2 G\, ds\right] + \int [\psi(s,\tau) - \psi(\sigma,\tau)]\, D^2 G\, ds,$$

where $(\sigma(\tau), \tau)$ is an arbitrary point in $S(\tau)$ for each τ. In particular, this representation of D^2J is valid when $\sigma(\tau) = \phi(\tau; x, y)$ for all τ in $(\eta + y)/2 \leqq \tau < y$. Thus we have

$$D^2 J = \int D^2 G\psi\, ds = \psi(\sigma,\tau)\sum_{j=1}^{4} I_j + I_5$$

where I_k denotes the k^{th} integral on the right-hand side of (5.8) with σ set equal to $\phi(\tau; x, y)$ after the indicated differentiations are performed.

For (s^*, τ) on $\partial S(\tau)$ we have, by Lemma 3.2, that

$$|z(x, y; s^*, \tau)| \geqq e^{-P(y''-y')}\Delta(x, y; s^*, \tau) \geqq \tfrac{1}{2} e^{-P(y''-y')} = \rho > 0.$$

Thus, in view of (4.18), there exists a constant $k_1 > 0$ such that

$$|I_1| \leqq K^{(1)}[\varepsilon(y - \tau)]^{-(n+1)/2} \exp\left\{-\frac{l\rho^2}{8\varepsilon(y - \tau)}\right\}\int_{\partial S(\tau)} ds^* \leqq k_1.$$

Similarly, since for (s, τ) in $E^n - S(\tau)$ we have $|z| \geqq \rho$,

$$|I_4| \leqq K^{(2)}[\varepsilon(y - \tau)]^{-1}\exp\left\{-\frac{l\rho^2}{16\varepsilon(y - \tau)}\right\}$$
$$\cdot \int [\varepsilon(y - \tau)]^{-n/2}\exp\left\{\frac{l\,|z|^2}{-16\varepsilon(y - \tau)}\right\} ds \leqq k_2.$$

To estimate I_2 we note that (3.2) implies

$$\left| \frac{\partial \phi}{\partial s_i}(y; s, \tau) - e_i \right| = \left| \frac{\partial \phi}{\partial s_i}(y; s, \tau) - \frac{\partial \phi}{\partial s_i}(\tau; s, \tau) \right| \leqq P \mid y - \tau \mid,$$

and hence, by 4.23 and Lemmas 4.2 and 3.3,

$$\left| \left(\frac{\partial \phi}{\partial s_i} - e_i \right)^T F(e_j G^\sigma + zDG^\sigma) \right| \leqq k_3 [\varepsilon(y - \tau)]^{-n/2} \exp\left\{ -\frac{l \mid z \mid^2}{16\varepsilon(y - \tau)} \right\},$$

where $k_3 > 0$ is constant. It follows that

$$\mid I_2 \mid \leqq k_4 \varepsilon^{-1}$$

for some constant $k_4 > 0$. Making use of Lemmas 4.4, 3.2, and 3.3, we find

$$\begin{aligned} \mid I_3 \mid &\leqq \hat{K}^{(2)} \int_{S(\tau)} \mid s - \sigma \mid^\gamma [\varepsilon(y - \tau)]^{-(n+2)/2} \exp\left\{ -\frac{l \mid z \mid^2}{8\varepsilon(y - \tau)} \right\} ds \\ &\leqq \hat{K}^{(2)} e^{\gamma P(y'' - y')} \int_{S(\tau)} \mid z \mid^\gamma [\varepsilon(y - \tau)]^{-(n+2)/2} \exp\left\{ -\frac{l \mid z \mid^2}{8\varepsilon(y - \tau)} \right\} ds \\ &\leqq k_5 [\varepsilon(y - \tau)]^{-1+\gamma/2} \end{aligned}$$

since $\sigma = \phi(\tau; x, y)$, where $k_5 > 0$ is a constant. Moreover, we have $\phi(y'; \sigma, \tau) = \phi(y'; x, y)$. Hence, in view of the fact that $(\eta + y)/2 \leqq \tau < y$, we obtain

$$\begin{aligned} \mid \psi(\sigma, \tau) \mid &\leqq Q \varepsilon^{-n/2+\gamma/2} (\tau - \eta)^{-(n+2)/2+\gamma/2} \exp\{ -\hat{r}\Delta^2(\sigma, \tau; \xi, \eta)/\varepsilon(\tau - \eta) \} \\ &\leqq Q \varepsilon^{-n/2+\gamma/2} [\tfrac{1}{2}(y - \eta)]^{-(n+2)/2+\gamma/2} \exp\{ -\hat{r}\Delta^2(x, y; \xi, \eta)/\varepsilon(y - \eta) \}. \end{aligned}$$

Together with the estimates for I_j, $j = 1, \cdots, 4$, this implies that there exists a constant $k_6 > 0$ such that

$$(5.9) \qquad \mid \psi(\sigma, \tau) \mid \left| \textstyle\sum_{j=1}^4 I_j \right| \leqq k_6 (y - \tau)^{-1+\gamma/2} \mathfrak{A}(\gamma/2 - 1, \gamma/2, \hat{r}).$$

According to Lemma 5.3 we have

$$\begin{aligned} \mid \psi(s, \tau) - \psi(\sigma, \tau) \mid &\leqq \hat{Q} \mid s - \sigma \mid^{\gamma/2} \varepsilon^{-n/2+\gamma/4} (\tau - \eta)^{-(n+2)/2+\gamma/4} \\ &\cdot [\exp\{ -\hat{r}\Delta^2(s, \tau; \xi, \eta)/8\varepsilon(\tau - \eta) \} + \exp\{ -\hat{r}\Delta^2(x, y; \xi, \eta)/8\varepsilon(\tau - \eta) \}] \\ &= T_1 + T_2, \end{aligned}$$

and by (3.5), (4.18), and Lemma 3.3 we have

$$\begin{aligned} \mid s - \sigma \mid^{\gamma/2} \mid D^2 G \mid &\leqq \mid z \mid^{\gamma/2} e^{(\gamma/2) P(y'' - y')} \mid D^2 G \mid \\ &\leqq k_7 [\varepsilon(y - \tau)]^{-(n+2)/2+\gamma/4} \exp\{ -l \mid z \mid^2 / 16\varepsilon(y - \tau) \} \\ &\leqq k_7 [\varepsilon(y - \tau)]^{-(n+2)/2+\gamma/4} \exp\{ -\hat{r}\Delta^2(x, y; s, \tau)/8\varepsilon(y - \tau) \} \end{aligned}$$

for some constant $k_7 > 0$. It follows from Lemma 5.1 and $(\eta + y)/2 \leqq$

$\tau < y$ that

$$\int |D^2G| \, T_1 \, ds \leqq k_8(y - \tau)^{-1+\gamma/4}(\tau - \eta)^{-1+\gamma/4}\mathcal{A}(\gamma/2 - 1, 1, \hat{r}/8)$$
$$\leqq 2k_8(y - \tau)^{-1+\gamma/4}\mathcal{A}(\gamma/2 - 1, \gamma/4, \hat{r}/8),$$

where $k_8 > 0$ is a constant. Moreover, we have

$$T_2 \leqq \hat{Q} \, | s - \sigma |^{\gamma/2}\varepsilon^{-n/2+\gamma/4}[\tfrac{1}{2}(y - \eta)]^{-(n+2)/2+\gamma/4}$$
$$\cdot \exp\{-\hat{r}\Delta^2(x, y; \xi, \eta)/8\varepsilon(y - \eta)\},$$

and hence

$$\int |D^2G| \, T_2 \, ds \leqq k_7 \hat{Q}\varepsilon^{-(n+2-\gamma)/2}[\tfrac{1}{2}(y - \eta)]^{-(n+2)/2+\gamma/4}(y - \tau)^{-1+\gamma/4}$$
$$\cdot \exp\left\{-\frac{\hat{r}\Delta^2(x, y; \xi, \eta)}{8\varepsilon(y - \eta)}\right\} \int [\varepsilon(y - \tau)]^{-n/2} \exp\left\{-\frac{l \, |z|^2}{16\varepsilon(y - \tau)}\right\} ds$$
$$= k_9(y - \tau)^{-1+\gamma/4}\mathcal{A}(\gamma/2 - 1, \gamma/4, \hat{r}/8).$$

Thus, in view of (5.9) and $| I_5 | \leqq \int | D^2G | \, (T_1 + T_2) \, ds$, there exists a constant $q_2^{(2)} > 0$ such that

$$| D^2J | \leqq q_2^{(2)}(y - \tau)^{-1+\gamma/4}\mathcal{A}(\gamma/2 - 1, \gamma/4, \hat{r}/8)$$

for $(\eta + y)/2 \leqq \tau < y$. This completes the proof of Lemma 5.5.

It follows from Lemma 5.5 that V_2 converges uniformly in $R \times R$ for $y > \eta$ and $\varepsilon > 0$. Moreover, there is a constant $\bar{q}^{(2)} > 0$ such that

$$| V_2 | \leqq \bar{q}^{(2)}\mathcal{A}(\gamma/2 - 1, \gamma/2, \hat{r}/8).$$

By Lemma 5.4, D^2J is uniformly continuous in $R \times R \times I$ for $\eta < \tau < y$ and $\varepsilon > 0$. Hence V_2 is uniformly continuous in $R \times R$ for $y > \eta$ and $\varepsilon > 0$. Since $DV = V_1$ is convergent, this implies that $D^2V = V_2$. Thus we have proved

LEMMA 5.6. *For* $m = 0, 1, 2$

$$D^mV(x, y; \xi, \eta; \varepsilon) = \int_\eta^y d\tau \int D^mG(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon) \, ds,$$

where D^mV is uniformly continuous in $R \times R$ for $y > \eta$ and $\varepsilon > 0$. Moreover, there exist constants $\bar{q}^{(m)} > 0$ such that

$$| D^mV(x, y; \xi, \eta; \varepsilon) | \leqq \bar{q}^{(m)}\mathcal{A}(\gamma/2 - m/2, \gamma/2 + (2 - m)/2, \hat{r}/8).$$

Since, for $m = 0, 1, 2$, D^mG and $\partial G/\partial y$ are related by $\Lambda_\varepsilon(G) = 0$, we can use the estimate for D^2J given in Lemma 5.5 to obtain an estimate for $\partial J/\partial y$ which will enable us to prove the uniform convergence of

$$V_y(x, y; \xi, \eta; \varepsilon) = \int_\eta^y \frac{\partial}{\partial y} J(x, y; \xi, \eta; \tau; \varepsilon) \, d\tau$$

and obtain an expression for $\partial V/\partial y$.

LEMMA 5.7. *In $R \times R$ for $y > \eta$ and $\varepsilon > 0$*

$$(5.10)\quad \begin{aligned} \frac{\partial}{\partial y} V(x, y; \xi, \eta; \varepsilon) &= \psi(x, y; \xi, \eta; \varepsilon) \\ &\quad + \int_\eta^y d\tau \int \frac{\partial}{\partial y} G(x, y; s, \tau; \varepsilon)\psi(s, \tau; \xi, \eta; \varepsilon)\, ds, \end{aligned}$$

where $\partial V/\partial y$ is uniformly continuous and there exists a constant $\hat{q} > 0$ such that

$$|(\partial/\partial y)V(x, y; \xi, \eta; \varepsilon)| \leqq \hat{q}\mathfrak{A}(\gamma/2 - 1/2, \gamma/2, \hat{r}/8).$$

Proof. In view of Lemmas 4.1 and 5.4 we have for $\eta < \tau < y$ and $\varepsilon > 0$

$$\frac{\partial J}{\partial y} = \int \frac{\partial G}{\partial y}\psi\, ds = \int [\varepsilon a_{jk} G_{,jk} + \{a_j + z_k a_{j,k}\}G_{,j} + bG]\psi\, ds,$$

where $a_{jk} = a_{jk}(y; s, \tau)$, etc. Since $a_{jk}(y; \sigma, \tau)G^\sigma_{,jk}$ is Hölder continuous in σ, and since a_j, $a_{j,k}$, and b are bounded, each of the terms in $\partial J/\partial y$ can be estimated in the same manner as the corresponding x-derivative of J. Thus there exist constants $\tilde{q}_1$, $\tilde{q}_2 > 0$ such that

$$(5.11)\quad \left|\frac{\partial J}{\partial y}\right| \leqq \begin{cases} \tilde{q}_1(\tau - \eta)^{-1+\gamma/2}\mathfrak{A}(\gamma/2 - 1/2, 0, \hat{r}) & \text{for } \eta < \tau \leqq (\eta + y)/2 \\ \tilde{q}_2(y - \tau)^{-1+\gamma/4}\mathfrak{A}(\gamma/2 - 1/2, \gamma/4, \hat{r}/8) & \text{for } (\eta + y)/2 \leqq \tau < y. \end{cases}$$

By virtue of Lemmas 5.1 and 5.4, it follows that V_y converges uniformly to a uniformly continuous function which has the asserted bound in $R \times R$ for $y > \eta$ and $\varepsilon > 0$.

For any $y > \eta$, let $\Delta y > 0$, and consider

$$\int_y^{y+\Delta y} V_y(x, t)\, dt = \int_y^{y+\Delta y} dt \int_\eta^t \frac{\partial}{\partial t} J(x, t)\, d\tau = I^*.$$

By using (5.11), it can be easily shown that the order of integrations in I^* can be interchanged and that

$$I^* = V(x, y + \Delta y) - V(x, y) - \int_y^{y+\Delta y} \lim_{t\to\tau+} J(x, t; \xi, \eta; \tau; \varepsilon)\, d\tau.$$

According to Lemma 5.3, ψ is uniformly Hölder continuous with respect to x in $R \times R$ for $y > \eta$ and $\varepsilon > 0$. Hence it follows from Corollary 4.5 that

$$I^* = V(x, y + \Delta y) - V(x, y) - \int_y^{y+\Delta y} \psi(x, \tau; \xi, \eta; \varepsilon)\, d\tau.$$

On the other hand, by the first Theorem of the Mean for integrals, there exists a $\bar{y}$, $y < \bar{y} < y + \Delta y$, such that

$$\int_y^{y+\Delta y} V_y(x, t)\, dt = V_y(x, \bar{y})\Delta y,$$

and therefore

$$\frac{V(x, y + \Delta y) - V(x, y)}{\Delta y} = V_y(x, \bar{y}) + \frac{1}{\Delta y}\int_y^{y+\Delta y} \psi(x, \tau; \xi, \eta; \varepsilon)\, d\tau.$$

A similar result holds for $\Delta y < 0$ provided that $y + \Delta y > \eta$. In view of the uniform continuity of V_y for $\bar{y} > \eta$, and of ψ for $\tau \geqq y > \eta$, the existence of $\partial V/\partial y$ and formula (5.10) follow immediately.

Lemmas 5.6 and 5.7 show that L_ε can be applied to V, and hence to Γ for $y > \eta$ and $\varepsilon > 0$. Indeed, we have

$$L_\varepsilon(\Gamma) = L_\varepsilon(G) + \int_\eta^y d\tau \int \{\varepsilon a_{jk}(x, y)G_{,jk} + a_j(x, y)G_{,j} + b(x, y)G - G_y\}\psi\, ds - \psi$$
$$= L_\varepsilon(G) - \psi + \int_\eta^y d\tau \int L_\varepsilon(G)\psi\, ds,$$

and, since ψ is the solution of (5.2), $L_\varepsilon(\Gamma) = 0$. Moreover, it follows from Lemmas 4.2, 5.6, and 5.7 that $D^m\Gamma$ $(m = 0, 1, 2)$ and $\partial\Gamma/\partial y$ are uniformly continuous in $R \times R$ for $y > \eta$, $\varepsilon > 0$, and that they satisfy (5.3). To complete the proof of Theorem I it remains only to be shown that (2.1) holds for suitable g. If g belongs to class $H(\gamma'; x; E^n)$, then in view of Corollary 4.5 it suffices to show that

$$\lim_{y \to \eta+} \int g(\xi)V(x, y; \xi, \eta; \varepsilon)\, d\xi = 0.$$

However, for bounded g this is obvious from the bound for V given in Lemma 5.6. In §6 we will show that (2.1) holds for a much wider class of initial functions g.

6. The initial value problem

The f.s. of (1.8) which we constructed in §5 permits us to solve the i.v.p.

(6.1) $L_\varepsilon(u) = f(x, y)$ for $y > t$, $u = g(x)$ for $y = t$ $(\varepsilon > 0, t \text{ in } I)$

and to study the behavior of its solution $u(x, y; t; \varepsilon)$ as $\varepsilon \to 0+$, under rather general conditions on f and g. The main result of this section is the following:

THEOREM II. *Suppose that $g(x)$ is continuous in E^n, and that $f(x, y)$ is continuous in R and Hölder continuous with respect to x uniformly for y in I. If $(\mathcal{H})$ holds, and if there exist constants $m_1 > 0$, $m_2 \geqq 0$ such that*

$$|g(x)| \leqq m_1 e^{m_2|x|^2} \quad \text{in } E^n \qquad \text{and} \qquad |f(x, y)| \leqq m_1 e^{m_2|x|^2} \quad \text{in } R,$$

then for any $\varepsilon > 0$ and t in I

$$u(x, y; t; \varepsilon) = \int \Gamma(x, y; \xi, t; \varepsilon)g(\xi)\, d\xi - \int_t^y d\eta \int \Gamma(x, y; \xi, \eta; \varepsilon)f(\xi, \eta)\, d\xi \tag{6.2}$$

is a solution of (6.1) *in* $R \times I$ *for* $0 \leqq y - t \leqq \hat{y}(\mu, t) = \min(y'' - t, (\hat{r} - \mu)/32\varepsilon m_2)$, *where* μ *is any number such that* $0 < \mu < \hat{r}$. *In* $R \times I$ *for* $0 < y - t \leqq \hat{y}(\mu, t)$

$$\lim_{\varepsilon \to 0+} u(x, y; t; \varepsilon) = v^*(x, y; t), \tag{6.3}$$

where v^* *is the weak solution of the i.v.p.*: $L_0(v) = f$ *for* $y > t$, $v = g$ *for* $y = t$. *Moreover,* (6.2) *is the only solution of* (6.1) *in the class of functions which are bounded by* $k_1 e^{k_2 |x|^2}$ *in* R *for some constants* k_1, $k_2 > 0$.

To prove Theorem II we will have to investigate the properties of the integrals which appear on the right-hand side of (6.2). This analysis will be similar to the discussion of J and V in §5, and we will omit many of the details. Throughout this section S will denote an arbitrary simply connected compact subregion of E^n.

LEMMA 6.1. *Let*

$$u^*(x, y; t; \varepsilon) = \int \Gamma(x, y; \xi, t; \varepsilon) g(\xi)\, d\xi,$$

where g *satisfies the conditions of Theorem II. Then for* $m = 0, 1, 2$

$$D^m u^* = \int D^m \Gamma g\, d\xi \quad \text{and} \quad \frac{\partial}{\partial y} u^* = \int \frac{\partial}{\partial y} \Gamma g\, d\xi,$$

where $D^m u^*$ *and* $\partial u^*/\partial y$ *are continuous in* $R \times I$ *for* $0 < y - t \leqq \hat{y}(\mu, t)$ *and* $\varepsilon > 0$, *uniformly for* x *in* S. *There exist constants* $k^{(m)}(\mu)$, $\tilde{k}(\mu) > 0$ *such that for* $m = 0, 1, 2$

$$\begin{aligned} &| D^m u^* | \leqq k^{(m)}(\mu)[\varepsilon(y - t)]^{-m/2} \exp\left\{ \frac{4m_2 \hat{r}}{\mu} | X |^2 \right\} \\ &\text{and} \quad \left| \frac{\partial}{\partial y} u^* \right| \leqq \tilde{k}(\mu) \varepsilon^{-1/2} (y - t)^{-1} \exp\left\{ \frac{4m_2 \hat{r}}{\mu} | X |^2 \right\}, \end{aligned} \tag{6.4}$$

where $X = \phi(y'; x, y)$.

Proof. Let $u_m^* = \int D^m \Gamma g\, d\xi$ $(m = 0, 1, 2)$ and $u_y^* = \int (\partial\Gamma/\partial y) g\, d\xi$. We will show that u_m^* and u_y^* satisfy the inequalities (6.4), i.e., that these integrals converge uniformly in $S \times I$ for $0 < y - t \leqq \hat{y}$ and $\varepsilon > 0$, where S is arbitrary. The lemma then follows in view of the continuity of $D^m\Gamma$, $\partial\Gamma/\partial y$, and g.

We assert that if $| g | \leqq m_1 \exp(m_2 | x |^2)$, then for any τ in I

$$| g(x) | \leqq m_3 \exp\{4m_2 | \phi(y'; x, \tau) |^2\}, \tag{6.5}$$

where $m_3 = m_1 \exp\{4m_2 P^2(y'' - y')^2\}$. For $| x | \leqq 2P(y'' - y')$, (6.5) is obvious. Suppose that $| x | \geqq 2P(y'' - y')$. In view of Lemma 3.1 we have $| x - \phi(y'; x, \tau) | = | \phi(\tau; x, \tau) - \phi(y'; x, \tau) | \leqq P | \tau - y' | \leqq P(y'' - y')$.

Thus $|\phi(y'; x, \tau)| \geqq |x| - |x - \phi| \geqq P(y'' - y')$ and

$$|x|^2 \leqq \{|x - \phi| + |\phi|\}^2$$
$$\leqq |\phi|^2 + 2P|\phi|(y'' - y') + P^2(y'' - y')^2 \leqq 4|\phi|^2.$$

Therefore, (6.5) holds for all x.

It follows from (6.5) with $\tau = \eta$ and Theorem I that

$$|u_m^*| \leqq m_3 Q^{(m)} \int [\varepsilon(y - t)]^{-(n+m)/2} \cdot \exp\left\{ 4m_2|\sigma|^2 - \frac{\hat{r}}{8\varepsilon(y - t)}|X - \sigma|^2 \right\} d\xi,$$

where $\sigma = \phi(y'; \xi, \eta)$. It is easy to verify that

$$4m_2|\sigma|^2 - \frac{\hat{r}}{8\varepsilon(y - t)}|X - \sigma|^2 = \frac{4m_2\hat{r}}{\hat{r} - 32\varepsilon m_2(y - t)}|X|^2 - \frac{\hat{r} - 32\varepsilon m_2(y - t)}{8\varepsilon(y - t)}|\sigma - \hat{X}|^2,$$

where

$$\hat{X} = \frac{\hat{r}}{\hat{r} - 32\varepsilon m_2(y - t)} X.$$

Thus if $0 < \varepsilon(y - t) \leqq (\hat{r} - \mu)/32m_2$ for some $0 < \mu < \hat{r}$, we have

$$|u_m^*| \leqq m_3 Q^{(m)} \exp\left\{ \frac{4m_2\hat{r}}{\mu}|X|^2 \right\} \cdot \int [\varepsilon(y - t)]^{-(n+m)/2} \exp\left\{ -\frac{\mu}{8\varepsilon(y - t)}|\sigma - \hat{X}|^2 \right\} d\xi.$$

Consider the change of variables

$$\nu = [\mu/8\varepsilon(y - t)]^{1/2}(\sigma - \hat{X}).$$

In view of Lemma 3.1 there exists a constant $k > 0$ such that

$$\det(\partial\nu_i/\partial\xi_j) = [\mu/8\varepsilon(y - t)]^{n/2} \det \phi_\xi(y'; \xi, \eta) \geqq (1/k)[\mu/8\varepsilon(y - t)]^{n/2} > 0.$$

Hence

$$|u_m^*| \leqq k m_3 Q^{(m)} \left(\frac{8}{\mu}\right)^{n/2} [\varepsilon(y - t)]^{-m/2} \exp\left\{ \frac{4m_2\hat{r}}{\mu}|X|^2 \right\} \int e^{-|\nu|^2} d\nu$$
$$= k^{(m)}(\mu)[\varepsilon(y - t)]^{-m/2} \exp\left\{ \frac{4m_2\hat{r}}{\mu}|X|^2 \right\}.$$

Similarly, u_y satisfies (6.4).

Remark. Let $w_1(x, y; \xi, \eta; \varepsilon)$ be continuous in $R \times R$ for $y > \eta$, $\varepsilon > 0$, and let $w_2(x, y)$ be continuous in R, where $|w_1| \leqq k\mathfrak{A}(\alpha, \beta, r)$ for $r > 0$

and $| w_2 | \leqq m_1 e^{m_2|x|^2}$. Then by the argument used above it follows that

$$(6.6)\quad \left| \int w_1(x, y; \xi, \eta; \varepsilon) w_2(\xi, \eta)\, d\xi \right| \leqq k(\mu)\varepsilon^{\alpha}(y - \eta)^{-1+\beta} \exp\left\{\frac{4m_2 r}{\mu} | X |^2\right\}$$

in $R \times I$ for $0 < y - \eta \leqq \min (y'' - \eta, (r - \mu)/4\varepsilon m_2)$ and $\varepsilon > 0$ $(0 < \mu < r)$.

Since $L_\varepsilon(\Gamma) = 0$, it follows from Lemma 6.1 that $L_\varepsilon(u^*) = 0$. Moreover, (6.4) implies that u^* is bounded as $y \to t+$. We now show that $u^* \to g(x)$ as $y \to t+$, i.e., that u^* satisfies (6.1) for $f = 0$.

LEMMA 6.2. *If $g(x)$ satisfies the conditions of Theorem II, then*

$$\lim_{y \to t+} u^*(x, y; t; \varepsilon) = \lim_{t \to y-} u^*(x, y; t; \varepsilon) = g(x)$$

in $R \times I$ for $\varepsilon > 0$, and

$$\lim_{\varepsilon \to 0+} u^*(x, y; t; \varepsilon) = g(\phi(t; x, y)) \exp \int_t^y b(\nu; x, y)\, d\nu$$

in $R \times I$ for $0 < y - t \leqq \hat{y}(\mu, t)$, where the convergence is uniform for x in S.

Proof. Write u^* in the form

$$u^* = g(\sigma) \int G\, d\xi + \int [g(\xi) - g(\sigma)] G\, d\xi + \int g(\xi) V\, d\xi,$$

where $\sigma = \phi(t; x, y)$. According to Lemma 4.5, we have

$$\int G\, d\xi = \exp \int_t^y b\, d\nu + O\{[\varepsilon(y - t)]^{\gamma/2}\},$$

and according to (6.6) and Lemma 5.6 we have

$$\left| \int gV\, d\xi \right| \leqq k(\mu)[\varepsilon(y - t)]^{\gamma/2} \exp\left\{ \frac{4m_2 \hat{r}}{\mu} | X |^2 \right\}.$$

Thus it suffices to show that

$$\mathcal{J} = \int [g(\xi) - g(\sigma)] G\, d\xi \to 0$$

in the various cases under consideration.

Consider first the case $y \to t+$. For any $0 < \delta \leqq \frac{1}{2}$, let $\mathcal{S}_\delta(x) = \{\xi \mid | x - \xi | \leqq 3\delta/2\}$ and write $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, where $\mathcal{J}_1$ is taken over $\mathcal{S}_\delta$ and $\mathcal{J}_2$ is taken over $E^n - \mathcal{S}_\delta$. It follows easily from Lemma 3.1 that $| x - \sigma | \leqq P(y - t)$. Thus if $0 \leqq y - t \leqq \delta/2P$, we have

$$(6.7)\quad | \xi - \sigma | \leqq | \xi - x | + | x - \sigma | \leqq 2\delta \qquad \text{for } \xi \text{ in } \mathcal{S}_\delta(x)$$

and

$$(6.7')\quad \begin{aligned} \Delta(x, y; \xi, t) &\geqq e^{-P(y''-y')}\{| \xi - x | - | x - \sigma |\} \\ &\geqq \delta e^{-P(y''-y')} = \hat{\delta} > 0 \qquad \text{for } \xi \text{ in } E^n - \mathcal{S}_\delta(x). \end{aligned}$$

Moreover, since S is compact, there exists a sphere S^* in E^n with radius at most $\frac{1}{2}$ diam $(S) + P(y'' - y')$ such that the (closed) cylinder $S^* \times I$ contains all points (σ, τ), where $\sigma = \phi(\tau; x, y)$ for arbitrary x in S and y, τ in I. Thus in view of (4.18) and (6.7) there exists a constant $k_1 > 0$ such that

$$| \mathcal{J}_1 | \leqq k_1 \max_{|x''-x'| \leqq 2\delta, x' \text{ in } S^*} | g(x'') - g(x') | \quad \text{for } 0 < y - t \leqq \delta/2P.$$

On the other hand, it follows from (4.18), (6.5), and (6.7′) that

$$| \mathcal{J}_2 | \leqq m_3 K^{(0)} \exp\left\{ - \frac{\hat{r}\hat{\delta}^2}{2\varepsilon(y - t)} \right\} \int \{\exp 4m_2 | \phi(y'; \xi, t) |^2$$
$$+ \exp 4m_2 | X |^2\}[\varepsilon(y - t)]^{-n/2} \exp\left\{ - \frac{\hat{r}\Delta^2(x, y; \xi, \eta)}{2\varepsilon(y - t)} \right\} d\xi$$

for $0 < y - t \leqq \delta/2P$. Thus, by (6.6) with $\mu = \frac{1}{2}\hat{r}$, there exists a constant $k_2(S) > 0$ depending only on S such that

$$| \mathcal{J}_2 | \leqq k_2(S) \exp \{-\hat{r}\hat{\delta}^2/2\varepsilon(y - t)\} \quad \text{for } x \text{ in } S \text{ and } 0 < y - t \leqq \delta/2P.$$

Hence, for any given $\theta > 0$, there exist $\delta(\theta, S) > 0$ and $\rho(\theta, S) > 0$, independent of x in S, such that $| \mathcal{J}_1 | \leqq \theta/2$ for all $0 < y - t \leqq \delta(\theta, S)/2P$ and $| \mathcal{J}_2 | \leqq \theta/2$ for all $0 < y - t \leqq \rho(\theta, S)$, where $\rho(\theta, S) \leqq \delta(\theta, S)/2P$, i.e., $\mathcal{J} \to 0$ as $y \to t+$ in $R \times I$ for $\varepsilon > 0$, uniformly in $S \times I$ for any S.

The cases $\varepsilon \to 0+$ and $t \to y-$ can be treated simultaneously in a similar manner. Let $\mathcal{S}_\delta(x, y) = \{\xi, t \mid \Delta(x, y; \xi, t) \leqq \delta e^{-P(y''-y')}\}$. Then for (ξ, t) in $\mathcal{S}_\delta$ we have $| \xi - \sigma | \leqq \delta$ and for (ξ, t) in $E^n - \mathcal{S}_\delta$ we have $\Delta(x, y; \xi, t) \geqq \hat{\delta} > 0$. Proceeding as in the previous paragraph we find

$$| \mathcal{J}_1 | \leqq k_1 \max_{|x''-x'| \leqq \delta, x' \epsilon S^*} | g(x'') - g(x') |$$
$$\text{and} \quad | \mathcal{J}_2 | \leqq k_2(S) \exp \{-\hat{r}\hat{\delta}^2/2\varepsilon(y - t)\} \quad \text{for } x \text{ in } S,$$

where both estimates hold for all $y - t > 0$ and $\varepsilon > 0$, and the assertion follows easily.

We now consider the second term on the right-hand side of (6.2).

LEMMA 6.3. *Let*

$$\bar{u}(x, y; t; \varepsilon) = - \int_t^y d\eta \int \Gamma(x, y; \xi, \eta; \varepsilon) f(\xi, \eta) \, d\xi,$$

where f satisfies the conditions of Theorem II. Then

$$L_\varepsilon[\bar{u}(x, y; t; \varepsilon)] = f(x, y)$$

in $R \times I$ for $0 < y - t \leqq \hat{y}(\mu, t)$ and $\varepsilon > 0$. Moreover

$$(6.8) \qquad | \bar{u}(x, y; t; \varepsilon) | \leqq k^{(0)}(\mu)(y - t) \exp \{(4m_2 \hat{r}/\mu)| X |^2\}.$$

Proof. Since $f(x, y)$ is a continuous function of x uniformly for y in I, it follows that the conclusions of Lemmas 6.1 and 6.2 apply to

$$\bar{u}^*(x, y; \eta; \varepsilon) = \int \Gamma(x, y; \xi, \eta; \varepsilon) f(\xi, \eta) \, d\xi$$

for any η in I such that $\eta < y$. Thus (6.8) follows immediately from (6.4). If we show that

$$\int_t^y D^2\bar{u}^* \, d\eta \quad \text{and} \quad \int_t^y \frac{\partial}{\partial y} \bar{u}^* \, d\eta$$

converge uniformly in $S \times I$ for $0 < y - t \leqq \hat{y}(\mu, t)$ and $\varepsilon > 0$, where S is arbitrary, then the lemma follows by essentially the same argument as that used to prove the existence of $L_\varepsilon(V)$ in §5. Hence it will suffice to obtain an estimate for $D^2\bar{u}^*$ analogous to the one for D^2J in Lemma 5.5, since $D^2\bar{u}^*$ and $\partial\bar{u}^*/\partial y$ are related by $L_\varepsilon(\bar{u}^*) = 0$.

Indeed, if we estimate $\int D^2Gf \, d\xi$ by the procedure employed in Lemma 5.5, we find that

$$| D^2\bar{u}^* | \leqq k_1 \varepsilon^{-1}(y - \eta)^{-1+\gamma/2} | f(\sigma, \eta) | + \int | f(\xi, \eta) - f(\sigma, \eta) | \, | D^2G | \, d\xi$$

$$+ \int | f(\xi, \eta) | \, | D^2V | \, d\xi = T_1 + T_2 + T_3 ,$$

for some constant $k_1 > 0$, where $\sigma = \phi(\eta; x, y)$. In view of (6.5), (6.6), and Lemma 5.6, there exists a constant $k_1(S, \mu) > 0$ such that

$$T_1 + T_3 \leqq k_1(S, \mu)\varepsilon^{-1}(y - \eta)^{-1+\gamma/2}$$

in $S \times I$ for $0 < y - \eta \leqq \hat{y}(\mu, \eta)$ and $\varepsilon > 0$. For x in S, σ is in the sphere S^*. Let S_1^* denote the sphere which is concentric with S^* and whose radius is one unit larger than that of S^*. Then if we write T_2 as the sum of two integrals, $T_2^{(1)}$ taken over S_1^* and $T_2^{(2)}$ taken over $E^n - S_1^*$, we have from (4.18), Lemma 3.2, and the Hölder continuity of f that

$$T_2^{(1)} \leqq H(S_1^*)k_2 \int | z |^{\bar{\gamma}}[\varepsilon(y - \eta)]^{-(n+2)/2} \exp\left\{ - \frac{l \, | z |^2}{8\varepsilon(y - \eta)} \right\} d\xi$$

$$\leqq k_3 \, H(S_1^*)[\varepsilon(y - \eta)]^{-1+\bar{\gamma}/2}$$

for x in S, where k_2, $k_3 > 0$ are absolute constants, $H(S_1^*) > 0$ is a constant depending on S_1^*, and $0 < \bar{\gamma} \leqq 1$ ($\bar{\gamma}$ may also depend on S_1^*). Since for ξ in $E^n - S_1^*$ we have $\Delta(x, y; \xi, \eta) \geqq e^{-P(y'' - y')} = \delta > 0$, it follows (cf. the estimates for $\mathfrak{J}_2$ in Lemma 6.2) that

$$T_2^{(2)} \leqq k_4(S, \mu)[\varepsilon(y - \eta)]^{-1} \exp\{-\hat{r}^2\delta^2/2\varepsilon(y - \eta)\}.$$

Thus there is a constant $k_5(S, \mu) > 0$ such that

$$| D^2\bar{u}^* | \leqq k_1(S, \mu)\varepsilon^{-1}(y - \eta)^{-1+\gamma/2} + k_5(S, \mu)[\varepsilon(y - \eta)]^{-1+\bar{\gamma}/2}$$

in $S \times I$ for $0 < y - \eta \leqq \hat{y}(\mu, \eta)$ and $\varepsilon > 0$. The proof of the lemma can now be completed as indicated above.

It follows from Lemmas 6.1, 6.2, and 6.3 that (6.2) is a solution of the i.v.p.

(6.1), and in view of (6.4)

$$\lim_{\varepsilon\to 0+} \int_t^y d\eta \int \Gamma(x, y; \xi, \eta; \varepsilon) f(\xi, \eta)\, d\xi$$

$$= \int_t^y \left\{ \lim_{\varepsilon\to 0+} \int \Gamma(x, y; \xi, \eta; \varepsilon) f(\xi, \eta)\, d\xi \right\} d\eta$$

$$= \int_t^y \left\{ f(\phi(\eta; x, y), \eta) \exp \int_\eta^y b(\nu; x, y)\, d\nu \right\} d\eta,$$

which together with Lemma 6.2 proves that (6.3) holds. Moreover, according to (6.4) and (6.8) there exists a constant $k(\mu) > 0$ such that

$$| u(x, y; t; \varepsilon) | \leqq k(\mu) \exp \{(4m_2 \hat{r}/\mu) | X |^2\}$$

in $R \times I$ for $0 \leqq y - t \leqq \hat{y}(\mu, t)$ and $\varepsilon > 0$. Thus the uniqueness of u in the class of functions which do not grow more rapidly than $k_1 e^{k_2|x|^2}$ in R follows from a theorem of M. Krzyżański [7; Theorem I]. This completes the proof of Theorem II.

Theorem II can be improved in two ways. The conclusion of the theorem remains valid if the Hölder continuity of f with respect to x is replaced by Dini continuity. In addition, it is possible to obtain an estimate of the difference $u(x, y; t; \varepsilon) - v^*(x, y; t)$ in $S \times I$ in terms of the bounds for f and g, and the moduli of continuity of f, g, and the coefficients of L_ε. To simplify the exposition we will carry this out only for the special case in which f and g are bounded and the various continuity conditions are uniform in R. For the general case of unbounded f and g the argument is essentially the same, although the details are considerably more complicated. The principal tool of this phase of the investigation is the following:

LEMMA 6.4. *If $g(x)$ is uniformly continuous and $| g | \leqq \hat{N}$ in E^n, then there exists a constant $k > 0$ independent of g such that for $\varepsilon(y - t) > 0$ sufficiently small and any δ which satisfies $0 < \delta < 1$*

$$\left| u^*(x, y; t; \varepsilon) - g(\phi(t; x, y)) \exp \int_t^y b(\nu; x, y)\, d\nu \right| \leqq k\{\hat{N}[\varepsilon(y - t)]^{\gamma/2} + \mathfrak{M}(g; [\varepsilon(y - t)]^{\delta/2})\} \tag{6.9}$$

uniformly in $R \times I$, where $y > t$, $\varepsilon > 0$ and $\mathfrak{M}(g; r)$ is the modulus of continuity of g.

Proof. It follows as in the proof of Lemma 6.2, by using (6.4) with $m_1 = \hat{N}$ and $m_2 = 0$, that

$$\left| u^* - g(\sigma) \exp \int_t^y b\, d\nu \right| \leqq \hat{N} k_1 [\varepsilon(y - t)]^{\gamma/2} + | \mathcal{G} |$$

for some constant $k_1 > 0$ (independent of g), where $\sigma = \phi(t; x, y)$. Since g

is uniformly continuous in E^n, there exist a constant $\rho > 0$ and a function $\mathfrak{M}(g; r)$ defined and positive for $0 < r \leqq \rho$ such that $\mathfrak{M}(g; r) \downarrow 0$ as $r \downarrow 0$ and

$$| g(x'') - g(x') | \leqq \mathfrak{M}(g; | x'' - x' |) \qquad \text{for } | x'' - x' | \leqq \rho.$$

In view of the boundedness of g we can extend the definition of $\mathfrak{M}$ to the range $r > \rho$ by the formula $\mathfrak{M}(g; r) \equiv 2\hat{N}$ for $r \geqq \rho$. Then in view of (4.18) and Lemma 3.2 we have

$$| \mathcal{G} | \leqq k_2 \int \mathfrak{M}(g; | \xi - \sigma |)[\varepsilon(y - t)]^{-n/2} \exp\left\{ - \frac{\hat{r} | \xi - \sigma |^2}{\varepsilon(y - t)} \right\} d\xi,$$

where $k_2 > 0$ is a constant (independent of g). Choose any δ such that $0 < \delta < 1$, and suppose that $[\varepsilon(y - t)]^{\delta/2} < \rho$. Then since $\mathfrak{M}(g; r)$ is an increasing function of r for $0 \leqq r \leqq \rho$ and $\mathfrak{M} \leqq 2\hat{N}$ for all $r \geqq 0$, we have

$$| \mathcal{G} | \leqq k_2 \mathfrak{M}(g; [\varepsilon(y - t)]^{\delta/2}) \int_{\mathcal{S}} [\varepsilon(y - t)]^{-n/2} \exp\left\{ - \frac{\hat{r} | \xi - \sigma |^2}{\varepsilon(y - t)} \right\} d\xi$$

$$+ 2\hat{N}k_2 \exp\left\{ -\frac{\hat{r}}{2} [\varepsilon(y - t)]^{-1+\delta} \right\} \int_{E^n - \mathcal{S}} [\varepsilon(y - t)]^{-n/2} \exp\left\{ -\frac{\hat{r} | \xi - \sigma |^2}{2\varepsilon(y - t)} \right\} d\xi,$$

where $\mathcal{S} = \{\xi \mid | \xi - \sigma | \leqq [\varepsilon(y - t)]^{\delta/2}\}$. Hence there exists a constant $k_3 > 0$ (independent of g) such that for $\varepsilon(y - t) > 0$ sufficiently small

$$(6.10) \qquad | \mathcal{G} | \leqq k_3 \mathfrak{M}(g; [\varepsilon(y - t)]^{\delta/2})$$

in $R \times I$, and (6.9) follows.

Remark. It is clear that if we make additional assumptions about the nature of $\mathfrak{M}$, the estimate (6.10) for $| \mathcal{G} |$ can be improved. For instance if $\mathfrak{M}(g; r) = Hr^{\gamma}$ for some $0 < \gamma \leqq 1$, then $| \mathcal{G} | \leqq k\mathfrak{M}(g; [\varepsilon(y - t)]^{1/2})$.

As a consequence of Lemma 6.4 we have

THEOREM III. *Suppose that $g(x)$ is uniformly continuous in E^n, and that $f(x, y)$ is uniformly continuous in R and uniformly Dini continuous with respect to x in R. If ($\mathcal{K}$) holds, and if there exist constants N_1, $N_2 > 0$ such that $| g | \leqq N_1$ in E^n and $| f | \leqq N_2$ in R, then the conclusion of Theorem II holds in $R \times I$ for $t \leqq y \leqq y''$ and $\varepsilon > 0$. Moreover, there exists a constant $k > 0$ independent of f and g such that for $\varepsilon(y - t) > 0$ sufficiently small and any δ which satisfies $0 < \delta < 1$*

$$(6.11) \qquad \begin{aligned} | u(x, y; t; \varepsilon) - v^*(x, y; t) | &\leqq k[N_1[\varepsilon(y - t)]^{\gamma/2} + \mathfrak{M}(g; [\varepsilon(y - t)]^{\delta/2}) \\ &+ (y - t)\{N_2[\varepsilon(y - t)]^{\gamma/2} + \mathfrak{M}(f; [\varepsilon(y - t)]^{\delta/2})\}] \end{aligned}$$

uniformly in $R \times I$, where $y > t$, $\varepsilon > 0$ and $\mathfrak{M}(f; r)$ is the modulus of continuity of f with respect to x.

Proof. In the proof of Theorem II the exact nature of the continuity of f as a function of x is used only in showing that $\int_t^y D^2\bar{u}^* \, d\eta$ and $\int_t^y \partial\bar{u}^*/\partial y \, d\eta$ converge, i.e., in estimating $D^2\bar{u}^*$. It is clear from the proof of Lemma 6.3

that it suffices to show that $\int_t^y T_2\, d\eta$ converges. Let $\mathfrak{M}(f; r)$ be the modulus of continuity of f as a function of x, where $\mathfrak{M}(f; r) \equiv 2N_2$ for $r \geqq \rho$. The assumption that f is Dini continuous means that for any finite $r > 0$

$$\int_0^r \mathfrak{M}(f; \tau)\tau^{-1}\, d\tau < \infty.$$

Note that if $0 < \alpha < 1$, then

$$\int_0^r \mathfrak{M}(f; \tau^\alpha)\tau^{-1}\, d\tau = \frac{1}{\alpha}\int_0^{r^\alpha} \mathfrak{M}(f, \nu)\nu^{-1}\, d\nu < \infty. \tag{6.12}$$

If we proceed as in the proof of Lemma 6.4, it is clear that there exist constants k_1, $k_2 > 0$ such that

$$| T_2 | \leqq k_1[\varepsilon(y - \eta)]^{-1}\mathfrak{M}(f; [\varepsilon(y - \eta)]^{\delta/2}) + N_2 k_2[\varepsilon(y - \eta)]^{-1} \exp\{-\tfrac{1}{2}\hat{r}[\varepsilon(y - \eta)]^{-1+\delta}\}$$

in $R \times I$ for $\varepsilon(y - \eta) > 0$ sufficiently small and $0 < \delta < 1$. Thus it follows from (6.12) that $\int_t^y T_2\, d\eta$ converges uniformly in $R \times I$ for $y > t$ and $\varepsilon > 0$. Therefore the conclusion of Theorem II holds in the present case. The estimate (6.11) follows immediately from (6.9).

Appendix

By the methods employed in the body of this paper we obtain the following result for the parabolic operator

$$L(u) \equiv \sum_{i,j=1}^n a_{ij}(x, y)u_{,ij} + \sum_{i=1}^n a_i(x, y)u_{,i} + b(x, y)u - u_{,y}.$$

THEOREM. *If* ($\mathcal{H}_1$) *and* ($\mathcal{H}_2$) *hold, then for all* (x, y), (ξ, η) *in* R *such that* $y > \eta$, *the f.s. of* $L(u) = 0$ *exists and can be written in the form*

$$\Gamma(x, y; \xi, \eta) = \bar{G}(x, y; \xi, \eta) + \int_\eta^y d\tau \int \bar{G}(x, y; s, \tau)\psi(s, \tau; \xi, \eta)\, ds,$$

where

$$\bar{G}(x, y; \xi, \eta) = 2^{-n}\left\{\pi^n \det \int_\eta^y A(\xi, \nu)\, d\nu\right\}^{-1/2} \cdot \exp\left\{\int_\eta^y b(\xi, \nu)\, d\nu - \tfrac{1}{4}(x - \xi + \alpha)^T\left(\int_\eta^y A(\xi, \nu)\, d\nu\right)^{-1}(x - \xi + \alpha)\right\},$$

$$\alpha = \alpha(y; \xi, \eta) = \int_\eta^y a(\xi, \nu)\, d\nu,$$

and ψ *is the solution of the integral equation*

$$\psi(x, y; \xi, \eta) = L[\bar{G}(x, y; \xi, \eta)] + \int_\eta^y d\tau \int L[\bar{G}(x, y; s, \tau)]\psi(s, \tau; \xi, \eta)\, ds.$$

If $g(x)$ *and* $f(x, y)$ *satisfy the conditions of Theorem II (where the Hölder* *continuity of f with respect to x can be replaced by Dini continuity), then*

$$u(x, y; t) = \int \Gamma(x, y; \xi, t) g(\xi)\, d\xi - \int_t^y d\eta \int \Gamma(x, y; \xi, \eta) f(\xi, \eta)\, d\xi$$

is the unique (in the sense of Theorem II) solution of the i.v.p.

$$L(u) = f(x, y) \quad \text{for} \quad y > t, \qquad u = g(x) \quad \text{for} \quad y = t$$

in $R \times I$ *for* $0 \leqq y - t \leqq \min(y'' - t, (l - \mu)/32m_2)$, *where* $0 < \mu < l$ *and* $l/(y - \eta)$ *is the lower bound for the eigenvalues of* $(\int_\eta^y A(x, \nu)\, d\nu)^{-1}$ *for* $y > \eta$.

The proof of this theorem can be carried out along the same lines as the proofs of Theorems I and II above. We will omit the details and content ourselves with a few remarks concerning the most important modifications of these proofs which must be made. The parametrix $\bar{G}(x, y; \xi, \eta)$ is a solution with singularity at (ξ, η) of the equation

$$\text{(A.1)} \quad \sum_{i,j=1}^n a_{ij}(\xi, y) u_{,ij} + \sum_{i=1}^n a_i(\xi, y) u_{,i} + b(\xi, y) u - u_{,y} = 0.$$

Moreover, since $|\alpha(y; \xi, \eta)| \leqq P(y - \eta)$ in R', it is easy to show that there exists a constant $k > 0$ depending only on $\hat{l}$, P, and $y'' - y'$ such that

$$\exp\left\{ -\tfrac{1}{4}(x - \xi + \alpha)^T \left(\int_\eta^y A(\xi, \nu)\, d\nu \right)^{-1} (x - \xi + \alpha) \right\} \leqq k \exp\left\{ -\frac{l\,|x - \xi|^2}{8(y - \eta)} \right\},$$

where

$$l \frac{\lambda^T \lambda}{y - \eta} \leqq \lambda^T \left(\int_\eta^y A(\xi, \nu)\, d\nu \right)^{-1} \lambda \leqq \hat{l} \frac{\lambda^T \lambda}{y - \eta}$$

in R for all $y > \eta$ and all real n-vectors λ. By using these two facts, together with the observation $|x - \xi + \alpha| \leqq |x - \xi| + P(y - \eta)$, the analogues of Lemmas 4.2 and 4.3 can be proved with little difficulty. Thus, for example, we obtain estimates of the form

$$|D^m \bar{G}(x, y; \xi, \eta)| \leqq K^{(m)} (y - \eta)^{-(n+m)/2} \exp\{-l|x - \xi|^2/16(y - \eta)\} \quad (m \geqq 1).$$

Since we no longer assume that the $a_i(x, y)$ are differentiable with respect to x, it is necessary to alter the definition of G^σ employed in §4 slightly in the present case in order to assure the differentiability of $\bar{G}^\sigma(x, y; \xi, \eta)$ with respect to ξ. In particular, for any σ in E^n we define

$$\bar{G}^\sigma(x, y; \xi, \eta) = 2^{-n} \left\{ \pi^n \det \int_\eta^y A(\sigma, \nu)\, d\nu \right\}^{-1/2} \cdot \exp\left\{ \int_\eta^y b(\sigma, \nu)\, d\nu - \tfrac{1}{4}[x - \xi + \alpha(y; \sigma, \eta)]^T \cdot \left(\int_\eta^y A(\sigma, \nu)\, d\nu \right)^{-1} [x - \xi + \alpha(y; \sigma, \eta)] \right\},$$

i.e., $\bar{G}^\sigma$ is the f.s. of (A.1) with coefficients evaluated at (σ, y). With $\bar{G}^\sigma$ so defined, the analogues of Lemmas 4.4 and 4.5 can be shown to hold. The remainder of the proof of the theorem proceeds by a systematic specialization of the relevant results in §§5 and 6. Since these results depend only on the properties of G and G^σ derived in §4, the theorem follows.

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