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FINITE SYMPLECTIC ACTIONS ON THE $K3$ LATTICE

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Abstract. In this paper, we study finite symplectic actions on $K3$ surfaces X, that is, actions of finite groups G on X which act on $H^{2,0}(X)$ trivially. We show that the action on the $K3$ lattice $H^2(X,\mathbb{Z})$ induced by a symplectic action of G on X depends only on G up to isomorphism, except for five groups.

Contents

§0. Introduction

A compact complex surface X is called a $K3$ *surface* if it is simply connected and has a nowhere vanishing holomorphic 2-form ω_X. (For properties on $K3$ surfaces, see [1].) An automorphism g of X is said to be *symplectic* if $g^*\omega_X = \omega_X$. Nikulin ([19], [20]) studied symplectic actions of finite groups on $K3$ surfaces. In particular, he showed the following result.

Theorem 0.1 ([20, Theorems 4.5, 4.7]). *There exist exactly* 14 *finite abelian groups G ($G = C_2, C_3, \dots$) which act on $K3$ surfaces faithfully and*

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symplectically. Moreover, for each G, the action of G on the $K3$ lattice induced by a faithful and symplectic action of G on a $K3$ surface is unique up to isomorphism.

In this paper, we prove that the above uniqueness holds for any finite groups except for five groups (see Theorem 8.10). We use the same notation for groups as in [29] (see Table 10.2).

Main Theorem. *Let G be a finite group such that $G \neq Q_8, T_{24}, \mathfrak{S}_5, L_2(7)$, $\mathfrak{A}_6$. Then the action of G on the $K3$ lattice induced by a faithful and symplectic action of G on a $K3$ surface is unique up to isomorphism. More precisely, if $G_i \cong G$ acts on a $K3$ surface X_i faithfully and symplectically ($i=1,2$), then there exists an isomorphism $\alpha : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ preserving the intersection forms such that $\alpha \circ G_1 \circ \alpha^{-1} = G_2$ in $\mathrm{GL}(H^2(X_2,\mathbb{Z}))$.*

As a corollary, we have the following by a similar argument in [20] (see [28] for a detailed argument).

Corollary 0.2. *Let G be a finite group which does not belong to the exceptional cases listed above. If G acts on $K3$ surfaces X_i faithfully and symplectically for $i=1,2$, then there exists a connected family $\mathcal{X}$ of $K3$ surfaces with an action of G which satisfies the following conditions:*

(1) *X_1, X_2 are fibers of $\mathcal{X}$;*
(2) *the restriction of the action of G on $\mathcal{X}$ to the fiber X_i coincides with the given one ($i=1,2$);*
(3) *the action of G on each fiber of $\mathcal{X}$ is symplectic.*

If two $K3$ surfaces X_1 and X_2 with actions of G satisfy the conclusions of Corollary 0.2, X_1 and X_2 are said to be *G-deformable*.

We recall known results on finite symplectic actions on $K3$ surfaces. After a result of Nikulin [20], Mukai [18] classified finite groups which act on $K3$ surfaces faithfully and symplectically by listing the 11 maximal groups (see Theorem 2.4). Xiao [29] gave another proof of Mukai's result by studying the singularities of the quotient $G\backslash X$ for a $K3$ surface X with a symplectic action of a finite group G. Moreover, he showed the following.

Theorem 0.3 ([29, Theorem 3]). *Let G be a finite group. Suppose that $G \neq Q_8, T_{24}$. Then, for any $K3$ surface X with a faithful and symplectic action of G, the quotient $G\backslash X$ has the same A-D-E configuration of the singularities.*

Considering his result, one may expect that the uniqueness as in Theorem 0.1 holds for most nonabelian finite groups as well. This paper is motivated by this expectation. We follow Kondō's approach (see [15]) with which he gave another proof of Mukai's result. (See Remark 3.4 for Nikulin's contribution to this other proof.) He embeds the coinvariant lattice $H^2(X,\mathbb{Z})_G = (H^2(X,\mathbb{Z})^G)^\perp$ into a Niemeier lattice N and describes a symplectic action as an action on N. Here a Niemeier lattice is a negative definite even unimodular lattice of rank 24 which is not isomorphic to the Leech lattice. By looking at this action more carefully, we prove the main theorem. For some finite groups, the uniqueness of their (symplectic) actions on $K3$ surfaces was studied by several authors (see [16], [23], [14], [22], [30], [28]). In the case where G is abelian, Garbagnati and Sarti ([10], [11]) and Garbagnati [7] computed the structure of $H^2(X,\mathbb{Z})^G$ and $H^2(X,\mathbb{Z})_G$ and corrected errors of computations of discriminant groups of $H^2(X,\mathbb{Z})_G$ in [20]. We use computer algebra systems GAP [12] and Maxima [17] for the computations of permutation groups and lattices.

The paper proceeds as follows. In Section 1, we recall basic facts on lattices, which are used through the paper. We recall results on finite symplectic actions on $K3$ surfaces in Section 2. Using these results, we take a lattice-theoretic approach to study finite symplectic actions on $K3$ surfaces. We introduce the notion of *finite symplectic actions on the* $K3$ *lattice* Λ, taking account of Nikulin's characterization of symplectic actions on $K3$ surfaces (see Definition 2.5 and Proposition 2.6). The set of finite symplectic actions $G \subset \mathrm{O}(\Lambda)$ on Λ is denoted by $\mathcal{L}$. For $G \in \mathcal{L}$, there exists a $K3$ surface X with a symplectic action of G such that we have a G-equivalent isomorphism $\Lambda \cong H^2(X,\mathbb{Z})$. Section 3 is the key of the paper. By Kondō's lemma (see Lemma 3.2), the coinvariant lattice Λ_G for $G \in \mathcal{L}$ is embedded into a Niemeier lattice N primitively. Since the action of G on Λ_G is extended to that on N such that $N_G = \Lambda_G$, we can study G as an automorphism group of N. Applying the classification of Niemeier lattices, we classify the primitive embeddings of Λ_G into Niemeier lattices. To prove the main theorem, we first prove the uniqueness of Λ_G and Λ^G. In Sections 4 and 6, we show the uniqueness of Λ_G and Λ^G, respectively, by using the result in Section 3. Next, we show the uniqueness of the gluing data of Λ^G and Λ_G to Λ. In Sections 5 and 7, we show that either $\overline{\mathrm{O}(\Lambda_G)} = \mathrm{O}(q(\Lambda_G))$ or $\overline{\mathrm{O}(\Lambda^G)} = \mathrm{O}(q(\Lambda^G))$ holds for any $G \in \mathcal{L}$. This implies the uniqueness of the gluing data. Finally, in Section 8, we prove the main theorem by using the results in the previous sections. Some applications of the main theorem are given in Section 9. We give the list of Niemeier lattices and the results of the computations in Section 10.

§1. Basic facts on lattices

1.1. Definitions

A lattice $L=(L,\langle\ ,\ \rangle)$ is a free $\mathbb{Z}$-module L of finite rank equipped with an integral symmetric bilinear form $\langle\ ,\ \rangle$. We identify a lattice L with its Gramian matrix $(\langle v_i, v_j\rangle)$ under an integral basis (v_i) of L. The discriminant $\operatorname{disc}(L)$ of L is defined as the determinant of the Gramian matrix of L. If $\operatorname{disc}(L)\neq 0$ (resp., $=\pm 1$), a lattice L is said to be *nondegenerate* (resp., *unimodular*). Let $t_{(+)}$ (resp., $t_{(-)}$) be the number of positive (resp., negative) eigenvalues of the Gramian matrix of L. We call $(t_{(+)}, t_{(-)})$ the signature of L and write

$$\operatorname{sign} L=(t_{(+)}, t_{(-)}). \tag{1.1}$$

If $\langle v,v\rangle\equiv 0 \bmod 2$ for all $v\in L$, a lattice L is said to be *even*. We denote by $L(\lambda)$ the $\mathbb{Z}$-module L equipped with λ times the bilinear form $\langle\ ,\ \rangle$, that is, $(L,\lambda\langle\ ,\ \rangle)$. A sublattice K of L is said to be *primitive* if L/K is torsion free. An automorphism of L is defined as a $\mathbb{Z}$-automorphism of L preserving the bilinear form $\langle\ ,\ \rangle$. We denote by $\mathrm{O}(L)$ the group of automorphisms of L. For a subset $S\subset L$, we define a subgroup $\mathrm{O}(L,S)$ of $\mathrm{O}(L)$ by

$$\mathrm{O}(L,S)=\bigl\{g\in \mathrm{O}(L)\bigm| g\cdot S=S\bigr\}. \tag{1.2}$$

Definition 1.1. A lattice L with an action of G is called a *G-lattice* if G is a subgroup of $\mathrm{O}(L)$ and is denoted by (G,L). For a G-lattice (G,L), we define the invariant lattice L^G and the coinvariant lattice L_G by

$$L^G=\bigl\{v\in L\bigm| g\cdot v=v(\forall g\in G)\bigr\},\qquad L_G=(L^G)^{\perp}_{L}. \tag{1.3}$$

A G-lattice (G,L) and a G'-lattice (G',L') are said to be *isomorphic* if there exists an isomorphism $\alpha: L\to L'$ such that

$$\alpha\circ G\circ\alpha^{-1}=G'. \tag{1.4}$$

We recall some basic properties on discriminant forms of lattices for the reader's convenience (see [21] for details). Let L be a nondegenerate even lattice. The discriminant group $A(L)$ is a finite abelian group defined by

$$A(L)=L^{\vee}/L,\qquad L^{\vee}=\bigl\{v\in L\otimes\mathbb{Q}\bigm|\langle v,L\rangle\subset\mathbb{Z}\bigr\}. \tag{1.5}$$

Here we extend the bilinear form $\langle\ ,\ \rangle$ on L to that on $L \otimes \mathbb{Q}$ linearly. We have

$$|A(L)| = |\operatorname{disc}(L)|. \tag{1.6}$$

The discriminant form $q(L)$ of L is defined by

$$q(L) : A(L) \to \mathbb{Q}/2\mathbb{Z}; \qquad x \bmod L \mapsto \langle x, x\rangle \bmod 2\mathbb{Z}. \tag{1.7}$$

We simply write $q(L)$ instead of $(A(L), q(L))$. For a prime number p, let $A(L)_p$ and $q(L)_p$ denote the p-components of $A(L)$ and $q(L)$, respectively. We have

$$A(L) = \bigoplus_p A(L)_p, \qquad q(L) = \bigoplus_p q(L)_p. \tag{1.8}$$

We can consider $q(L)_p$ as the discriminant form of $L \otimes \mathbb{Z}_p$. (The discriminant group and form for a nondegenerate even lattice over $\mathbb{Z}_p$ are similarly defined. Note that any lattice over $\mathbb{Z}_p$ is even if $p \neq 2$.) An automorphism of $q(L)$ is defined as an automorphism of a finite abelian group $A(L)$ preserving $q(L)$. We denote the group of automorphisms of $q(L)$ by $\mathrm{O}(q(L))$. An automorphism $\varphi \in \mathrm{O}(L)$ induces an automorphism $\overline{\varphi} \in \mathrm{O}(q(L))$. This correspondence gives the natural homomorphism

$$\mathrm{O}(L) \to \mathrm{O}\bigl(q(L)\bigr). \tag{1.9}$$

We define

$$\mathrm{O}_0(L) = \operatorname{Ker}\bigl(\mathrm{O}(L) \to \mathrm{O}(q(L))\bigr) \tag{1.10}$$

and

$$\overline{\mathrm{O}(L)} = \operatorname{Im}\bigl(\mathrm{O}(L) \to \mathrm{O}(q(L))\bigr). \tag{1.11}$$

1.2. Facts

We use the following facts (for details, see [21]).

Lemma 1.2. *Let L_1, L_2 be nondegenerate even lattices. We define*

$$\operatorname{Isom}\bigl(q(L_1), -q(L_2)\bigr) = \bigl\{\gamma : q(L_1) \xrightarrow{\sim} q(L_2)\bigr\}. \tag{1.12}$$

If $\gamma \in \operatorname{Isom}(q(L_1), -q(L_2))$, the lattice Γ_γ, defined by

$$\Gamma_\gamma = \bigl\{x \oplus y \in L_1^\vee \oplus L_2^\vee \bigm| \gamma(x \bmod L_1) = y \bmod L_2\bigr\}, \tag{1.13}$$

is an even unimodular lattice which contains L_1 and L_2 primitively. This correspondence gives a one-to-one correspondence between $\mathrm{Isom}(q(L_1), -q(L_2))$ *and the set of even unimodular lattices $\Gamma \subset L_1^\vee \oplus L_2^\vee$ which contain L_1 and L_2 primitively. Moreover, let $\gamma' \in \mathrm{Isom}(q(L_1), -q(L_2))$, and let $\varphi_i \in \mathrm{O}(L_i)$. Then, $\varphi_1 \oplus \varphi_2 \in \mathrm{O}(\mathrm{L}_1 \oplus \mathrm{L}_2)$ is extended to an isomorphism $\Gamma_\gamma \to \Gamma_{\gamma'}$ if and only if $\gamma' \circ \overline{\varphi}_1 \circ \gamma^{-1} = \overline{\varphi}_2$ in $\mathrm{O}(q(L_2))$.*

LEMMA 1.3. *Let Γ be a nondegenerate even lattice, and let L be a nondegenerate primitive sublattice of Γ.*

(1) *If $g \in \mathrm{O}_0(L)$, the action of g on L is extended to that on Γ whose restriction to $(L)^\perp_\Gamma$ is trivial.*
(2) *Suppose that Γ is unimodular. If G is a subgroup of $\mathrm{O}(\Gamma, L)$ and the action of G on $(L)^\perp_\Gamma$ is trivial, then the induced action of G on $A(L)$ is trivial.*
(3) *Suppose that Γ is unimodular. If a group G acts on Γ and Γ_G is nondegenerate, then the induced action of G on $A(\Gamma_G)$ is trivial.*

A lattice over $\mathbb{Z}_p$ is defined as a free $\mathbb{Z}_p$-module of finite rank with a $\mathbb{Z}_p$-valued symmetric bilinear form $\langle\ ,\ \rangle$. First we consider the case $p \neq 2$. In this case, any lattice can be diagonalized over $\mathbb{Z}_p$.

PROPOSITION 1.4 (see [5, Chapter 15, Section 7]). *Let p be an odd prime, and let $\varepsilon_p \in \mathbb{Z}_p^\times$ be a nonsquare p-adic unit. If $L^{(p)}$ is a nondegenerate lattice over $\mathbb{Z}_p$, we have*

$$L^{(p)} \cong \bigoplus_{k \geq 0} (\langle p^k \rangle^{\oplus n_k} \oplus \langle \varepsilon_p p^k \rangle^{\oplus m_k}), \tag{1.14}$$

where $n_k \geq 0$ and $m_k \in \{0, 1\}$ are uniquely determined. Hence,

$$q(L^{(p)}) \cong \bigoplus_{k \geq 1} \big(q_+^{(p)}(p^k)^{\oplus n_k} \oplus q_-^{(p)}(p^k)^{\oplus m_k}\big), \tag{1.15}$$

where

$$q_+^{(p)}(p^k) = \langle 1/p^k \rangle \quad \text{on } \mathbb{Z}/p^k\mathbb{Z}, \tag{1.16}$$

$$q_-^{(p)}(p^k) = \langle \varepsilon_p/p^k \rangle \quad \text{on } \mathbb{Z}/p^k\mathbb{Z}. \tag{1.17}$$

In (1.15), n_k and m_k are also uniquely determined.

Let L be a nondegenerate lattice. We can determine $q(L)_p$ as follows. Let $\mathbb{Z}_{(p)}$ be a localization of $\mathbb{Z}$ by the prime ideal (p), which is considered as a subring of $\mathbb{Z}_p$. Then L can be diagonalized over $\mathbb{Z}_{(p)}$. Then we can write

$$L \cong \bigoplus_{k \geq 0} L_k^{(p)}(p^k) \tag{1.18}$$

over $\mathbb{Z}_{(p)}$, where $L_k^{(p)}$ are lattices over $\mathbb{Z}_{(p)}$ such that $L_k^{(p)} = 0$ or $\operatorname{disc}(L_k^{(p)}) \in \mathbb{Z}_{(p)}^{\times}/(\mathbb{Z}_{(p)}^{\times})^2$. (The discriminant of a lattice over a ring R is defined modulo $(R^{\times})^2$.) The n_k and m_k for $L \otimes \mathbb{Z}_p$ in the above proposition are determined by

$$(n_k, m_k) = \begin{cases} (0,0) & \text{if } L_k^{(p)} = 0, \\ (\operatorname{rank} L_k^{(p)}, 0) & \text{if } \operatorname{disc}(L_k^{(p)}) \in (\mathbb{Z}_p^{\times})^2/(\mathbb{Z}_{(p)}^{\times})^2, \\ (\operatorname{rank} L_k^{(p)} - 1, 1) & \text{otherwise.} \end{cases} \tag{1.19}$$

Next we consider the more complicated case $p = 2$.

PROPOSITION 1.5 (see [5, Chapter 15, Section 7]). *Let $L^{(2)}$ be a nondegenerate lattice over $\mathbb{Z}_2$. Then $L^{(2)}$ can be written as an orthogonal sum of the following lattices:*

$$\langle \varepsilon 2^k \rangle, \qquad \begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}, \tag{1.20}$$

where $k \geq 0$ and $\varepsilon \in \{1, 3, 5, 7\}$. Hence, if $L^{(2)}$ is even, $q(L^{(2)})$ can be written as an orthogonal sum of the following:

$$q_{\varepsilon}^{(2)}(2^k) = \langle \varepsilon / 2^k \rangle \quad \text{on } \mathbb{Z}/2^k\mathbb{Z}, \tag{1.21}$$

$$u^{(2)}(2^k) = \begin{pmatrix} 0 & 1/2^k \\ 1/2^k & 0 \end{pmatrix} \quad \text{on } (\mathbb{Z}/2^k\mathbb{Z})^{\oplus 2}, \tag{1.22}$$

$$v^{(2)}(2^k) = \begin{pmatrix} 1/2^{k-1} & 1/2^k \\ 1/2^k & 1/2^{k-1} \end{pmatrix} \quad \text{on } (\mathbb{Z}/2^k\mathbb{Z})^{\oplus 2}. \tag{1.23}$$

In the case $p = 2$, the uniqueness as in Proposition 1.4 does not hold. Although there is a complete system of invariants of a nondegenerate lattice over $\mathbb{Z}_2$ (see [5]), we only recall the unimodular case.

Proposition 1.6 (see [5, Chapter 15, Section 7]). *For a nondegenerate lattice* $L^{(2)}$ *over* $\mathbb{Z}_2$ *with* $\operatorname{disc}(L^{(2)}) \in \mathbb{Z}_2^\times$*, a quadruple* (r, d, t, e) *defined as follows is a complete system of invariants of* $L^{(2)}$*. If*

$$L^{(2)} \cong \bigoplus_i \langle \varepsilon_i \rangle \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus n} \oplus \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{\oplus m}, \tag{1.24}$$

the invariants r, d, t, e *are defined by*

$$r = \operatorname{rank} L^{(2)}, \tag{1.25}$$

$$d = \begin{cases} +1 & \textit{if } \operatorname{disc}(L^{(2)}) \in \pm(\mathbb{Z}_2^\times)^2/(\mathbb{Z}_2^\times)^2, \\ -1 & \textit{otherwise;} \end{cases} \tag{1.26}$$

$$t = \sum_i \varepsilon_i \bmod 8\mathbb{Z}_2 \in \mathbb{Z}_2/8\mathbb{Z}_2; \tag{1.27}$$

$$e = \begin{cases} \mathrm{I} & \textit{if } L^{(2)} \textit{ is odd,} \\ \mathrm{II} & \textit{otherwise.} \end{cases} \tag{1.28}$$

For example, we can directly check that

$$\langle 1 \rangle^{\oplus 3} \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus \langle 3 \rangle \tag{1.29}$$

over $\mathbb{Z}_2$. We actually have $(r, d, t, e) = (3, +1, \overline{3}, \mathrm{I})$ for both lattices. Using Proposition 1.6, we can determine $q(L)_2$ for a nondegenerate even lattice L similarly to the case $p \neq 2$. We can find an orthogonal decomposition

$$L \cong \bigoplus_{k \geq 0} L_k^{(2)}(2^k) \tag{1.30}$$

over $\mathbb{Z}_2$, where $L_k^{(2)}$ is of the form (1.24). Then we can write $q(L)_2$ as the corresponding orthogonal sum of (1.21)–(1.23). (For relations among (1.21)–(1.23), see [21].)

For a finite abelian group A, let $l(A)$ denote the minimum number of generators of A. Let L be a nondegenerate even lattice. Since $\operatorname{rank} L^\vee = \operatorname{rank} L$ (see (1.5)), we have

$$l\big(A(L)\big) \leq \operatorname{rank} L. \tag{1.31}$$

The following theorem is a reformulation of Eichler's result in a viewpoint of discriminant forms.

THEOREM 1.7 ([21, Theorem 1.13.2]). *Let L, L' be indefinite (nondegenerate) even lattices of* $\operatorname{rank} \geq 3$. *Suppose that the following conditions are satisfied*

(1) *For each $p \neq 2$, either* $\operatorname{rank} L \geq l(A(L)_p) + 2$, *or $n_k + m_k \geq 2$ for some k in the orthogonal decomposition (1.15); that is,*

$$q(L)_p \cong q_p \oplus q_{\pm}^{(p)}(p^k) \oplus q_{\pm}^{(p)}(p^k) \tag{1.32}$$

for some q_p and $k > 0$.

(2) *Either* $\operatorname{rank} L \geq l(A(L)_p) + 2$, *or*

$$q(L)_2 \cong q_2 \oplus q_2' \tag{1.33}$$

for some q_2 and q_2', where q_2' is one of the following:

$$u^{(2)}(2^k), \quad k > 0, \tag{1.34}$$

$$v^{(2)}(2^k), \quad k > 0, \tag{1.35}$$

$$q_{\varepsilon_1}^{(2)}(2^k) \oplus q_{\varepsilon_2}^{(2)}(2^k), \quad \varepsilon_i \in \mathbb{Z}_2^{\times}, k > 0. \tag{1.36}$$

(3) *We have* $\operatorname{sign} L = \operatorname{sign} L'$ *and $q(L) \cong q(L')$.*

Then L is isomorphic to L'.

We use the following facts in Section 7.

THEOREM 1.8 ([21, Theorem 1.14.2]). *Let L be an indefinite even lattice of* $\operatorname{rank} \geq 3$. *If the following conditions are satisfied, $\overline{\mathrm{O}(L)} = \mathrm{O}(q(L))$.*

(1) *For each $p \neq 2$,* $\operatorname{rank} L \geq l(A(L)_p) + 2$.

(2) *Either* $\operatorname{rank} L \geq l(A(L)_p) + 2$, *or*

$$q(L)_2 \cong q_2 \oplus u^{(2)}(2) \text{ or } q_2 \oplus v^{(2)}(2) \tag{1.37}$$

for some q_2.

REMARK 1.9. The conditions of Theorem 1.8 are stronger than those of Theorem 1.7.

THEOREM 1.10 ([21, Theorem 1.9.5]). *If $L^{(p)}$ is a nondegenerate even lattice over $\mathbb{Z}_p$, we have $\overline{\mathrm{O}(L^{(p)})} = \mathrm{O}(q(L^{(p)}))$.*

§2. Finite symplectic actions on the $K3$ lattice Λ

A compact complex surface X is called a $K3$ *surface* if it is simply connected and has a nowhere vanishing holomorphic 2-form ω_X.

DEFINITION 2.1. For a $K3$ surface X, an automorphism g of X is said to be symplectic if $g^*\omega_X = \omega_X$.

We study faithful and symplectic actions of finite groups on $K3$ surfaces.

NOTATION 2.2. We use a Fraktur letter (e.g., $\mathfrak{G}$) for an abstract group, and we use a Latin letter (e.g., G) for a group with an action on an object (a lattice, a finite set, etc.). The abstract group structure of G is denoted by $[G]$.

DEFINITION 2.3. We denote by $\mathfrak{G}_{K3}^{\mathrm{symp}}$ the set of finite abstract groups $\mathfrak{G} \neq 1$ which can be realized as faithful and symplectic actions of groups on $K3$ surfaces.

Mukai [18] determined $\mathfrak{G}_{K3}^{\mathrm{symp}}$ by listing the eleven maximal groups in $\mathfrak{G}_{K3}^{\mathrm{symp}}$.

THEOREM 2.4 ([18, Theorem 0.6]). *A finite abstract group $\mathfrak{G} \neq 1$ is an element in $\mathfrak{G}_{K3}^{\mathrm{symp}}$ if and only if $\mathfrak{G}$ is a subgroup of the following 11 groups:*

$$T_{48}, N_{72}, M_9, \mathfrak{S}_5, L_2(7), H_{192}, T_{192}, \mathfrak{A}_{4,4}, \mathfrak{A}_6, F_{384}, M_{20}.$$

There are exactly 79 groups in $\mathfrak{G}_{K3}^{\mathrm{symp}}$. See Table 10.2 for all elements in $\mathfrak{G}_{K3}^{\mathrm{symp}}$. We use Xiao's notation (see [29]).

For a $K3$ surface X, the second integral cohomology group $H^2(X,\mathbb{Z})$ with its intersection form is isomorphic to the $K3$ lattice Λ defined by

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3} \oplus E_8(-1)^{\oplus 2}, \tag{2.1}$$

which is the unique even unimodular lattice of signature $(3,19)$ up to isomorphism (see Theorem 1.7; see also [26]). Here E_8 is the root lattice of type E_8. The Néron–Severi group $\mathrm{NS}(X)$ of X is considered as a sublattice of $H^2(X,\mathbb{Z})$. If a group G acts on X, the action of G induces a left action on $H^2(X,\mathbb{Z})$ by

$$g \cdot v = (g^{-1})^* v, \quad g \in G, v \in H^2(X,\mathbb{Z}). \tag{2.2}$$

Note that if the action of G is faithful, so is the induced action of G on $H^2(X,\mathbb{Z})$ by the global Torelli theorem (see [24], [3], [1]). Hence, if we take an isomorphism $\alpha : H^2(X,\mathbb{Z}) \to \Lambda$, the action of G on X induces a subgroup $\alpha \circ G \circ \alpha^{-1} \subset \mathrm{O}(\Lambda)$, which is isomorphic to G as an abstract group.

We define the notion of *finite symplectic actions on the $K3$ lattice.*

Definition 2.5. A finite subgroup $G \neq 1$ of $\mathrm{O}(\Lambda)$ is called a *finite symplectic action on the $K3$ lattice* Λ if the following conditions are satisfied:

(1) Λ_G is negative definite;
(2) $\langle v, v\rangle \neq -2$ for all $v \in \Lambda_G$.

We denote the set of finite symplectic actions on the $K3$ lattices Λ by $\mathcal{L}$. Note that the finiteness of G follows from condition (1).

Definition 2.5 is justified due to the following.

Proposition 2.6 ([20, Lemma 4.2, Theorem 4.3]). *If a finite group G acts on a $K3$ surface X faithfully and symplectically, then $H^2(X,\mathbb{Z})_G \subset \mathrm{NS}(X)$ and the induced subgroup of $\mathrm{O}(\Lambda)$ is an element in $\mathcal{L}$. Conversely, any element in $\mathcal{L}$ is induced by a symplectic action of a finite group on a $K3$ surface.*

A $K3$ surface which admits a symplectic action of a finite group is characterized by coinvariant lattices Λ_G of $G \in \mathcal{L}$.

Proposition 2.7 ([20, Theorem 4.15]). *Let $\mathfrak{G} \in \mathfrak{G}_{K3}^{\mathrm{symp}}$. A $K3$ surface X admits a symplectic action of $\mathfrak{G}$ if and only if there exists a primitive embedding $\Lambda_G \hookrightarrow \mathrm{NS}(X)$ for some $G \in \mathcal{L}$ such that $[G] = \mathfrak{G}$.*

Now we consider extensions of symplectic actions.

Proposition 2.8. *Suppose that a finite group G acts on a $K3$ surface X faithfully and symplectically. Then the action of G on X is extended to a faithful and symplectic action of $G' := \mathrm{O}_0(H^2(X,\mathbb{Z})_G)$.*

Proof (see [20]). By Lemma 1.3(1), the action of G on $H^2(X,\mathbb{Z})$ is extended to that of G' such that

$$H^2(X,\mathbb{Z})^G = H^2(X,\mathbb{Z})^{G'}. \tag{2.3}$$

By the definition of a symplectic action, we have $\omega_X \in H^2(X,\mathbb{C})^G$. Since G is a finite group, there exists a G-invariant Kähler $(1,1)$-form $\kappa \in H^2(X,\mathbb{R})^G$. By (2.3), the action of G' also fixes ω_X and κ. By the global Torelli theorem for $K3$ surfaces, the action of G' on $H^2(X,\mathbb{Z})$ is induced by that on X. Since the action of G' fixes ω_X, the action of G' on X is symplectic. ∎

Definition 2.9. For $G \in \mathcal{L}$, we define $\mathrm{Clos}(G)$ by

$$\mathrm{Clos}(G) = \mathrm{O}_0(\Lambda_G). \tag{2.4}$$

By Lemma 1.3(1), the action of G on Λ is extended to that of $\mathrm{Clos}(G)$ such that $\Lambda_G = \Lambda_{\mathrm{Clos}(G)}$, and $\mathrm{Clos}(G)$ is considered as an element in $\mathcal{L}$ (see Definition 2.5). We define the subset $\mathcal{L}_{\mathrm{clos}}$ of $\mathcal{L}$ by

$$\mathcal{L}_{\mathrm{clos}} = \left\{ G \in \mathcal{L} \;\middle|\; \mathrm{Clos}(G) = G \right\}. \tag{2.5}$$

By the following proposition, $\operatorname{rank} \Lambda_G$ depends only on the structure of G as an abstract group.

Proposition 2.10 ([20, Proposition 7.1], [18, Proposition 1.2]). *Let g be an element in $\mathrm{O}(\Lambda)$ such that the group $\langle g \rangle$ generated by g is an element in $\mathcal{L}$. Then* $\mathrm{ord}(g) \le 8$ *and* $\mathrm{Tr}(g; \Lambda) = \chi(g) - 2$, *where*

$$\chi(g) = 24, 8, 6, 4, 4, 2, 3, 2 \quad \textit{if } \mathrm{ord}(g) = 1, 2, 3, 4, 5, 6, 7, 8. \tag{2.6}$$

Hence, for $G \in \mathcal{L}$,

$$\operatorname{rank} \Lambda_G = c(G) := 24 - \frac{1}{|G|} \sum_{g \in G} \chi(g). \tag{2.7}$$

In particular, $c(G) = c(\mathrm{Clos}(G))$.

§3. Embeddings of Λ_G into Niemeier lattices

In this paper, a Niemeier lattice is a negative definite even unimodular lattice of rank 24 which is not isomorphic to the negative Leech lattice. Here the negative Leech lattice is the unique negative definite even unimodular lattice of rank 24 which has no vector v such that $\langle v, v \rangle = -2$ (see [5]). In this section, we study primitive embeddings of Λ_G into Niemeier lattices.

Definition 3.1. Let $\mathcal{N}$ denote the set of isomorphism classes of G-lattices (G, N) which satisfy the following conditions:

(1) $G \neq 1$ and N is a Niemeier lattice;
(2) there exists a vector $v \in N^G$ such that $\langle v, v \rangle = -2$;
(3) there exists no vector $v \in N_G$ such that $\langle v, v \rangle = -2$;
(4) there exists a primitive embedding $N_G \hookrightarrow \Lambda$.

LEMMA 3.2 ([15, Lemma 5]). *For any $G \in \mathcal{L}$, $(G, \Lambda_G) \cong (G', N_{G'})$ for some $(G', N) \in \mathcal{N}$. Conversely, if $(G', N) \in \mathcal{N}$, then there exists an element $G \in \mathcal{L}$ such that $(G, \Lambda_G) \cong (G', N_{G'})$.*

REMARK 3.3. In Lemma 3.2, we write (G, Λ_G) instead of $(G|_{\Lambda_G}, \Lambda_G)$ (see Definition 1.1). We use the same notation in what follows.

REMARK 3.4. Lemma 3.2 is a direct consequence of Nikulin's work (see [20], [21]). Moreover, Nikulin pointed out that lattices such as Λ_G ("Leech-type" lattices) can be classified by embedding them into even unimodular lattices (the latter part of [20, Section 1.14]).

By Lemma 3.2, the study of (G, Λ_G) for $G \in \mathcal{L}$ is reduced to that of $\mathcal{N}$. In the following sections, we present how to make a complete list of $\mathcal{N}$. Some consequences from the list are given in Section 3.4.

3.1. Some facts on Niemeier lattices

The following theorem is standard.

THEOREM 3.5 (see [5, Chapter 16]). *There exist exactly 23 isomorphism classes of Niemeier lattices. The isomorphism class of a Niemeier lattice N is determined by the root sublattice of N, whose type is given in Table 10.1. Here the root sublattice of N is the sublattice generated by vectors $v \in N$ such that $\langle v, v \rangle = -2$.*

Let N be a Niemeier lattice. A vector $d \in N$ is called a *root* if $\langle d, d \rangle = -2$. Let Δ denote the set of roots of N. A Weyl chamber $\mathcal{C}$ is a connected component of $N \otimes \mathbb{R} - \bigcup_{d \in \Delta} d^{\perp}$. The set of positive roots Δ^+ corresponding to $\mathcal{C}$ is defined by

$$\Delta^+ = \{ d \in \Delta \mid \langle d, \mathcal{C} \rangle \subset \mathbb{R}_{>0} \}. \tag{3.1}$$

We have $\Delta = \Delta^+ \sqcup -\Delta^+$. The set of simple roots $R(N, \Delta^+)$ corresponding to Δ^+ is the set of positive roots $d \in \Delta^+$ such that there exists no decomposition $d = d_1 + d_2$ with $d_i \in \Delta^+$. It is known that $R(N, \Delta^+)$ becomes a Dynkin diagram of rank 24. The automorphism group of the Dynkin diagram $R(N, \Delta^+)$ is denoted by $\mathrm{Aut}(R(N, \Delta^+))$. Let $W(N)$ denote the subgroup of $\mathrm{O}(N)$ generated by reflections of $d \in \Delta$. The action of $W(N)$ on the set of Weyl chambers is free and transitive. The group $\mathrm{O}(N, \Delta^+)$ (see (1.2)) is considered as a subgroup of $\mathrm{Aut}(R(N, \Delta^+))$. We have $\mathrm{O}(N) = W \rtimes \mathrm{O}(N, \Delta^+)$.

3.2. Method for making the list of $\mathcal{N}$

We use the above result to construct a complete list of $\mathcal{N}$. For the proof of the following lemma, see [15].

LEMMA 3.6 ([15, Lemma 6]). *Let N be a Niemeier lattice, and let G be a subgroup of $\mathrm{O}(N)$. Then Definition 3.1(3) is satisfied if and only if there exists a G-invariant set of positive roots.*

Let $N_1, \dots, N_{23}$ be all Niemeier lattices, and let Δ_i^+ be a set of positive roots of N_i. Let $G \subset \mathrm{O}(N_i)$ be a subgroup satisfying Definition 3.1(3). By Lemma 3.6, we may assume that G preserves Δ_i^+ by replacing G by $\gamma G \gamma^{-1}$ for some $\gamma \in W(N_i)$ if necessary. Hence, we may only consider subgroups of $\mathrm{O}(N_i, \Delta_i^+)$. Using GAP [12], we can make a complete list of subgroups $G_{i1}, \dots, G_{ij_i}$ of $\mathrm{O}(N_i, \Delta_i^+)$ such that $[G_{ij}] \in \mathfrak{G}_{K3}^{\mathrm{symp}}$ up to conjugacy.* Since $\mathrm{O}(N_i, \Delta_i^+)$ is realized as a subgroup of $\mathrm{Aut}(R(N_i, \Delta_i^+))$, so is G_{ij}. To decide whether $(G_{ij}, N_i) \in \mathcal{N}$, we should check conditions (2)–(4) in Definition 3.1 for (G_{ij}, N_i).

Condition (2) can be checked directly. For example, if N_i is of type $A_1^{\oplus 24}$, condition (2) is equivalent to the existence of a G_{ij}-fixed element in $R(N_i, \Delta_i^+)$. By Lemma 3.6, condition (3) is already satisfied.

To confirm condition (4), it is sufficient to show that there exists an even lattice L such that

$$\operatorname{sign} L = \big(3, 19 - c(G_{ij})\big), \qquad q(L) \cong -q(N_{G_{ij}}) \tag{3.2}$$

by Lemma 1.2 and Proposition 2.10. We can compute the Gramian matrix of $N^{G_{ij}}$ by using the orbit decomposition of $R(N_i, \Delta_i^+)$ which is obtained from the list of (G_{ij}, N_i). From the Gramian matrix of $N^{G_{ij}}$, we can determine $A(N^{G_{ij}})$ and $q(N^{G_{ij}})$ (see Section 1). Since $q(N_{G_{ij}}) \cong -q(N^{G_{ij}})$ by Lemma 1.2, we obtain the list of $q(N_{G_{ij}})$. From the list, we have the following.

LEMMA 3.7. *For (G_{ij}, N_i) satisfying Definition 3.1(2), condition (4) is equivalent to the inequality*

$$l\big(A(N^{G_{ij}})\big) \leq 22 - c(G_{ij}) = \operatorname{rank} N^{G_{ij}} - 2. \tag{3.3}$$

Here $l(A)$ denotes the minimum number of generators of a finite abelian group A.

---

*Note that conjugacy in $\mathrm{O}(N_i, \Delta_i^+)$ is equivalent to conjugacy in $\mathrm{O}(N_i)$, which is a property of semidirect product groups.

Proof. For each case satisfying inequality (3.3), we can find a lattice L satisfying (3.2). (See Tables 10.2 and 10.3 for $q(N_{G_{ij}})$ and L in each case, respectively.) Conversely, the existence of L implies that

$$l\big(A(N^{G_{ij}})\big) = l\big(A(N_{G_{ij}})\big) = l\big(A(L)\big) \le \operatorname{rank} L = 22 - c(G_{ij}) \tag{3.4}$$

by Lemma 1.2 and (1.31). ∎

By the above argument, the set which consists of (G_{ij}, N_i) satisfying Definition 3.2(2) and the inequality (3.3) becomes a complete list of $\mathcal{N}$.

3.3. Example

We consider the case of the cyclic group C_8 of order 8 as an example. We make the list of $(G, N) \in \mathcal{N}$ with $[G] = C_8$. Since $c(C_8) = 18$, we have $\operatorname{rank} N_G = 18$ and $\operatorname{rank} N^G = 6$. Using GAP [12], we can make a complete list of subgroups $G \subset \mathrm{O}(N, \Delta^+)$ such that $[G] = C_8$ up to conjugacy for each Niemeier lattice N. The result is as follows.

Case	(I)	(II)	(III)	(IV)	(V)	(VI)
Root type of N	$E_6^{\oplus 4}$	$A_5^{\oplus 4} \oplus D_4$	$A_3^{\oplus 8}$	$A_2^{\oplus 12}$	$A_2^{\oplus 12}$	$A_1^{\oplus 24}$
Number of stable components of $R(N, \Delta^+)$	0	1	0	2	0	2
$(G, N) \in \mathcal{N}$?	No	Yes	No	Yes	No	Yes

If Definition 3.1(2) holds, then at least one component of the Dynkin diagram $R(N, \Delta^+)$ is stable under the action of G. In the case (I), the action of G as a permutation group of the components E_6 of $R(N, \Delta^+)$ is transitive. Therefore, we have $(G, N) \notin \mathcal{N}$ in the case (I). Similarly, we have $(G, N) \notin \mathcal{N}$ in the cases (III) and (V). In fact, we have $(G, N) \in \mathcal{N}$ in the cases (II), (IV), and (VI), as we will see below. Let g be a generator of G.

The case (II). There exists a numbering of $R(N, \Delta^+) = \{v_1, \ldots, v_{24}\}$ as in Figure 1 such that

$$g \cdot v_i = v_{\sigma(i)}, \tag{3.5}$$

where

$$\begin{aligned} \sigma = {} & (1, 6, 11, 16, 5, 10, 15, 20)(2, 7, 12, 17, 4, 9, 14, 19) \\ & (3, 8, 13, 18)(23, 24). \end{aligned} \tag{3.6}$$

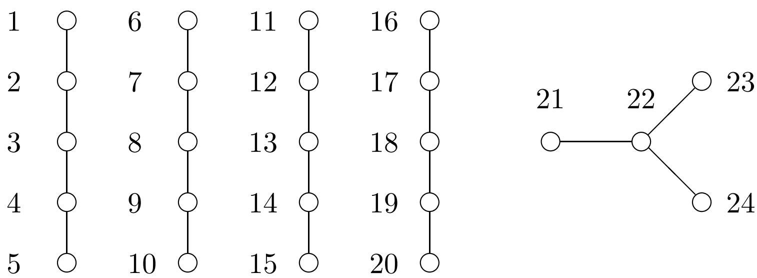


Figure 1: $A_5^{\oplus 4} \oplus D_4$

Hence, $N^G \otimes \mathbb{Q}$ is generated by

$$\begin{aligned} w_1 &= \sum_{i=0}^{3}(v_{1+5i} + v_{5+5i}), \qquad w_2 = \sum_{i=0}^{3}(v_{2+5i} + v_{4+5i}), \\ w_3 &= \sum_{i=0}^{3} v_{3+5i}, \qquad w_4 = v_{21}, \qquad w_5 = v_{22}, \qquad w_6 = v_{23} + v_{24} \end{aligned} \tag{3.7}$$

over $\mathbb{Q}$. From the explicit description of $G \subset \mathrm{O}(N, \Delta^+)$, we find that N^G is generated by the above vectors and $(w_1 + w_3)/2$ over $\mathbb{Z}$. Therefore,

$$w_1, w_2, (w_1 + w_3)/2, w_4, w_5, w_6 \tag{3.8}$$

form a basis of N^G over $\mathbb{Z}$. The Gramian matrix of N^G under the basis (3.8) is

$$\begin{pmatrix} -16 & 8 & 0 & 0 & 0 & 0 \\ 8 & -16 & 8 & 0 & 0 & 0 \\ 0 & 8 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 2 & -4 \end{pmatrix}. \tag{3.9}$$

We can determine $A(N^G)$ and $q(N^G)$ from (3.9) is (see Section 1):

$$A(N^G) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^{\oplus 2}, \tag{3.10}$$

Figure 2: $A_2^{\oplus 12}$

$$q(N^G) \cong \langle 1/2 \rangle \oplus \langle 1/4 \rangle \oplus \begin{pmatrix} 0 & 1/8 \\ 1/8 & 0 \end{pmatrix}. \tag{3.11}$$

Since $q(N_G) \cong -q(N^G)$ by Lemma 1.2, we have

$$q(N_G) \cong \langle -1/2 \rangle \oplus \langle -1/4 \rangle \oplus \begin{pmatrix} 0 & 1/8 \\ 1/8 & 0 \end{pmatrix}. \tag{3.12}$$

The case (IV). Similarly, there exists a numbering of $R(N, \Delta^+)$ as in Figure 2 such that $g \cdot v_i = v_{\sigma(i)}$, where

$$\begin{aligned} \sigma = {} & (3,4)(5,7,6,8)(9,11,13,15,17,19,21,23) \\ & (10,12,14,16,18,20,22,24). \end{aligned} \tag{3.13}$$

Moreover, $N^G \otimes \mathbb{Q}$ is generated by

$$\begin{aligned} & w_1 = v_1, \qquad w_2 = v_2, \qquad w_3 = v_3 + v_4, \qquad w_4 = \sum_{i=5}^{8} v_i, \\ & w_5 = \sum_{i=0}^{7} v_{9+2i}, \qquad w_6 = \sum_{i=0}^{7} v_{10+2i} \end{aligned} \tag{3.14}$$

over $\mathbb{Q}$, and N^G is generated by

$$w_1, w_2, w_3, w_4, w_5, \frac{1}{3}(w_1 - w_2 + w_5 - w_6) \tag{3.15}$$

$$1 \ \bigcirc \qquad 2 \ \bigcirc \qquad \cdots \qquad 23 \ \bigcirc \qquad 24 \ \bigcirc$$

Figure 3: $A_1^{\oplus 24}$

over $\mathbb{Z}$. The Gramian matrix of N^G under the basis (3.15) is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & -1 \\ 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -16 & -8 \\ -1 & 1 & 0 & 0 & -8 & -6 \end{pmatrix}. \tag{3.16}$$

From (3.16), we can check that $q(N_G)$ is isomorphic to (3.12).

The case (VI). There exists a numbering of $R(N, \Delta^+)$ as in Figure 3 such that $g \cdot v_i = v_{\sigma(i)}$, where

$$\begin{aligned} \sigma = {} & (3,4)(5,6,7,8)(9,10,11,12,13,14,15,16) \\ & (17,18,19,20,21,22,23,24). \end{aligned} \tag{3.17}$$

Moreover, $N^G \otimes \mathbb{Q}$ is generated by

$$\begin{aligned} & w_1 = v_1, \qquad w_2 = v_2, \qquad w_3 = \sum_{i=3}^{4} v_i, \qquad w_4 = \sum_{i=5}^{8} v_i, \\ & w_5 = \sum_{i=9}^{16} v_i, \qquad w_6 = \sum_{i=17}^{24} v_i \end{aligned} \tag{3.18}$$

over $\mathbb{Q}$, and N^G is generated by

$$w_1, w_2, w_3, \frac{1}{2}(w_1 + w_2 + w_3 + w_4), \frac{1}{2}(w_4 + w_5), \frac{1}{2}(w_4 + w_6) \tag{3.19}$$

over $\mathbb{Z}$. The Gramian matrix of N^G under the basis (3.19) is

$$\begin{pmatrix} -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & -2 & 0 & 0 \\ -1 & -1 & -2 & -4 & -2 & -2 \\ 0 & 0 & 0 & -2 & -6 & -2 \\ 0 & 0 & 0 & -2 & -2 & -6 \end{pmatrix}. \tag{3.20}$$

From (3.20), we can check that $q(N_G)$ is isomorphic to (3.12).

The type of the root sublattice of N^G, that is, the sublattice generated by vectors $v \in N^G$ such that $\langle v, v\rangle = -2$, in each case is as follows:

(3.21)

Case	(II)	(IV)	(VI)
Root type	A_3	$A_1 \oplus A_2$	$A_1^{\oplus 2}$

Hence, Definition 3.1(2) is satisfied. Condition (3) is satisfied by Lemma 3.6. By the above argument, we have

$$q(N_G) \cong \langle -1/2\rangle \oplus \langle -1/4\rangle \oplus \begin{pmatrix} 0 & 1/8 \\ 1/8 & 0 \end{pmatrix} \tag{3.22}$$

in each case. Let L be a lattice defined by

$$L = \langle 2\rangle \oplus \langle 4\rangle \oplus \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}. \tag{3.23}$$

Then we have $\operatorname{sign} L = (3,1)$ and $q(L) \cong -q(N_G)$. By Lemma 1.2, there exists a primitive embedding $N_G \hookrightarrow \Lambda$ such that $(N_G)^{\perp}_{\Lambda} \cong L$. Thus, condition (4) is satisfied. Therefore, we have $(G,N) \in \mathcal{N}$ in the cases (II), (IV), and (VI).

3.4. Consequences from the list of $\mathcal{N}$

Let $\mathcal{Q}$ denote a set defined by

$$\mathcal{Q} = \{(\mathfrak{G}, q) \mid \exists G \in \mathcal{L} \text{ such that } \mathfrak{G} = [G], q \cong q(\Lambda_G)\}. \tag{3.24}$$

By Lemma 3.2, we have

$$\mathcal{Q} = \{(\mathfrak{G}, q) \mid \exists (G,N) \in \mathcal{N} \text{ such that } \mathfrak{G} = [G], q \cong q(N_G)\}. \tag{3.25}$$

We introduce an equivalence relation $\sim$ on $\mathcal{Q}$ by

$$(3.26)\qquad (\mathfrak{G},q)\sim(\mathfrak{G}',q')\Leftrightarrow \mathfrak{G}=\mathfrak{G}' \qquad \text{and} \qquad q\cong q'.$$

By (3.25) and the list of $q((N_i)^{G_{ij}})$ for $(G_{ij},N_i)\in\mathcal{N}$, we have the following.

PROPOSITION 3.8. *For $\mathfrak{G}\in\mathfrak{G}_{K3}^{\mathrm{symp}}$, we have*

$$(3.27)\qquad \sharp\big(\{q \mid (\mathfrak{G},q)\in\mathcal{Q}\}/\mathrm{isom}\big)=\begin{cases}1 & \text{if } \mathfrak{G}\neq Q_8,T_{24},\\ 2 & \text{if } \mathfrak{G}=Q_8,T_{24}.\end{cases}$$

REMARK 3.9. From Xiao's list (see [29]), we have $\sharp\mathfrak{G}_{K3}^{\mathrm{symp}}=79$. By the above proposition, $\sharp(\mathcal{Q}/\sim)=81$. In Table 10.2, we list a complete representative $\{(\mathfrak{G}_n,q_n)\}$ of $\mathcal{Q}/\sim$. Our numbering coincides with that in [29].

By (3.25), we have the natural map

$$(3.28)\qquad \pi:\mathcal{N}\to\mathcal{Q};\qquad (G,N)\mapsto\big([G],q(N_G)\big).$$

In Table 10.6, the type of the root sublattice of N^G for each $(G,N)\in\mathcal{N}$ is given. From the table, we have the following.

PROPOSITION 3.10. *Let $\mathcal{Q}^\circ$ denote the subset of $\mathcal{Q}$ defined by*

$$(3.29)\qquad \mathcal{Q}^\circ=\big\{(\mathfrak{G},q)\in\mathcal{Q}\;\big|\;\mathfrak{G}\neq\mathfrak{G}_{58}\big\}.$$

There exists a section $\sigma:\mathcal{Q}^\circ\to\pi^{-1}(\mathcal{Q}^\circ)$ of π with the following conditions. We set $\mathcal{N}'=\sigma(\mathcal{Q}^\circ)$.

(1) *Let $(G,N)\in\mathcal{N}$, and let $(G',N')\in\mathcal{N}'$. If $\pi(G,N)=\pi(G',N')$ and $N^G\cong(N')^{G'}$, then $(G,N)\cong(G',N')$.*
(2) *Let $(G,N)\in\mathcal{N}'$. If $[G]\neq\mathfrak{G}_3$, then N is of type $A_1^{\oplus24}$.*

Proof. For each $(\mathfrak{G},q)\in\mathcal{Q}^\circ$, we can choose $\sigma(\mathfrak{G},q)\in\mathcal{N}$ case by case. For example, we consider the case of $C_8=\mathfrak{G}_{14}$ (see Section 3.3). By the table (3.21), the root types of N^G for $(G,N)\in\mathcal{N}$ with $[G]=C_8$ are different from each other. Therefore, N^G are not isomorphic to each other. Hence, we can choose (G,N) of the case (VI), in which N is of type $A_1^{\oplus24}$, as $\sigma(\mathfrak{G}_{14},q_{14})$. Similarly, for $(G,N)\in\mathcal{N}$ with $\pi(G,N)=(\mathfrak{G}_n,q_n)$, the isomorphism classes of N^G can be distinguished by looking at the root types except for the cases $n=32,41,56,63$. For the cases $n=32,41,56,63$, we can distinguish them by looking at the root types and the numbers of vectors $v\in N^G$ such that $\langle v,v\rangle=-4$ or -6. As a consequence, we can choose (G,N) enclosed by boxes in Table 10.6. The choice of σ is not unique. $\square$

§4. Uniqueness of coinvariant lattices Λ_G

Let $\mathcal{S}$ denote a set of G-lattices defined by

$$\mathcal{S} = \big\{(G,S) \,\big|\, \exists G' \in \mathcal{L} \text{ such that } (G,S) \cong (G', \Lambda_{G'})\big\}. \tag{4.1}$$

For $(G,S) \in \mathcal{S}$, we have $G \subset \mathrm{O}_0(S)$ by Lemma 1.3(3). In this section, we apply the results in the previous section to prove the following.

THEOREM 4.1. *The natural map $\varphi : \mathcal{S}/\mathrm{isom} \to \mathcal{Q}/\sim$ is bijective.*

Proof. The surjectivity of φ is trivial. We will show the injectivity. Let $(\mathfrak{G}, q) \in \mathcal{Q}$. Suppose that $(G,S) \in \mathcal{S}$, $[G] = \mathfrak{G}$ and $q(S) \cong q$. We show that (G,S) is uniquely determined up to isomorphism.

(1) The case $\mathfrak{G} \neq \mathfrak{G}_{58}$. By Proposition 3.10, there exists an element $(\Gamma, N) \in \mathcal{N}'$ such that $[\Gamma] = \mathfrak{G}$ and $q(N_\Gamma) \cong q$. We show that $(G,S) \cong (\Gamma, N_\Gamma)$. By Lemma 1.2, $q(S) \cong q \cong q(N_\Gamma) \cong -q(N^\Gamma)$. Again by Lemma 1.2, there exists a primitive embedding $S \hookrightarrow N'$ of S into a Niemeier lattice N' such that $(S)^\perp_{N'} \cong N^\Gamma$. By Lemma 1.3, the action of G on S is extended to that on N' such that $(N')_G = S$ and $(N')^G \cong N^\Gamma$. Thus, $(G, N') \in \mathcal{N}$ (see Definition 3.1). By Proposition 3.10, we have $(G, N') \cong (\Gamma, N)$. Hence, $(G,S) = (G, (N')_G) \cong (\Gamma, N_\Gamma)$.

(2) The case $\mathfrak{G} = \mathfrak{G}_{58}$. From Table 10.4, we find that $\mathfrak{G}_{43} \subsetneq \mathfrak{G}_{58}$ and $c(\mathfrak{G}_{43}) = c(\mathfrak{G}_{58})$. Hence, there exists a subgroup G'_{43} of G such that $[G'_{43}] = \mathfrak{G}_{43}$. Since $c(\mathfrak{G}_{43}) = c(\mathfrak{G}_{58})$, we have $(G'_{43}, S) \in \mathcal{S}$. Let $G_{43} \in \mathcal{L}$ be as in Lemma 8.7. By (1) and Proposition 3.8, $(G', S') \in \mathcal{S}$ such that $[G'] = \mathfrak{G}_{43}$ is unique up to isomorphism. Therefore, we have $(G'_{43}, S) \cong (G_{43}, \Lambda_{G_{43}})$. By Lemma 8.7(2), there exists a unique subgroup G_{58} of $\mathrm{O}_0(\Lambda_{G_{48}})$ such that $[G_{58}] = \mathfrak{G}_{58}$ up to conjugacy in $\mathrm{O}(\Lambda_{G_{48}})$. Hence, $(G,S) \cong (G_{58}, \Lambda_{G_{43}})$. □

DEFINITION 4.2. Let $(\mathfrak{G}, q) \in \mathcal{Q}$. By Theorem 4.1, there exists a unique element $(G,S) \in \mathcal{S}$ such that $([G], q(S)) \sim (\mathfrak{G}, q)$; that is, $[G] = \mathfrak{G}$ and $q(S) \cong q$ up to isomorphism. The lattice S determined by this condition is denoted by $S(\mathfrak{G}, q)$. Since $G \subset \mathrm{O}_0(S)$, $\mathfrak{G}$ is a subgroup of $[\mathrm{O}_0(S(\mathfrak{G}, q))]$.

By the definition of $S(\mathfrak{G}, q)$, we have

$$\Lambda_G \cong S\big([G], q(\Lambda_G)\big) \tag{4.2}$$

for $G \in \mathcal{L}$.

COROLLARY 4.3. *Let $(\mathfrak{G}, q), (\mathfrak{G}', q') \in \mathcal{Q}$. If $\mathfrak{G} \subset \mathfrak{G}'$, $q \cong q'$, and $c(\mathfrak{G}) = c(\mathfrak{G}')$, then $S(\mathfrak{G}, q) \cong S(\mathfrak{G}', q')$.*

Proof. Let $G' \in \mathcal{L}$ such that $[G'] = \mathfrak{G}'$ and $q(\Lambda_{G'}) \cong q'$. Then $\Lambda_{G'} \cong S(\mathfrak{G}', q')$. Let G be the subgroup of G' which corresponds to the subgroup $\mathfrak{G}$ of $\mathfrak{G}'$. Since $c(G) = c(G')$, we have $S(\mathfrak{G}, q) \cong \Lambda_G = \Lambda_{G'} \cong S(\mathfrak{G}', q')$. ∎

Remark 4.4. In Table 10.4, we give the trees of

$$T_S = \left\{\mathfrak{G}_n \,\middle|\, S(\mathfrak{G}_n, q_n) \cong S\right\} \tag{4.3}$$

for T_S with $\sharp T_S \geq 2$. From Tables 10.2 and 10.4, we find that there exist exactly 40 isomorphism classes of lattices $S(\mathfrak{G}_n, q_n)$ (or Λ_G for $G \in \mathcal{L}$). Also, we can check that the natural map

$$\left\{S(\mathfrak{G}, q) \,\middle|\, (\mathfrak{G}, q) \in \mathcal{Q}\right\}/\text{isom} \to \left\{q \,\middle|\, \exists \mathfrak{G} \text{ such that } (\mathfrak{G}, q) \in \mathcal{Q}\right\}/\text{isom} \tag{4.4}$$

is bijective.

Definition 4.5. Let $(\mathfrak{G}, q) \in \mathcal{Q}$. We define $\mathrm{Clos}(\mathfrak{G}, q)$ by

$$\mathrm{Clos}(\mathfrak{G}, q) = \big(\big[\mathrm{O}_0\big(S(\mathfrak{G}, q)\big)\big], q\big). \tag{4.5}$$

Note that $\mathfrak{G}$ is a subgroup of $[\mathrm{O}_0(S(\mathfrak{G}, q))]$ (see Definition 4.2).

For $(\mathfrak{G}, q) \in \mathcal{Q}$, there exists an element $G \in \mathcal{L}$ such that $([G], q(\Lambda_G)) \sim (\mathfrak{G}, q)$. Since $S([G], q(\Lambda_G)) \cong \Lambda_G$, we have

$$\mathrm{Clos}(\mathfrak{G}, q) = \big([\mathrm{O}_0(\Lambda_G)], q\big) = \big([\mathrm{Clos}(G)], q\big) \tag{4.6}$$

(see Definition 2.9). In particular, we have $\mathrm{Clos}(\mathfrak{G}, q) \in \mathcal{Q}$. Let $\mathcal{Q}_{\mathrm{clos}}$ denote a subset of $\mathcal{Q}$ defined by

$$\mathcal{Q}_{\mathrm{clos}} = \left\{(\mathfrak{G}, q) \in \mathcal{Q} \,\middle|\, \mathrm{Clos}(\mathfrak{G}, q) = (\mathfrak{G}, q)\right\}. \tag{4.7}$$

For $G \in \mathcal{L}$, $G \in \mathcal{L}_{\mathrm{clos}}$ if and only if $([G], q(\Lambda_G)) \in \mathcal{Q}_{\mathrm{clos}}$.

Corollary 4.6. *The map*

$$\mathcal{Q}_{\mathrm{clos}}/\sim \longrightarrow \{\Lambda_G \mid G \in \mathcal{L}\}/\text{isom} \tag{4.8}$$

which is induced by the correspondence $(\mathfrak{G}, q) \mapsto S(\mathfrak{G}, q)$ *is bijective.*

Proof. The inverse map of (4.8) is the map induced by the correspondence $S \mapsto ([\mathrm{O}_0(S)], q(S))$. ∎

COROLLARY 4.7. *Let* $(\mathfrak{G}, q) \in \mathcal{Q}$. *Then we have* $\mathrm{Clos}(\mathfrak{G}, q) = (\mathfrak{G}', q)$, *where* $\mathfrak{G}'$ *is the unique maximal element in*

$$\left\{\mathfrak{G}'' \in \mathfrak{G}_{K3}^{\mathrm{symp}} \;\middle|\; (\mathfrak{G}'', q'') \in \mathcal{Q}, \mathfrak{G} \subset \mathfrak{G}'', q \cong q'', c(\mathfrak{G}) = c(\mathfrak{G}'')\right\}. \tag{4.9}$$

Moreover, we have the following.

(1) *If* $\mathfrak{G} \in \{Q_8, T_{24}\}$, *that is,* $(\mathfrak{G}, q) \sim (\mathfrak{G}_n, q_n)$ *for* $n \in \{12, 13, 37, 38\}$, *then we have the following table:*

n	$\mathfrak{G} = \mathfrak{G}_n$	m	$\mathfrak{G}' = \mathfrak{G}_m$
12	Q_8	12	Q_8
13	Q_8	40	$Q_8 * Q_8$
37	T_{24}	77	T_{192}
38	T_{24}	54	T_{48}

Here m *is determined by* $(\mathfrak{G}_m, q_m) \sim \mathrm{Clos}(\mathfrak{G}, q)$.

(2) *If* $\mathfrak{G} \notin \{Q_8, T_{24}\}$, *then* $\mathfrak{G}'$ *is the unique maximal element in*

$$\left\{\mathfrak{G}'' \in \mathfrak{G}_{K3}^{\mathrm{symp}} \;\middle|\; \mathfrak{G} \subset \mathfrak{G}'', c(\mathfrak{G}) = c(\mathfrak{G}'')\right\}. \tag{4.10}$$

Proof. For any element $\mathfrak{G}''$ in (4.9), we have $S(\mathfrak{G}, q) \cong S(\mathfrak{G}'', q'')$ by Corollary 4.3. Hence, $\mathfrak{G}'' \subset \mathfrak{G}' = [\mathrm{O}_0(S(\mathfrak{G}, q))]$. Therefore, the former part of the corollary follows. We can directly check the latter part by Proposition 3.8 and Table 10.4. ∎

§5. Property $\overline{\mathrm{O}(\Lambda_G)} = \mathrm{O}(q(\Lambda_G))$

This section is devoted to prove the following theorem, which gives a sufficient condition for $G \in \mathcal{L}$ such that $\overline{\mathrm{O}(\Lambda_G)} = \mathrm{O}(q(\Lambda_G))$ (see (1.11)).

THEOREM 5.1. *Let* $G \in \mathcal{L}$ *with* $c(G) = \operatorname{rank} \Lambda_G \geq 17$ *(see Proposition 2.10). The group* $\overline{\mathrm{O}(\Lambda_G)}$ *is equal to* $\mathrm{O}(q(\Lambda_G))$ *if and only if* $[\mathrm{Clos}(G) \notin \{\mathfrak{G}_{48}, \mathfrak{G}_{51}\}$. *In particular, if* $c(G) = \operatorname{rank} \Lambda_G = 19$, *then* $\overline{\mathrm{O}(\Lambda_G)} = \mathrm{O}(q(\Lambda_G))$.

Since $c(\mathfrak{G}_{48}) = c(\mathfrak{G}_{51}) = 18$ by Table 10.2, the latter part of the theorem follows from the former part.

5.1. Criterion of the property $\overline{\mathrm{O}(L)} = \mathrm{O}(q(L))$

We prepare for a criterion of the property $\overline{\mathrm{O}(L)} = \mathrm{O}(q(L))$.

LEMMA 5.2. *Let* H *be a group, and let* K_1, K_2 *be subgroups of* H. *If* $K_1 \subset K_2$ *and* $\sharp K_1 \backslash H / K_2 = 1$, *then* $K_2 = H$.

Proof. By the second assumption, any element in H is of the form k_1k_2 with $k_i \in K_i$. Hence, $K_2 = H$ by the first assumption. ∎

PROPOSITION 5.3. *Let L_1 be a nondegenerate even lattice. The group $\overline{\mathrm{O}(L_1)}$ is equal to $\mathrm{O}(q(L_1))$ if and only if there exists a nondegenerate even lattice L_2 satisfying the following conditions.*

(1) *There exists an essentially unique even unimodular lattice $\Gamma \subset L_1^\vee \oplus L_2^\vee$ which contains L_i primitively. Here the uniqueness of Γ means that for another Γ', there exist isomorphisms $\varphi_i \in \mathrm{O}(L_i)$ for $i = 1,2$ such that $\varphi_1 \oplus \varphi_2$ induces an isomorphism $\Gamma \to \Gamma'$.*

(2) *The restriction map $\mathrm{O}(\Gamma, L_2) \to \mathrm{O}(L_2)$ is surjective (see (1.2)).*

Proof. Assume that there exists L_2 satisfying conditions (1) and (2). Let $\gamma \in \mathrm{Isom}(q(L_1), -q(L_2))$ be the isomorphism corresponding to Γ (see Lemma 1.2). Condition (1) implies that

$$\begin{aligned}(5.1)\qquad &\overline{\mathrm{O}(L_2)} \backslash \mathrm{Isom}\big(q(L_1), -q(L_2)\big) / \overline{\mathrm{O}(L_1)} \\ &\quad \cong \gamma^{-1} \circ \overline{\mathrm{O}(L_2)} \circ \gamma \backslash \mathrm{O}\big(q(L_1)\big) / \overline{\mathrm{O}(L_1)}\end{aligned}$$

is a one point set by Lemma 1.2. On the other hand, condition (2) implies that for any $\varphi_2 \in \mathrm{O}(L_2)$, there exists an automorphism $\varphi_1 \in \mathrm{O}(L_1)$ such that $\gamma \circ \overline{\varphi}_1 \circ \gamma^{-1} = \overline{\varphi}_2$ by Lemma 1.2. Hence, $\gamma^{-1} \circ \overline{\mathrm{O}(L_2)} \circ \gamma \subset \overline{\mathrm{O}(L_1)}$. By Lemma 5.2, we have $\overline{\mathrm{O}(L_1)} = \mathrm{O}(q(L_1))$.

Conversely, assume that $\overline{\mathrm{O}(L_1)} = \mathrm{O}(q(L_1))$. Then any nondegenerate even lattice L_2 with $q(L_2) \cong -q(L_1)$ satisfies conditions (1) and (2) by Lemma 1.2. For example, we can take $L_1(-1)$ as L_2. ∎

5.2. Proof of Theorem 5.1

Now we apply Proposition 5.3 to prove Theorem 5.1. Let $G_0 \in \mathcal{L}$ with $c(G_0) \geq 17$. By Corollary 4.6, $\Lambda_{G_0} \cong S(\mathfrak{G}_n, q_n)$ for some $(\mathfrak{G}_n, q_n) \in \mathcal{Q}_{\mathrm{clos}}$. Since $n \neq 58$ (see Table 10.4), we have

$$(5.2)\qquad \Lambda_{G_0} \cong S(\mathfrak{G}_n, q_n) \cong N_G, \qquad \big([G], q(N_G)\big) \sim (\mathfrak{G}_n, q_n) \in \mathcal{Q}_{\mathrm{clos}}$$

for some $(G, N) \in \mathcal{N}'$ by Proposition 3.10. Since $c(\mathfrak{G}_3) = 12 < 17$, N is of type $A_1^{\oplus 24}$ by Proposition 3.10. To prove Theorem 5.1, it is sufficient to show that conditions (1) and (2) in Proposition 5.3 are satisfied for $L_1 = N_G$ and $L_2 = N^G$ if and only if $n \neq 48, 51$.

We check that for $(G, N) \in \mathcal{N}'$ satisfying (5.2), condition (1) is satisfied as follows. Let $N' \subset (N_G)^\vee \oplus (N^G)^\vee$ be a Niemeier lattice which contains N_G

and N^G primitively. By Lemma 1.3, the action of G on N_G is extended to that on N' such that $(N')^G = N^G$. We have $(G, N') \in \mathcal{N}$ by Definition 3.1. By Proposition 3.10, $(G, N) \cong (G, N')$. The uniqueness of N is shown.

Before showing condition (2), we prepare for a couple of lemmas.

LEMMA 5.4. *For $(G, N) \in \mathcal{N}'$ satisfying (5.2), let π denote the restriction map*

$$\pi : \mathrm{O}(N, N^G) \to \mathrm{O}(N^G). \tag{5.3}$$

Then we have $\mathrm{Ker}(\pi) = G$. *In particular,* $G \lhd \mathrm{O}(N, N^G)$.

Proof. Clearly, we have $G \subset \mathrm{Ker}(\pi)$. Let $g \in \mathrm{Ker}(\pi)$. Then $g|_{N_G} \in \mathrm{O}_0(N_G)$ by Lemma 1.3(3). Since $(\mathfrak{G}_n, q_n) \in \mathcal{Q}_{\mathrm{clos}}$, that is, $\mathrm{Clos}(\mathfrak{G}_n, q_n) = (\mathfrak{G}_n, q_n)$, we have $g \in G$ (see Definition 4.5). Hence, $\mathrm{Ker}(\pi) \subset G$. ∎

Let Δ^+ be a set of positive roots of N which is stable under the action of G (see Section 3.1). Since N is of type $A_1^{\oplus 24}$, $\mathrm{O}(N, \Delta^+)$ is isomorphic to the Mathieu group M_{24} of degree 24 and the Weyl group $W(N)$ of N is isomorphic to C_2^{24}. We have $\mathrm{O}(N) = W(N) \rtimes M_{24}$.

LEMMA 5.5. *For $(G, N) \in \mathcal{N}'$ satisfying (5.2), we have*

$$\mathrm{O}(N, N^G) = C_2^m \rtimes N_{M_{24}}(G), \tag{5.4}$$

where $m = \mathrm{rank}\, N^G = 24 - c(G)$ and where $N_{M_{24}}(G)$ is the normalizer subgroup of G in M_{24}. In particular, we have $|\,\mathrm{O}(N, N^G)| = 2^m |N_{M_{24}}(G)|$.

Proof. Set $\{v_1, \ldots, v_{24}\} = R(N, \Delta^+)$, and set $W' = \mathrm{O}(N, N^G) \cap W$. The action of G decomposes $R(N, \Delta^+)$ into n orbits $O_1, \ldots, O_m$. The invariant lattice N^G is generated by $\sum_{v \in O_j} v$ $(j = 1, \ldots, m)$ over $\mathbb{Q}$. Let $w \in W$. Then w is of the form

$$w = \prod_{i=1}^{24} T(v_i)^{e_i}, \quad e_i \in \{0, 1\}, \tag{5.5}$$

where $T(v)$ is the reflection of v. Since

$$w \cdot \sum_{i=1}^{24} a_i v_i = \sum_{i=1}^{24} (-1)^{e_i} a_i v_i, \quad a_i \in \mathbb{Q}, \tag{5.6}$$

W' is generated by $\prod_{v \in O_j} T(v)$ $(j = 1, \ldots, m)$; thus, $W' \cong C_2^m$. By Lemma 5.4, we have an injection $\iota : \mathrm{O}(N, N^G)/W' \to N_{M_{24}}(G)$. For $g \in N_{M_{24}}(G)$, we have $gG \cdot v_i = Gg \cdot v_i$. Therefore, for any j, we have $g \cdot O_j = O_{j'}$ for some j'. Hence, we have $N_{M_{24}}(G) \subset \mathrm{O}(N, N^G)$, and ι is an isomorphism. The assertion follows from this. ∎

Now we show that for $(G,N) \in \mathcal{N}'$ satisfying (5.2), condition (2) is satisfied. By Lemma 5.5, we can determine the order of $\mathrm{O}(N, N^G)$ from the order of $N_{M_{24}}(G)$. We can compute the order of $N_{M_{24}}(G)$ by using GAP [12]. On the other hand, we can also determine the order of $\mathrm{O}(N^G)$ as follows.

Let $B = (b_{ij}) \in M_m(\mathbb{Z})$ be the Gramian matrix of N^G. Then $\mathrm{O}(N^G)$ is identified with the matrix group H consisting of $P \in M_m(\mathbb{Z})$ such that ${}^tPBP = B$. Let S denote the set consisting of column vectors $v \in \mathbb{Z}^m$ such that ${}^tvBv = b_{ii}$ for some i. Then any element $P \in H$ is of the form $(v_1 \cdots v_m)$ with $v_i \in S$. Since N^G is negative definite, we can enumerate all elements in S and H in finite steps. Practically, we take B with smaller $|b_{ii}|$. Since the rank of N^G is less than or equal to $24 - 17 = 7$ by the assumption of Theorem 5.1, we can determine the order of $\mathrm{O}(N^G)$ in practical time by this method. The author used Maxima [17] for this computation. The result is the following.

Lemma 5.6. *For $(G,N) \in \mathcal{N}'$ satisfying (5.2), we have $[\mathrm{O}(N,N^G) : G] = |\,\mathrm{O}(N^G)|$ if and only if $[G] \neq \mathfrak{G}_{48}, \mathfrak{G}_{51}$.*

For example, we consider the case $n = 80$ ($[G] = \mathfrak{G}_{80} = F_{384}$). There exists exactly one element $(G,N) \in \mathcal{N}$ such that $[G] = F_{384}$. The Niemeier lattice N is of type $A_1^{\oplus 24}$. We have $[N_{M_{24}}(G) : G] = 2$ and $|\,\mathrm{O}(N^G)| = 64$. Since $c(G) = 19$, we have $|\,\mathrm{O}(N^G)| = [\mathrm{O}(N,N^G) : G] = 2^{24-19} \cdot 2 = 64$ by Lemma 5.5.

Similarly, for other cases except $n \neq 48, 51$, we have $[\mathrm{O}(N,N^G) : G] = |\,\mathrm{O}(N^G)|$. The following is the table of $k(G) = [N_{M_{24}}(G) : G]$:

n	12	26	32	33	34	39	40	46	49	54	55	56	61
$k(G)$	48	4	2	6	8	2	24	4	120	2	6	12	2

n	62	63	65	70	74	75	76	77	78	79	80	81
$k(G)$	2	6	24	1	2	24	2	4	4	2	2	24

On the other hand, we have $[\mathrm{O}(N,N^G) : G] < |\,\mathrm{O}(N^G)|$ for the cases $n = 48, 51$, as follows:

n	48	51
$k(G)$	2	2
$\mathrm{O}(N^G)/2^m$	6	6

We will finish the proof of Theorem 5.1. We already checked that condition (1) is satisfied. By Lemma 5.4, the restriction map $\pi : \mathrm{O}(N, N^G) \to \mathrm{O}(N^G)$ induces an injection $\mathrm{O}(N, N^G)/G \hookrightarrow \mathrm{O}(N^G)$. By Lemma 5.6, this map is an isomorphism if and only if $n \neq 48, 51$. Therefore, condition (2), that is, the surjectivity of π, is satisfied if and only if $n \neq 48, 51$. By Proposition 5.3, $\overline{\mathrm{O}(N_G)} = \mathrm{O}(q_{N_G})$ if and only if $n \neq 48, 51$.

§6. Uniqueness of invariant lattices Λ^G

This section is devoted to prove the following.

PROPOSITION 6.1. *Set $E = \{\mathfrak{S}_5, L_2(7), \mathfrak{A}_6\}$. For $(\mathfrak{G}, q) \in \mathcal{Q}$ (see (3.24)), we have*

$$\sharp\big(\{\Lambda^G \mid G \in \mathcal{L}, [G] = \mathfrak{G}, q(\Lambda_G) \cong q\}/\mathrm{isom}\big) = \begin{cases} 2 & \textit{if } \mathfrak{G} \in E, \\ 1 & \textit{otherwise.} \end{cases} \tag{6.1}$$

The Gramian matrices of Λ^G are given in Table 10.3.

Proof. Let $G \in \mathcal{L}$ such that $[G] = \mathfrak{G}$ and $q(\Lambda_G) \cong q$. By Lemma 1.2, $q(\Lambda^G) \cong -q(\Lambda_G) \cong -q$.

First we consider the case $\operatorname{rank} \Lambda^G > 3$. Since $\operatorname{sign} \Lambda = (3, 19)$ and Λ_G is negative definite, Λ^G is indefinite in this case. From Table 10.3, we can check that conditions (1) and (2) in Theorem 1.7 for Λ^G are satisfied. Hence, the assertion follows from Theorem 1.7. We can directly find the Gramian matrices of Λ^G with the given signature and discriminant form for each case.

Next we consider the case $\operatorname{rank} \Lambda^G = 3$. In this case, Λ^G is positive definite. From the table of definite ternary forms (see [25]), we can check that there exists a unique positive definite even lattice K of rank 3 such that $q(K) \cong -q$ up to isomorphism, except for the cases $\mathfrak{G} = \mathfrak{S}_5, L_2(7), \mathfrak{A}_6$. If $\mathfrak{G} = \mathfrak{S}_5, L_2(7), \mathfrak{A}_6$, there exist exactly two positive definite even lattices K_1, K_2 of rank 3 such that $q(K_i) \cong -q$ up to isomorphism. For each $i = 1, 2$, there exists a primitive embedding $\Lambda_G \to \Lambda$ such that $(\Lambda_G)^{\perp}_{\Lambda} \cong K_i$ by Lemma 1.2. By Lemma 1.3, the action of G on Λ_G is extended to that on Λ such that $\Lambda^G \cong K_i$. This action is an element in $\mathcal{L}$ by Definition 2.5. Therefore, the assertion follows. ∎

§7. Property $\overline{\mathrm{O}(\Lambda^G)} = \mathrm{O}(q(\Lambda^G))$

This section is devoted to prove the following.

THEOREM 7.1. *Let $G \in \mathcal{L}$. If* $\operatorname{rank} \Lambda^G \geq 4$ *or, equivalently,* $c(G) \leq 18$ *(see Proposition 2.10), then* $\overline{\mathrm{O}(\Lambda^G)} = \mathrm{O}(q(\Lambda^G))$.

We may assume that $G \in \mathcal{L}_{\mathrm{clos}}$ by replacing G by $\mathrm{Clos}(G)$ if necessary. Then $\Lambda_G \cong S(\mathfrak{G}_n, q_n)$ for some $(\mathfrak{G}_n, q_n) \in \mathcal{Q}_{\mathrm{clos}}$ (see Section 4). We can check that Λ^G satisfies conditions (1) and (2) in Theorem 1.8 from Table 10.3, except for the following nine cases:

$$n = 26, 30, 32, 33, 40, 46, 48, 56, 61. \tag{7.1}$$

Hence, we have $\overline{\mathrm{O}(\Lambda^G)} = \mathrm{O}(q(\Lambda^G))$ except for these nine cases.

For example, in the case $n = 65$, we find that

$$\Lambda^G \cong \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus \langle 4 \rangle \oplus \langle -8 \rangle, \tag{7.2}$$

$$q(\Lambda^G) \cong -q_{65} \cong v^{(2)}(2) \oplus q_1^{(2)}(4) \oplus q_7^{(2)}(8) \oplus q_+^{(3)}(3) \tag{7.3}$$

from Table 10.3. Since

$$\operatorname{rank} \Lambda^G = 4 > l\big(A(\Lambda^G)_3\big) + 2 = 3, \tag{7.4}$$

condition (1) is satisfied. On the other hand, since $v^{(2)}(2)$ appears in the orthogonal decomposition (7.3) of $q(\Lambda^G)$, condition (2) is satisfied.

7.1. Preparation for the cases (7.1)

Before studying the nine cases (7.1), we recall some properties of the spinor norm (see, e.g., [4]). Let L be a nondegenerate lattice. For any $\varphi \in \mathrm{O}(L \otimes \mathbb{Q})$, φ is written as a composition of reflections:

$$\varphi = \prod_{i=1}^{r} T(v_i), \quad v_i \in L \otimes \mathbb{Q}, \langle v_i, v_i \rangle \neq 0. \tag{7.5}$$

Here $T(v) \in \mathrm{O}(L \otimes \mathbb{Q})$ is the reflection of v, which is defined by

$$T(v) \cdot w = w - \frac{2\langle v, w \rangle}{\langle v, v \rangle} v. \tag{7.6}$$

The spinor norm $\theta(\varphi)$ of φ is defined by

$$\theta(\varphi) = \prod_{i=1}^{r} \langle v_i, v_i \rangle \bmod (\mathbb{Q}^\times)^2 \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2, \tag{7.7}$$

which is independent of the choice of expression (7.5). We define a map f and a subgroup $\mathrm{O}'(L) \subset \mathrm{O}(L)$ by

$$f = \det \times \theta : \mathrm{O}(L) \to \{\pm 1\} \times \mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \tag{7.8}$$

and $\mathrm{O}'(L) = \mathrm{Ker}(f)$. Note that if $L = L_1 \oplus L_2$, then $f(\mathrm{O}(L_i)) \subset f(\mathrm{O}(L))$. We can define the spinor norm $\theta_p(\varphi_p) \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ of $\varphi_p \in \mathrm{O}(L \otimes \mathbb{Q}_p)$ in a similar way. Moreover, we define

$$f_p = \det \times \theta_p : \mathrm{O}(L_p) \to \{\pm 1\} \times \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 \tag{7.9}$$

and $\mathrm{O}'(L_p) = \mathrm{Ker}(f_p)$, where $L_p = L \otimes \mathbb{Z}_p$.

To deal with the cases (7.1), we use the following proposition, which is a consequence of the strong approximation theorem of quadratic forms (see [4]).

PROPOSITION 7.2. *Let L be an indefinite even lattice of* rank ≥ 3. *We set* $\mathrm{O}_0(L_p) = \mathrm{Ker}\big(\mathrm{O}(L_p) \to \mathrm{O}(q(L_p))\big)$ *and* $d = \mathrm{disc}(L)$. *If the natural map*

$$\mathrm{O}(L) \to \prod_{p|d} \frac{f_p(\mathrm{O}(L_p))}{f_p(\mathrm{O}_0(L_p))} \tag{7.10}$$

is surjective, then $\overline{\mathrm{O}(L)} = \mathrm{O}(q(L))$.

Proof. We have a natural commutative diagram

$$\begin{array}{ccccccccc} 1 \to & \mathrm{O}'(L) & \to & \mathrm{O}(L) & \to & f(\mathrm{O}(L)) & \to 1 \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ 1 \to & \prod_{p|d} \dfrac{\mathrm{O}'(L_p)}{\mathrm{O}'_0(L_p)} & \to & \prod_{p|d} \dfrac{\mathrm{O}(L_p)}{\mathrm{O}_0(L_p)} & \to & \prod_{p|d} \dfrac{f_p(\mathrm{O}(L_p))}{f_p(\mathrm{O}_0(L_p))} & \to 1, \end{array} \tag{7.11}$$

where $\mathrm{O}'_0(L_p) = \mathrm{O}'(L_p) \cap \mathrm{O}_0(L_p)$. The rows in (7.11) are exact. Since

$$\mathrm{O}\big(q(L)\big) = \prod_{p|d} \mathrm{O}\big(q(L)_p\big) \cong \prod_{p|d} \frac{\mathrm{O}(L_p)}{\mathrm{O}_0(L_p)} \tag{7.12}$$

by Theorem 1.10, it is sufficient to show that β is surjective. Since $[\mathrm{O}'(L_p) : \mathrm{O}'_0(L_p)] < \infty$, each coset of $\mathrm{O}'(L_p)/\mathrm{O}'_0(L_p)$ is an open dense subset of $\mathrm{O}'(L_p)$ in p-adic topology. By the strong approximation theorem of quadratic forms (see [4]), the image of $\mathrm{O}'(L)$ in $\prod_{p|d} \mathrm{O}'(L_p)$ is dense. Therefore, α is surjective. On the other hand, γ is surjective by the assumption. By chasing the diagram, β is surjective. ∎

For $f(\mathrm{O}(L))$ and $f_p(\mathrm{O}_0(L_p))$, we have the following.

Lemma 7.3. *Let $L^{(p)}$ be a nondegenerate even lattice over $\mathbb{Z}_p$.*

(1) *If $v \in L^{(p)}$ satisfies $a = \langle v, v\rangle \in \mathbb{Z}_p^\times \cup 2\mathbb{Z}_p^\times$, then $T(v) \in \mathrm{O}_0(L^{(p)})$ and $f_p(T(v)) = (-1, \overline{a}) \in f_p(\mathrm{O}_0(L_p))$.*
(2) *If $L^{(p)}$ contains $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as a sublattice, then*

$$f_p\big(\mathrm{O}_0(L^{(p)})\big) \supset \begin{cases} J_2 := \langle (1, \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2), (-1, \overline{2})\rangle & \textit{if } p = 2, \\ J_p := \{\pm 1\} \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 & \textit{otherwise.} \end{cases} \tag{7.13}$$

(3) *If $p = 2$ and $L^{(2)}$ contains $V = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ as a sublattice, then*

$$f_2\big(\mathrm{O}_0(L^{(2)})\big) \supset J_2. \tag{7.14}$$

Proof. Let v, a be as in (1). Since $T(v) \cdot w = w - (2\langle v, w\rangle/a)v$ and $2/a \in \mathbb{Z}_p^\times$, we have $T(v) \cdot w \in L^{(p)}$ for $w \in L^{(p)}$. Hence, $T(v) \in \mathrm{O}(L^{(p)})$. If $w \in (L^{(p)})^\vee$, then $\langle v, w\rangle \in \mathbb{Z}_p$; thus $T(v) \cdot w \equiv w \bmod L^{(p)}$. Hence, $T(v) \in \mathrm{O}_0(L^{(p)})$. Since the determinant of any reflection is equal to -1, we have $f_p(T(v)) = (-1, \overline{a})$. This proves (1).

Let (e_1, e_2) be a basis of U such that $\langle e_i, e_i\rangle = 0$ and $\langle e_1, e_2\rangle = 1$. For $x \in \mathbb{Z}_p^\times$, set $v_x = e_1 + xe_2$. We have $\langle v_x, v_x\rangle = 2x \in 2\mathbb{Z}_p^\times$. By (1), $T(v_x) \in \mathrm{O}_0(L^{(p)})$ and $f_p(T(v_x)) = (-1, \overline{2x})$. We can check that the group generated by elements of the form $(-1, \overline{2x})$ is J_2 (resp., J_p) if $p = 2$ (resp., $p \neq 2$).

The proof of (3) is similar to that of (2), and we omit it. ∎

Lemma 7.4. *Let L be a nondegenerate even lattice.*

(1) *We have $f(-1_L) = ((-1)^{\operatorname{rank} L}, \overline{\operatorname{disc}(L)})$.*
(2) *If $L \cong U(t) \oplus L'$ for some L', then $f(\mathrm{O}(L)) \supset \langle (-1, \pm\overline{2t})\rangle$, where $U(t) = \left(\begin{smallmatrix} 0 & t \\ t & 0 \end{smallmatrix}\right)$.*

Proof. Let $(e_1, \ldots, e_r)$ be an orthogonal basis of $L \otimes \mathbb{Q}$, where $r = \operatorname{rank} L$. Then, $-1_L = \prod_{i=1}^r T(e_i)$ and $\prod_{i=1}^r \langle e_i, e_i\rangle \equiv \operatorname{disc}(L) \bmod (\mathbb{Q}^\times)^2$. Therefore, $f(-1_L) = ((-1)^r, \overline{\operatorname{disc}(L)})$. This proves (1).

Let (e_1, e_2) be a basis of $U(t)$ such that $\langle e_i, e_i\rangle = 0$ and $\langle e_1, e_2\rangle = t$. Then, $\mathrm{O}(U(t)) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is generated by $T(e_1 \pm e_2)$. Therefore, $f(\mathrm{O}(U(t))) = \langle(-1, \pm\overline{2t})\rangle$. This proves (2). ∎

7.2. Proof of Theorem 7.1 for the cases (7.1)

We set $L = \Lambda^G$, $r = \operatorname{rank} L$, and $d = \operatorname{disc}(L)$. We will show that the map (7.10) is surjective in each case in (7.1). In other words, we show that $\prod_{p|d} f_p(\mathrm{O}(L_p))$ is generated by the images of $\mathrm{O}(L)$ and $\prod_{p|d} f_p(\mathrm{O}_0(L_p))$. As is shown below, we have $f_p(\mathrm{O}(L_p)) = N_p$ except for the cases $n = 46, 61$, where

$$N_p = \{\pm 1\} \times \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2. \tag{7.15}$$

Recall that the map $(a, b, c) \mapsto (-1)^a 3^b 2^c$ induces an isomorphism $(\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$. Moreover, the map $(a, b) \mapsto \varepsilon_p^a p^b$ induces an isomorphism $(\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ if $p \neq 2$, where ε_p is a nonsquare p-adic unit. Let $(e_1, \ldots, e_r)$ be a basis of L whose Gramian matrix is given by Table 10.3. We say that a is *represented* by L if there exists a vector $v \in L$ such that $\langle v, v\rangle = a$. We denote $f(\mathrm{O}(L))$ and $f_p(\mathrm{O}_0(L_p))$ by I and I_p, respectively.

(1) The case $n = 26$. We have

$$L \cong \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix} \oplus \langle 2\rangle \oplus \langle 4\rangle, \qquad d = -2^9. \tag{7.16}$$

Since 2 and 6 are represented by L, we have $(-1, \overline{2}), (-1, \overline{6}) \in I_2$ by Lemma 7.3(1). By Lemma 7.4(2), $(-1, \pm\overline{16}) = (-1, \pm\overline{1}) \in I$. We can check that the images of these four elements generate N_2. (In what follows, we omit "the image(s) of" for simplicity.)

(2) The case $n = 30$. We have

$$L \cong \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}^{\oplus 2} \oplus \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}, \qquad d = -3^6. \tag{7.17}$$

By Lemma 7.4(2), $(-1, \pm\overline{6}) \in I$. Since $T(e_5) \in \mathrm{O}(L)$, we have $f(T(e_5)) = (-1, \overline{2}) \in I$. We can check that these three elements generate N_3.

(3) The case $n = 32$. We have

$$L \cong \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix} \oplus \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}, \qquad d = -2^2 \cdot 5^3. \tag{7.18}$$

Since L_2 contains U, we have $J_2 \subset I_2$ by Lemma 7.3(2). Since 4 is represented by L, we have $(-1, \overline{4}) = (-1, \overline{1}) \in I_5$ by Lemma 7.3(1). By Lemma 7.4(2), $(-1, \pm\overline{10}) \in I$. Since $T(e_3) \in \mathrm{O}(L)$, we have $f(T(e_1)) = (-1, \overline{4}) = (-1, \overline{1}) \in I$. Let $L' = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 6 \end{smallmatrix}\right)$. By Lemma 7.4(1), $f(-1_{L'}) = (1, \overline{20}) = (1, \overline{5}) \in I$. Therefore, the images of I, I_2, I_5 contain the following elements:

	Image in $N_2 \times N_5$
I_2	$(1, \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2) \times (1, \overline{1}), (-1, \overline{2}) \times (1, \overline{1})$
I_5	$(1, \overline{1}) \times (-1, \overline{1})$
I	$(-1, \pm\overline{10}) \times (-1, \pm\overline{10}), (-1, \overline{1}) \times (-1, \overline{1}), (1, \overline{5}) \times (1, \overline{5})$

From this, we can check that I, I_2, I_5 generate $N_2 \times N_5$.

(4) The case $n = 33$. We have

$$L \cong \begin{pmatrix} 0 & 7 \\ 7 & 0 \end{pmatrix} \oplus \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \qquad d = -7^3. \tag{7.19}$$

By Lemma 7.4(2), $(-1, \pm\overline{14}) \in I$. Since $T(e_3) \in \mathrm{O}(L)$, we have $(-1, \overline{2}) \in I$. We can check that these three elements generate N_7.

(5) The case $n = 40$. We have

$$L \cong \langle 4 \rangle^{\oplus 3} \oplus \langle -4 \rangle^{\oplus 2}, \qquad d = 2^{10}. \tag{7.20}$$

Let $\varphi = T(e_1)T(e_1 + 2e_2) \in \mathrm{O}(L_2)$. Then, modulo L_2, we have

$$\varphi \cdot \frac{e_1}{4} = T(e_1) \cdot \Big(\frac{e_1}{4} - \frac{2}{20}(e_1 + 2e_2)\Big) \equiv T(e_1) \cdot \frac{3}{4} e_1 \equiv \frac{e_1}{4}, \tag{7.21}$$

$$\varphi \cdot \frac{e_2}{4} = T(e_1) \cdot \Big(\frac{e_2}{4} - \frac{4}{20}(e_1 + 2e_2)\Big) \equiv T(e_1) \cdot \frac{e_2}{4} = \frac{e_2}{4}. \tag{7.22}$$

Hence, $\varphi \in \mathrm{O}_0(L_2)$ and $f_2(\varphi) = (-1, \overline{4}) \cdot (-1, \overline{20}) = (1, \overline{5}) \in I_2$. Since $T(e_1)$, $T(e_4), T(e_1 + e_2) \in \mathrm{O}(L)$, we have $(-1, \pm\overline{4}), (-1, \overline{8}) \in I$. We can check that these four elements generate N_2.

(6) The case $n = 46$. We have

$$L \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus \langle 6 \rangle \oplus \langle -18 \rangle, \qquad d = -2^2 \cdot 3^4. \tag{7.23}$$

Since L_2 contains V, we have $J_2 \subset I_2$ by Lemma 7.3(3). By [6, Theorem 3.14(i)], we have $f_2(\mathrm{O}(L_2)) = J_2$; thus, $I_2 = f_2(\mathrm{O}(L_2)) = J_2$. Since

$T(e_1), T(e_3), T(e_4) \in \mathrm{O}(L)$, we have $(-1, \overline{2}), (-1, \overline{6}), (-1, -\overline{18}) \in I$. From this, we can check that I, I_2 generate $f_2(\mathrm{O}(L_2)) \times N_3$.

(7) The case $n = 48$. We have

$$L \cong \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}, \qquad d = -2^2 \cdot 3^5. \tag{7.24}$$

Since L_2 contains U, we have $J_2 \subset I_2$ by Lemma 7.3(2). By Lemma 7.4(2), $(-1, \pm\overline{6}) \in I$. Since $T(e_3), T(e_3 + e_4) \in \mathrm{O}(L)$, we have $(-1, \overline{12}), (-1, \overline{36}) \in I$. Therefore, the images of I, I_2 contain the following elements:

	Image in $N_2 \times N_3$
I_2	$(1, \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2) \times (1, \overline{1}), (-1, \overline{2}) \times (1, \overline{1})$
I	$(-1, \pm\overline{6}) \times (-1, \pm\overline{6}), (-1, \overline{3}) \times (-1, \overline{3}), (-1, \overline{1}) \times (-1, \overline{1})$

From this, we can check that I, I_2 generate $N_2 \times N_3$.

(8) The case $n = 56$. We have

$$L \cong \langle 4 \rangle^{\oplus 3} \oplus \langle -8 \rangle, \qquad d = -2^9. \tag{7.25}$$

By the argument in the case $n = 40$, $\varphi = T(e_1)T(e_1 + 2e_2) \in \mathrm{O}_0(L_2)$ and $f_2(\varphi) = (1, \overline{5}) \in I_2$. Since $T(e_1), T(e_4), T(e_1 + e_2) \in \mathrm{O}(L)$, we have $(-1, \overline{4})$, $(-1, -\overline{8}), (-1, \overline{8}) \in I$. We can check that these four elements generate N_2.

(9) The case $n = 61$. We have

$$L \cong \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}, \qquad d = -2^4 \cdot 3^3. \tag{7.26}$$

Since L_2 contains U, we have $J_2 \subset I_2$ by Lemma 7.3(2). By [6, Theorem 3.14(i)], $f_2(\mathrm{O}(L_2)) = J_2$; thus, $I_2 = f_2(\mathrm{O}(L_2)) = J_2$. Since $T(e_3) \in \mathrm{O}(L)$, $(-1, \overline{8}) = (-1, \overline{2}) \in I$. By Lemma 7.4(2), $(-1, \pm\overline{6}) \in I$. From this, we can check that I, I_2 generate $f_2(\mathrm{O}(L_2)) \times N_3$.

Now we have proved Theorem 7.1. $\square$

§8. Uniqueness of symplectic actions on the $K3$ lattice

In this section, we use the results in the previous sections to prove the main theorem.

8.1. Case $c(G) \leq 18$

Proposition 8.1. *The natural map*

$$\{G \in \mathcal{L} \mid c(G) \leq 18\}/\mathrm{conj} \to \{(G,S) \in \mathcal{S} \mid c(G) \leq 18\}/\mathrm{isom} \tag{8.1}$$

is bijective.

Proof. The surjectivity follows from the definition of $\mathcal{S}$ (see (4.1)). Let $(G,S) \in \mathcal{S}$ such that $c(G) \leq 18$. Suppose that $G_i \in \mathcal{L}$ and $(G_i, \Lambda_{G_i}) \cong (G,S)$ for $i = 1,2$. To prove the injectivity, it is sufficient to show that G_1 and G_2 are conjugate in $\mathrm{O}(\Lambda)$. By Proposition 6.1, $\Lambda^{G_1} \cong \Lambda^{G_2}$. By Theorem 7.1, $\overline{\mathrm{O}(\Lambda^{G_1})} = \mathrm{O}(q(\Lambda^{G_1}))$. Therefore, a primitive embedding $\Lambda_{G_1} \to \Lambda$ such that $(\Lambda_{G_1})^{\perp}_{\Lambda} \cong \Lambda^{G_1}$ is unique up to isomorphism, and the restriction map

$$\pi : \mathrm{O}(\Lambda, \Lambda_{G_1}) \to \mathrm{O}(\Lambda_{G_1}) \tag{8.2}$$

is surjective by Lemma 1.2. Hence, we may assume that $\Lambda_{G_1} = \Lambda_{G_2}$ by replacing G_2 by $\varphi G_2 \varphi^{-1}$ for some $\varphi \in \mathrm{O}(\Lambda)$ if necessary. Since $(G_1, \Lambda_{G_1}) \cong (G_2, \Lambda_{G_2}) \cong (G,S)$, G_1 and G_2 are conjugate as subgroups of $\mathrm{O}(\Lambda_{G_1})$. Since π is surjective, G_1 and G_2 are conjugate in $\mathrm{O}(\Lambda)$. □

8.2. Case $c(G) = 19$

Lemma 8.2. *Let $G_1, G_2 \in \mathcal{L}$ such that $[G_1] = [G_2]$, $\mathrm{Clos}(G_1) = \mathrm{Clos}(G_2)$, and $c(G_i) = 19$. If $[\mathrm{Clos}(G_i)] \neq \mathfrak{A}_{4,4}, F_{384}$, then G_1 and G_2 are conjugate in $\mathrm{Clos}(G_i)$.*

Proof. It is sufficient to consider the case $G_i \subsetneq \mathrm{Clos}(G_i)$. By Tables 10.2 and 10.4, we find that $\mathfrak{H} := [\mathrm{Clos}(G_i)] = T_{48}, H_{192}, T_{192}, M_{20}$. Using GAP [12], we can check that there exists a unique subgroup $\mathfrak{G}$ of $\mathfrak{H}$ up to conjugacy in $\mathfrak{H}$ such that $\mathfrak{G} = [G_i]$ (see Appendix). The assertion follows from this. □

Now we consider subgroups $\mathfrak{G}$ of $\mathfrak{A}_{4,4}$ or F_{384} such that $c(\mathfrak{G}) = 19$. Mukai [18] constructed $K3$ surfaces with maximal finite symplectic actions. We use two $K3$ surfaces with symplectic actions of $\mathfrak{A}_{4,4}$ or F_{384} from [18].

Let X be a surface in $\mathbb{P}^5$ defined by the following equations:

$$x^2 + y^2 + z^2 = \sqrt{3}u^2, \tag{8.3}$$

$$x^2 + \zeta y^2 + \zeta^2 z^2 = \sqrt{3}v^2, \tag{8.4}$$

$$x^2 + \zeta^2 y^2 + \zeta z^2 = \sqrt{3}w^2, \tag{8.5}$$

where $\zeta = \exp(2\pi\sqrt{-1}/3)$ and where x, y, z, u, v, w are homogeneous coordinates of $\mathbb{P}^5$. Since X is a smooth complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$, X is a $K3$ surface. Let G denote a subgroup of $\mathrm{PGL}(6, \mathbb{C})$ generated by

$$(x : y : z : u : v : w) \mapsto (-x : -y : z : u : v : w), \tag{8.6}$$

$$(x : y : z : u : v : w) \mapsto (x : y : z : -u : -v : w), \tag{8.7}$$

$$(x : y : z : u : v : w) \mapsto (y : z : x : u : \zeta v : \zeta^2 w), \tag{8.8}$$

$$(x : y : z : u : v : w) \mapsto (x : \zeta^2 y : \zeta z : v : w : u), \tag{8.9}$$

$$(x : y : z : u : v : w) \mapsto (-x : -z : -y : u : w : v). \tag{8.10}$$

Then G acts on X symplectically, and $[G] = \mathfrak{A}_{4,4}$. Moreover, let $\widetilde{G}$ denote the group generated by G and

$$g : (x : y : z : u : v : w) \mapsto (u : v : w : x : z : y). \tag{8.11}$$

Then $\widetilde{G}$ acts on X, and $g^*\omega_X = \sqrt{-1}\omega_X$. Using GAP, we can show the following (see Appendix).

LEMMA 8.3. *Suppose that $\mathfrak{G} \in \mathfrak{G}_{K3}^{\mathrm{symp}}$ is a subgroup of $\mathfrak{A}_{4,4}$ and that $c(\mathfrak{G}) = 19$. Then there exists a unique subgroup K of G such that $[K] = \mathfrak{G}$ up to conjugacy in $\widetilde{G}$.*

Let Y be a surface in $\mathbb{P}^3$ defined by the following equation:

$$x^4 + y^4 + z^4 + t^4 = 0, \tag{8.12}$$

where x, y, z, t are homogeneous coordinates of $\mathbb{P}^3$. Since Y is a smooth quartic surface in $\mathbb{P}^3$, Y is a $K3$ surface. Let H denote a subgroup of $\mathrm{PGL}(4, \mathbb{C})$ generated by

$$(x : y : z : t) \mapsto (ix : -iy : z : t), \tag{8.13}$$

$$(x : y : z : t) \mapsto (y : -x : z : t), \tag{8.14}$$

$$(x : y : z : t) \mapsto (y : z : t : -x), \tag{8.15}$$

where $i = \sqrt{-1}$. Then H acts on Y symplectically, and $[H] = F_{384}$. Moreover, let $\widetilde{H}$ denote the group generated by H and

$$h : (x : y : z : t) \mapsto (ix : y : z : t). \tag{8.16}$$

Then $\widetilde{H}$ acts on Y, and $h^*\omega_Y = i\omega_Y$. Again using GAP, we can show the following (see Appendix).

LEMMA 8.4. *Suppose that* $\mathfrak{G} \in \mathfrak{G}_{K3}^{\mathrm{symp}}$ *is a subgroup of* F_{384} *and that* $c(\mathfrak{G}) = 19$. *Then there exists a unique subgroup* K *of* H *such that* $[K] = \mathfrak{G}$ *up to conjugacy in* $\widetilde{H}$.

REMARK 8.5. We can show that the projective automorphism groups of X and Y are $\widetilde{G}$ and $\widetilde{H}$, respectively (see [13]). However, since X and Y have Picard number 20, the automorphism groups of X and Y are infinite groups by [27].

By considering induced actions on $H^2(X,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$, which are isomorphic to Λ, we have the following.

LEMMA 8.6. *Consider* G *(resp.,* H*) as a subgroup of* $\mathrm{O}(\Lambda)$. *Suppose that* $\mathfrak{G}$ *is a subgroup of* $\mathfrak{A}_{4,4}$ *(resp.,* F_{384}*) such that* $c(\mathfrak{G}) = 19$. *Then there exists a unique subgroup* K *of* G *(resp.,* H*) up to conjugacy in* $\mathrm{O}(\Lambda)$ *such that* $[K] = \mathfrak{G}$.

We use the following lemma in the proof of Theorem 4.1.

LEMMA 8.7. *There exists an element* $G_{43} \in \mathcal{L}$ *which satisfies the following.*

(1) *We have* $[G_{43}] = \mathfrak{G}_{43}$.
(2) *There exists a unique subgroup* G_{58} *of* $\mathrm{O}_0(\Lambda_{G_{43}})$ *such that* $[G_{58}] = \mathfrak{G}_{58}$ *up to conjugacy in* $\mathrm{O}(\Lambda_{G_{43}})$.

Proof. We fix an identification $H^2(Y,\mathbb{Z}) = \Lambda$. By Table 10.4, there exists a subgroup G_{43} of H such that $[G_{43}] = \mathfrak{G}_{43}$. Since $c(\mathfrak{G}_{43}) = c(H) = 19$, we have $\Lambda_{G_{43}} = \Lambda_H$. Since $[H] = F_{384}$ is a maximal element in $\mathfrak{G}_{K3}^{\mathrm{symp}}$, we have $[\mathrm{O}_0(\Lambda_H)] = [H]$. Since $H \triangleleft \widetilde{H}$, we have $\widetilde{H} \subset \mathrm{O}(\Lambda, \Lambda^H)$. By Lemma 8.4 and Table 10.4, condition (2) is satisfied. ∎

We have the following by the above lemmas.

PROPOSITION 8.8. *Set* $E = \{\mathfrak{S}_5, L_2(7), \mathfrak{A}_6\} \subset \mathfrak{G}_{K3}^{\mathrm{symp}}$. *The natural map*

$$\begin{aligned} &\big\{G \in \mathcal{L} \mid c(G) = 19, [G] \notin E\big\}/\mathrm{conj} \\ &\qquad \to \big\{(G,S) \in \mathcal{S} \mid c(G) = 19, [G] \notin E\big\}/\mathrm{isom} \end{aligned} \tag{8.17}$$

is bijective.

Proof. The surjectivity follows from the definition of $\mathcal{S}$ (see (4.1)). Let $(G,S) \in \mathcal{S}$ such that $c(G) = 19$ and $[G] \notin E$. Suppose that $G_i \in \mathcal{L}$ and

$(G_i, \Lambda_{G_i}) \cong (G, S)$ for $i = 1, 2$. To prove the injectivity, it is sufficient to show that G_1 and G_2 are conjugate in $\mathrm{O}(\Lambda)$. By Proposition 6.1, $\Lambda^{G_1} \cong \Lambda^{G_2}$. By Theorem 5.1, $\overline{\mathrm{O}(\Lambda_{G_1})} = \mathrm{O}(q(\Lambda_{G_1}))$. Therefore, a primitive embedding $\Lambda_{G_1} \to \Lambda$ such that $(\Lambda_{G_1})^{\perp}_{\Lambda} \cong \Lambda^{G_1}$ is unique up to isomorphism by Lemma 1.2. Hence, we may assume that $\Lambda_{G_1} = \Lambda_{G_2}$ by replacing G_2 by $\varphi G_2 \varphi^{-1}$ for some $\varphi \in \mathrm{O}(\Lambda)$ if necessary. Thus, $[\mathrm{Clos}(G_1)] = [\mathrm{Clos}(G_2)]$.

(1) The case $[\mathrm{Clos}(G_i)] \neq \mathfrak{A}_{4,4}, F_{384}$. By Lemma 8.2, G_1 and G_2 are conjugate in $\mathrm{Clos}(G_i)$ $(\subset \mathrm{O}(\Lambda))$.

(2) The case $[\mathrm{Clos}(G_i)] = \mathfrak{A}_{4,4}$ (resp., F_{384}). By the above argument, we have $\Lambda_{G_i} = \Lambda_G$ (resp., Λ_H) for some identification $\Lambda = H^2(X, \mathbb{Z})$ (resp., $H^2(Y, Z)$). Hence, $\mathrm{Clos}(G_i) = G$ (resp., H). By Lemma 8.6, G_1 and G_2 are conjugate in $\mathrm{O}(\Lambda)$. ∎

Proposition 8.9. *For $\mathfrak{G} = \mathfrak{S}_5, L_2(7), \mathfrak{A}_6$, there exist exactly two elements G_1, G_2 in $\mathcal{L}$ up to conjugacy in $\mathrm{O}(\Lambda)$ such that $[G_i] = \mathfrak{G}$. We have $\Lambda_{G_1} \cong \Lambda_{G_2}$, $q(\Lambda^{G_1}) \cong q(\Lambda^{G_2})$ and $\Lambda^{G_1} \not\cong \Lambda^{G_2}$.*

Proof. By Proposition 3.8 and Theorem 4.1, there exists a unique element $(G_0, S) \in \mathcal{S}$ up to isomorphism such that $[G_0] = \mathfrak{G}$. Since $\mathfrak{G}$ is a maximal element in $\mathfrak{G}_{K3}^{\mathrm{symp}}$, we have $\mathrm{O}_0(S) = G_0$. By Theorem 5.1, $\overline{\mathrm{O}(S)} = \mathrm{O}(q(S))$. By Lemma 1.2 and Proposition 6.1, there exist exactly two primitive sublattices S_1, S_2 of Λ such that $S_i \cong S$ up to $\mathrm{O}(\Lambda)$. The action of $G_i := \mathrm{O}_0(S_i)$ on S_i is extended to that on Λ such that $\Lambda_{G_i} = S_i$ $(i = 1, 2)$. Let $G \in \mathcal{L}$ such that $[G] = \mathfrak{G}$. Then $\Lambda_G \cong S$. Hence, we may assume that $\Lambda_G = S_i$ $(i = 1, 2)$ by replacing G by $\varphi G \varphi^{-1}$ for some $\varphi \in \mathrm{O}(\Lambda)$ if necessary. Then we have $G = G_i$. This implies the assertion. ∎

8.3. Proof of the main theorem

Theorem 8.10. *Let $\mathfrak{G} \in \mathfrak{G}_{K3}^{\mathrm{symp}}$.*

(1) *If $\mathfrak{G} = Q_8, T_{24}$, there exist exactly two elements $G_1, G_2 \in \mathcal{L}$ such that $[G_i] = \mathfrak{G}$ up to conjugacy in $\mathrm{O}(\Lambda)$. We have the following table, by changing the numbering of G_1, G_2 if necessary (see Corollary 4.7):*

$\mathfrak{G}$	n	$[\mathrm{Clos}(G_1)]$	$\mathrm{disc}(\Lambda_{G_1})$	n	$[\mathrm{Clos}(G_2)]$	$\mathrm{disc}(\Lambda_{G_2})$
Q_8	12	Q_8	-512	40	$Q_8 * Q_8$	-1024
T_{24}	77	T_{192}	-192	54	T_{48}	-384

Here n is determined by $([G_i], q(\Lambda_{G_i})) \sim (\mathfrak{G}_n, q_n)$.

(2) *If* $\mathfrak{G} = \mathfrak{S}_5, L_2(7), \mathfrak{A}_6$*, there exist exactly two elements* $G_1, G_2 \in \mathcal{L}$ *such that* $[G_i] = \mathfrak{G}$ *up to conjugacy in* $\mathrm{O}(\Lambda)$*. We have* $\Lambda_{G_1} \cong \Lambda_{G_2}$*,* $q(\Lambda^{G_1}) \cong q(\Lambda^{G_2})$*, and* $\Lambda^{G_1} \not\cong \Lambda^{G_2}$*.*

(3) *Otherwise, there exists a unique* $G \in \mathcal{L}$ *such that* $[G] = \mathfrak{G}$ *up to conjugacy in* $\mathrm{O}(\Lambda)$*.*

Proof. By Theorem 4.1, $(G,S) \in \mathcal{S}$ is determined uniquely by $[G]$ and $q(S)$ up to isomorphism. Assertions (1) and (3) follow from Propositions 8.1 and 8.8 and Table 10.2. Assertion (2) is the same as Proposition 8.9. ∎

§9. Applications

Combining Xiao's result (Theorem 0.3), the following theorem is a consequence of Theorem 8.10 and the global Torelli theorem for $K3$ surfaces (see [20]).

THEOREM 9.1. *Let* G *be a group such that* $[G] \in \mathfrak{G}_{K3}^{\mathrm{symp}}$ *(see Notation 2.2). Set* $E_1 = \{Q_8, T_{24}\}$*,* $E_2 = \{\mathfrak{S}_5, L_2(7), \mathfrak{A}_6\}$*.*

(1) *If* $[G] \notin E_1 \cup E_2$*, then the moduli space of* $K3$ *surfaces with faithful and symplectic* G*-actions is connected.*

(2) *If* $[G] \in E_1 \cup E_2$*, then the moduli space of* $K3$ *surfaces with faithful and symplectic* G*-actions has exactly two connected components.*

(3) *If* X_i *is a* $K3$ *surface with a faithful and symplectic* G_i*-action for* $i = 1, 2$ *such that* $[G_i] \notin E_2$ *and* $G_1 \backslash X_1, G_2 \backslash X_2$ *have the same A-D-E configuration of the singularities, then* $[G_1] = [G_2] =: G$ *and* X_1, X_2 *are* G*-deformable (see Section 0).*

(4) *If* X *is a* $K3$ *surface with a faithful and symplectic action of* G *of type* $(\mathfrak{G}, q) \in \mathcal{Q}$*, that is,* $([G], q(H^2(X,\mathbb{Z})_G)) \sim (\mathfrak{G}, q)$*, then the action is extended to that of type* $\mathrm{Clos}(\mathfrak{G}, q)$ *(see Section 4 and Table 10.4).*

Assertion (4) for some cases was pointed out and studied in detail by Garbagnati ([8], [9]).

§10. Tables

10.1. Niemeier lattices

We give the list of Niemeier lattices N (see [5, Chapter 16]). Let Δ^+ be a set of positive roots of N. We denote by $\mathrm{O}(N,\Delta^+)_1$ the group which consists of $g \in \mathrm{O}(N,\Delta^+)$ preserving each connected component of the Dynkin diagram $R(N,\Delta^+)$. We set $\mathrm{O}(N,\Delta^+)_2 = \mathrm{O}(N,\Delta^+)/\mathrm{O}(N,\Delta^+)_1$. The group $\mathrm{O}(N,\Delta^+)_2$ acts on the set of connected components of $R(N,\Delta^+)$.

Table 10.1

i	Root type	$\lvert \mathrm{O}(N_i,\Delta_i^+)_1 \rvert$	$\mathrm{O}(N_i,\Delta_i^+)_2$	$\lvert \mathrm{O}(N_i,\Delta_i^+) \rvert$
1	D_{24}	1	1	1
2	$D_{16}\oplus E_8$	1	1	1
3	$E_8^{\oplus 3}$	1	$\mathfrak{S}_3$	6
4	A_{24}	2	1	2
5	$D_{12}^{\oplus 2}$	1	$\mathfrak{S}_2$	2
6	$A_{17}\oplus E_7$	2	1	2
7	$D_{10}\oplus E_7^{\oplus 2}$	1	$\mathfrak{S}_2$	2
8	$A_{15}\oplus D_9$	2	1	2
9	$D_8^{\oplus 3}$	1	$\mathfrak{S}_3$	6
10	$A_{12}^{\oplus 2}$	2	$\mathfrak{S}_2$	4
11	$A_{11}\oplus D_7\oplus E_6$	2	1	2
12	$E_6^{\oplus 4}$	2	$\mathfrak{S}_4$	48
13	$A_9^{\oplus 2}\oplus D_6$	2	$\mathfrak{S}_2$	4
14	$D_6^{\oplus 4}$	1	$\mathfrak{S}_4$	24
15	$A_8^{\oplus 3}$	2	$\mathfrak{S}_3$	12
16	$A_7^{\oplus 2}\oplus D_5^{\oplus 2}$	2	$\mathfrak{S}_2\times\mathfrak{S}_2$	8
17	$A_6^{\oplus 4}$	2	$\mathfrak{A}_4$	24
18	$A_5^{\oplus 4}\oplus D_4$	2	$\mathfrak{S}_4$	48
19	$D_4^{\oplus 6}$	3	$\mathfrak{S}_6$	2160
20	$A_4^{\oplus 6}$	2	$\mathfrak{S}_5$	240
21	$A_3^{\oplus 8}$	2	$\mathbb{F}_2^3\rtimes\mathrm{GL}(3,\mathbb{F}_2)$	2688
22	$A_2^{\oplus 12}$	2	M_{12}	190080
23	$A_1^{\oplus 24}$	1	M_{24}	244823040

10.2. Abstract groups and discriminant forms

We give the list of a complete representative $\{(\mathfrak{G}_n,q_n)\}$ of $\mathcal{Q}/\sim$. Recall that

$$\begin{aligned}\mathcal{Q}&=\big\{(\mathfrak{G},q)\,\big|\,\exists G\in\mathcal{L}\text{ such that }\mathfrak{G}=[G],q\cong q(\Lambda_G)\big\}\\&=\big\{(\mathfrak{G},q)\,\big|\,\exists (G,N)\in\mathcal{N}\text{ such that }\mathfrak{G}=[G],q\cong q(N_G)\big\}\end{aligned}$$

and $(\mathfrak{G},q)\sim(\mathfrak{G}',q')$ if and only if $\mathfrak{G}=\mathfrak{G}'$, $q\cong q'$ (see Section 3.4). For $q:A(q)\to\mathbb{Q}/2\mathbb{Z}$, we denote the order of $A(q)$ by $|q|$. We use the following notation (see [5]):

$$a^{+n} = q_+^{(p)}(a)^{\oplus n}, \qquad a^{-n} = q_+^{(p)}(a)^{\oplus n-1} \oplus q_-^{(p)}(a),$$
$$b_{\mathrm{II}}^{+n} = u^{(2)}(b)^{\oplus n}, \qquad b_{\mathrm{II}}^{-n} = u^{(2)}(b)^{\oplus n-1} \oplus v^{(2)}(b), \qquad b_t^{dr} = q\big(L^{(2)}_{r,d,t,\mathrm{I}}(b)\big),$$

where p is an odd prime, $a = p^k$, $b = 2^k$, and $L^{(2)}_{r,d,t,e}$ is a (unique) unimodular lattice over $\mathbb{Z}_2$ which has the invariants r, d, t, e defined in Proposition 1.6 (see Section 1). For example,

$$A(q_{63}) \cong (\mathbb{Z}/2)^{\oplus 3} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z},$$
$$q_{63} \cong \langle -1/2 \rangle \oplus \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \oplus \langle 2/3 \rangle \oplus \langle 2/9 \rangle.$$

In the list, for example, q_5 is isomorphic to q_{16}. The column i indicates the catalog number of $\mathfrak{G}_n$ in GAP (see Appendix).

Table 10.2

n	$\lvert\mathfrak{G}_n\rvert$	i	$\mathfrak{G}_n$	$\lvert q_n\rvert$	q_n	$c(\mathfrak{G}_n)$
1	2	1	C_2	256	2_{II}^{+8}	8
2	3	1	C_3	729	3^{+6}	12
3	4	2	C_2^2	1024	$2_{\mathrm{II}}^{-6}, 4_{\mathrm{II}}^{-2}$	12
4	4	1	C_4	1024	$2_2^{+2}, 4_{\mathrm{II}}^{+4}$	14
5	5	1	C_5	625	$\sharp 16$	16
6	6	1	D_6	972	$2_{\mathrm{II}}^{-2}, 3^{+5}$	14
7	6	2	C_6	1296	$\sharp 18$	16
8	7	1	C_7	343	$\sharp 33$	18
9	8	5	C_2^3	1024	$2_{\mathrm{II}}^{+6}, 4_2^{+2}$	14
10	8	3	D_8	1024	4_1^{+5}	15
11	8	2	$C_2 \times C_4$	1024	$\sharp 22$	16
12	8	4	Q_8	512	$2_7^{-3}, 8_{\mathrm{II}}^{-2}$	17
13	8	4	Q_8	512	$\sharp 40$	17
14	8	1	C_8	512	$\sharp 26$	18
15	9	2	C_3^2	729	$\sharp 30$	16
16	10	1	D_{10}	625	5^{+4}	16
17	12	3	$\mathfrak{A}_4$	576	$2_{\mathrm{II}}^{-2}, 4_{\mathrm{II}}^{-2}, 3^{+2}$	16
18	12	4	D_{12}	1296	$2_{\mathrm{II}}^{+4}, 3^{+4}$	16

Table 10.2

n	$\lvert\mathfrak{G}_n\rvert$	i	$\mathfrak{G}_n$	$\lvert q_n\rvert$	q_n	$c(\mathfrak{G}_n)$
19	12	5	$C_2 \times C_6$	1728	$\sharp 61$	18
20	12	1	Q_{12}	432	$\sharp 61$	18
21	16	14	C_2^4	512	$2_{\mathrm{II}}^{+6}, 8_1^{+1}$	15
22	16	11	$C_2 \times D_8$	1024	$2_{\mathrm{II}}^{+2}, 4_0^{+4}$	16
23	16	3	$\Gamma_2 c_1$	512	$\sharp 39$	17
24	16	13	$Q_8 * C_4$	1024	$\sharp 40$	17
25	16	2	C_4^2	1024	$\sharp 75$	18
26	16	8	SD_{16}	512	$2_7^{+1}, 4_7^{+1}, 8_{\mathrm{II}}^{+2}$	18
27	16	12	$C_2 \times Q_8$	256	$\sharp 75$	18
28	16	6	$\Gamma_2 d$	256	$\sharp 80$	19
29	16	9	Q_{16}	256	$\sharp 80$	19
30	18	4	$\mathfrak{A}_{3,3}$	729	$3^{+4}, 9^{-1}$	16
31	18	3	$C_3 \times D_6$	972	$\sharp 48$	18
32	20	3	$\mathrm{Hol}(C_5)$	500	$2_6^{-2}, 5^{+3}$	18
33	21	1	$C_7 \rtimes C_3$	343	7^{+3}	18
34	24	12	$\mathfrak{S}_4$	576	$4_3^{+3}, 3^{+2}$	17
35	24	13	$C_2 \times \mathfrak{A}_4$	576	$\sharp 51$	18
36	24	8	$C_3 \rtimes D_8$	432	$\sharp 61$	18
37	24	3	T_{24}	384	$\sharp 77$	19
38	24	3	T_{24}	384	$\sharp 54$	19
39	32	27	$2^4 C_2$	512	$2_{\mathrm{II}}^{+2}, 4_0^{+2}, 8_7^{+1}$	17
40	32	49	$Q_8 * Q_8$	1024	4_7^{+5}	17
41	32	6	$\Gamma_7 a_1$	512	$\sharp 56$	18
42	32	31	$\Gamma_4 c_2$	256	$\sharp 75$	18
43	32	7	$\Gamma_7 a_2$	256	$\sharp 80$	19
44	32	11	$\Gamma_3 e$	256	$\sharp 80$	19
45	32	44	$\Gamma_6 a_2$	256	$\sharp 80$	19
46	36	9	$3^2 C_4$	324	$2_6^{-2}, 3^{+2}, 9^{-1}$	18
47	36	11	$C_3 \times \mathfrak{A}_4$	432	$\sharp 61$	18
48	36	10	$\mathfrak{S}_{3,3}$	972	$2_{\mathrm{II}}^{-2}, 3^{+3}, 9^{-1}$	18
49	48	50	$2^4 C_3$	384	$2_{\mathrm{II}}^{-4}, 8_1^{+1}, 3^{-1}$	17
50	48	3	$4^2 C_3$	256	$\sharp 75$	18
51	48	48	$C_2 \times \mathfrak{S}_4$	576	$2_{\mathrm{II}}^{+2}, 4_2^{+2}, 3^{+2}$	18
52	48	49	$2^2(C_2 \times C_6)$	288	$\sharp 78$	19

(*continued*)

Table 10.2

n	$\|\mathfrak{G}_n\|$	i	$\mathfrak{G}_n$	$\|q_n\|$	q_n	$c(\mathfrak{G}_n)$
53	48	30	$2^2 Q_{12}$	288	$\sharp 78$	19
54	48	29	T_{48}	384	$2_7^{+1}, 8_{II}^{-2}, 3^{-1}$	19
55	60	5	$\mathfrak{A}_5$	300	$2_{II}^{-2}, 3^{+1}, 5^{-2}$	18
56	64	138	$\Gamma_{25}a_1$	512	$4_5^{+3}, 8_1^{+1}$	18
57	64	242	$\Gamma_{13}a_1$	256	$\sharp 75$	18
58	64	32	$\Gamma_{22}a_1$	256	$\sharp 80$	19
59	64	35	$\Gamma_{23}a_2$	256	$\sharp 80$	19
60	64	136	$\Gamma_{26}a_2$	256	$\sharp 80$	19
61	72	43	$\mathfrak{A}_{4,3}$	432	$4_{II}^{-2}, 3^{-3}$	18
62	72	40	N_{72}	324	$4_1^{+1}, 3^{+2}, 9^{-1}$	19
63	72	41	M_9	216	$2_7^{-3}, 3^{-1}, 9^{-1}$	19
64	80	49	$2^4 C_5$	160	$\sharp 81$	19
65	96	227	$2^4 D_6$	384	$2_{II}^{-2}, 4_7^{+1}, 8_1^{+1}, 3^{-1}$	18
66	96	70	$2^4 C_6$	384	$\sharp 76$	19
67	96	64	$4^2 D_6$	256	$\sharp 80$	19
68	96	195	$2^3 D_{12}$	288	$\sharp 78$	19
69	96	204	$(Q_8 * Q_8) \rtimes C_3$	192	$\sharp 77$	19
70	120	34	$\mathfrak{S}_5$	300	$4_3^{-1}, 3^{+1}, 5^{-2}$	19
71	128	931	F_{128}	256	$\sharp 80$	19
72	144	184	$\mathfrak{A}_4^2$	288	$\sharp 78$	19
73	160	234	$2^4 D_{10}$	160	$\sharp 81$	19
74	168	42	$L_2(7)$	196	$4_1^{+1}, 7^{+2}$	19
75	192	1023	$4^2 \mathfrak{A}_4$	256	$2_{II}^{-2}, 8_6^{-2}$	18
76	192	955	H_{192}	384	$4_4^{-2}, 8_7^{+1}, 3^{-1}$	19
77	192	1493	T_{192}	192	$4_7^{-3}, 3^{+1}$	19
78	288	1026	$\mathfrak{A}_{4,4}$	288	$2_{II}^{+2}, 8_1^{+1}, 3^{+2}$	19
79	360	118	$\mathfrak{A}_6$	180	$4_5^{-1}, 3^{+2}, 5^{+1}$	19
80	384	18135	F_{384}	256	$4_7^{+1}, 8_6^{+2}$	19
81	960	11357	M_{20}	160	$2_{II}^{-2}, 8_1^{+1}, 5^{-1}$	19

10.3. Invariant lattices Λ^G

For $G \in \mathcal{L}$, there exists a number n such that $([G], q(\Lambda_G)) \sim (\mathfrak{G}_n, q_n)$ (see Table 10.2). Here we give the invariant lattices Λ^G for each n. We set

$$r = \operatorname{rank} \Lambda^G = 22 - c(G), \qquad d = \operatorname{disc} \Lambda^G, \qquad q = -q_n \cong q(\Lambda^G).$$

In the table, we set

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad D_4 = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix},$$

and E_8 denotes the root lattice of type E_8, as usual. For abelian $G \in \mathcal{L}$, the Gramian matrices of Λ^G were determined in [11].

Table 10.3

n	r	d	q	Gramian matrix
1	14	-256	2_{II}^{+8}	$U^{\oplus 3} \oplus E_8(-2)$
2	10	-729	3^{+6}	$U \oplus U(3)^{\oplus 2} \oplus A_2(-1)^{\oplus 2}$
3	10	-1024	$2_{\mathrm{II}}^{-6}, 4_{\mathrm{II}}^{-2}$	$U \oplus U(2)^{\oplus 2} \oplus D_4(-2)$
4	8	-1024	$2_6^{+2}, 4_{\mathrm{II}}^{+4}$	$U \oplus U(4)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2}$
6	8	-972	$2_{\mathrm{II}}^{-2}, 3^{-5}$	$U(3) \oplus A_2(2) \oplus A_2(-1)^{\oplus 2}$
9	8	-1024	$2_{\mathrm{II}}^{+6}, 4_6^{+2}$	$U(2)^{\oplus 3} \oplus \langle -4 \rangle^{\oplus 2}$
10	7	1024	4_7^{+5}	$U \oplus \langle 4 \rangle^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 3}$
12	5	512	$2_1^{-3}, 8_{\mathrm{II}}^{-2}$	$\left(\begin{smallmatrix} 6 & 2 & 2 \\ 2 & 6 & -2 \\ 2 & -2 & 6 \end{smallmatrix}\right) \oplus \langle -2 \rangle^{\oplus 2}$
16	6	-625	5^{+4}	$U \oplus U(5)^{\oplus 2}$
17	6	-576	$2_{\mathrm{II}}^{-2}, 4_{\mathrm{II}}^{-2}, 3^{+2}$	$U \oplus A_2(2) \oplus A_2(-4)$
18	6	-1296	$2_{\mathrm{II}}^{+4}, 3^{+4}$	$U \oplus U(6)^{\oplus 2}$
21	7	512	$2_{\mathrm{II}}^{+6}, 8_7^{+1}$	$U(2)^{\oplus 3} \oplus \langle -8 \rangle$
22	6	-1024	$2_{\mathrm{II}}^{+2}, 4_0^{+4}$	$U(2) \oplus \langle 4 \rangle^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$
26	4	-512	$2_1^{+1}, 4_1^{+1}, 8_{\mathrm{II}}^{+2}$	$U(8) \oplus \langle 2 \rangle \oplus \langle 4 \rangle$
30	6	-729	$3^{+4}, 9^{+1}$	$U(3)^{\oplus 2} \oplus \left(\begin{smallmatrix} 2 & 3 \\ 3 & 0 \end{smallmatrix}\right)$
32	4	-500	$2_2^{-2}, 5^{+3}$	$U(5) \oplus \left(\begin{smallmatrix} 4 & 2 \\ 2 & 6 \end{smallmatrix}\right)$
33	4	-343	7^{-3}	$U(7) \oplus \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)$
34	5	576	$4_5^{+3}, 3^{+2}$	$U \oplus A_2(4) \oplus \langle -12 \rangle$
39	5	512	$2_{\mathrm{II}}^{+2}, 4_0^{+2}, 8_1^{+1}$	$U(2) \oplus \langle 4 \rangle \oplus \langle -4 \rangle \oplus \langle 8 \rangle$
40	5	1024	4_1^{+5}	$\langle 4 \rangle^{\oplus 3} \oplus \langle -4 \rangle^{\oplus 2}$
46	4	-324	$2_2^{-2}, 3^{+2}, 9^{+1}$	$A_2 \oplus \langle 6 \rangle \oplus \langle -18 \rangle$
48	4	-972	$2_{\mathrm{II}}^{-2}, 3^{-3}, 9^{+1}$	$U(3) \oplus A_2(6)$
49	5	384	$2_{\mathrm{II}}^{-4}, 8_7^{+1}, 3^{+1}$	$U(2) \oplus A_2(2) \oplus \langle -8 \rangle$

(*continued*)

Table 10.3

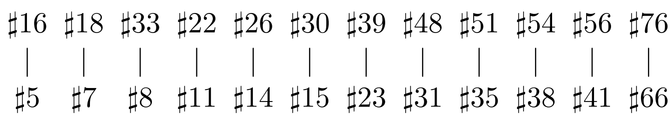
n	r	d	q	Gramian matrix
51	4	-576	$2_{\mathrm{II}}^{+2}, 4_6^{+2}, 3^{+2}$	$U(2) \oplus \langle 12 \rangle^{\oplus 2}$
54	3	384	$2_1^{+1}, 8_{\mathrm{II}}^{-2}, 3^{+1}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 16 & 8 \\ 0 & 8 & 16 \end{pmatrix}$
55	4	-300	$2_{\mathrm{II}}^{-2}, 3^{-1}, 5^{-2}$	$U \oplus A_2(10)$
56	4	-512	$4_3^{+3}, 8_7^{+1}$	$\langle 4 \rangle^{\oplus 3} \oplus \langle -8 \rangle$
61	4	-432	$4_{\mathrm{II}}^{-2}, 3^{+3}$	$U(3) \oplus A_2(4)$
62	3	324	$4_7^{+1}, 3^{+2}, 9^{+1}$	$\begin{pmatrix} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 12 \end{pmatrix}$
63	3	216	$2_1^{-3}, 3^{+1}, 9^{+1}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & 6 \\ 0 & 6 & 12 \end{pmatrix}$
65	4	-384	$2_{\mathrm{II}}^{-2}, 4_1^{+1}, 8_7^{+1}, 3^{+1}$	$A_2(2) \oplus \langle 4 \rangle \oplus \langle -8 \rangle$
70	3	300	$4_5^{-1}, 3^{-1}, 5^{-2}$	$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 20 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 2 \\ 2 & 6 & 1 \\ 2 & 1 & 16 \end{pmatrix}$
74	3	196	$4_7^{+1}, 7^{+2}$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 28 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 2 \\ 2 & 8 & 1 \\ 2 & 1 & 8 \end{pmatrix}$
75	4	-256	$2_{\mathrm{II}}^{-2}, 8_2^{-2}$	$\begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 2 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}$
76	3	384	$4_4^{-2}, 8_1^{+1}, 3^{+1}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$
77	3	192	$4_1^{-3}, 3^{-1}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix}$
78	3	288	$2_{\mathrm{II}}^{+2}, 8_7^{+1}, 3^{+2}$	$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 2 \\ 4 & 2 & 8 \end{pmatrix}$
79	3	180	$4_3^{-1}, 3^{+2}, 5^{+1}$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}, \begin{pmatrix} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 8 \end{pmatrix}$
80	3	256	$4_1^{+1}, 8_2^{+2}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$
81	3	160	$2_{\mathrm{II}}^{-2}, 8_7^{+1}, 5^{-1}$	$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 12 \end{pmatrix}$

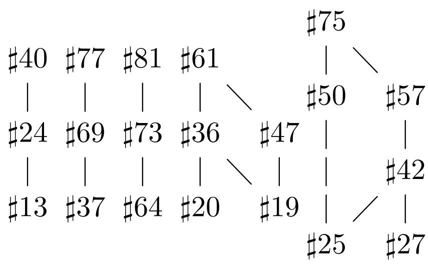
10.4. Trees of groups with common invariant lattices

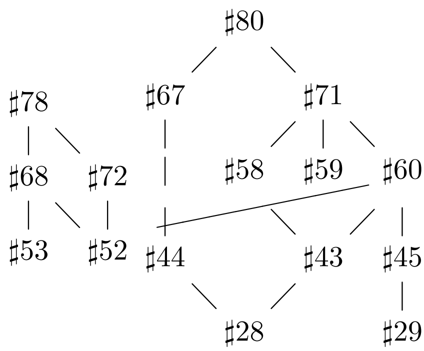
We give the trees of

$$T_S = \{ \mathfrak{G}_n \mid S(\mathfrak{G}_n, q_n) \cong S \} = \{ \mathfrak{G}_n \mid q_n \cong q(S) \}$$

for T_S with $\sharp T_S \geq 2$. In the table below, $\sharp n$ denotes $\mathfrak{G}_n$. The maximal element in each T_S corresponds to an element in $\mathcal{Q}_{\mathrm{clos}}$ defined by (4.7). The extensions $\sharp 5 - \sharp 16$, $\sharp 7 - \sharp 18$, $\sharp 11 - \sharp 22$, and $\sharp 15 - \sharp 30$ are studied in detail by Garbagnati ([8], [9]).







10.5. Extensions of $G \in \mathcal{L}$

We give the list of possible extensions of $G \in \mathcal{L}_{\mathrm{clos}}$. For example, let $G \in \mathcal{L}$ of type $(\mathfrak{G}_{55}, q_{55})$; that is, $([G], q(\Lambda_G)) \sim (\mathfrak{G}_{55}, q_{55})$. Then, for $i = 1, 2$, there exists an element $G' \in \mathcal{L}_{\mathrm{clos}}$ of type $(\mathfrak{G}_{79}, q_{79})$ such that $G \subset G'$ and G' is conjugate to G_i in Theorem 8.10(2). We omit the 11 maximal cases, $n = 54, 62, 63, 70, 74, 76, 77, 78, 79, 80, 81$, for there is no proper extension.

Table 10.5

n	Extensions
1	3, 4, 6, 9, 10, 12, 16, 17, 18, 21, 22, 26, 30, 32, 34, 39, 40, 46, 48, 49, 51, 54, 55, 56, 61, 62, 63, 65, 70, 74, 75, 76, 77, 78, 79, 80, 81
2	6, 17, 18, 30, 33, 34, 46, 48, 49, 51, 54, 55, 61, 62, 63, 65, 70, 74, 75, 76, 77, 78, 79, 80, 81
3	9, 10, 17, 18, 21, 22, 26, 34, 39, 40, 48, 49, 51, 54, 55, 56, 61, 62, 65, 70, 74, 75, 76, 77, 78, 79, 80, 81
4	10, 12, 22, 26, 32, 34, 39, 40, 46, 51, 54, 56, 61, 62, 63, 65, 70, 74, 75, 76, 77, 78, 79, 80, 81
6	18, 30, 34, 46, 48, 51, 54, 55, 61, 62, 63, 65, 70, 74, 76, 77, 78, 79, 80, 81
9	21, 22, 39, 40, 49, 51, 56, 65, 75, 76, 77, 78, 80, 81
10	22, 26, 34, 39, 40, 51, 54, 56, 61, 62, 65, 70, 74, 75, 76, 77, 78, 79, 80, 81
12	26, 54, 63, 75, 80, 81
16	32, 55, 70, 79, 81
17	34, 49, 51, 55, 61, 65, 70, 74, 75, 76, 77, 78, 79, 80, 81
18	48, 51, 54, 61, 62, 70, 76, 77, 78
21	39, 49, 56, 65, 75, 76, 77, 78, 80, 81
22	39, 40, 51, 56, 65, 75, 76, 77, 78, 80, 81
26	54, 80
30	46, 48, 61, 62, 63, 78, 79
32	70
33	74
34	51, 61, 65, 70, 74, 76, 77, 78, 79, 80, 81
39	56, 65, 75, 76, 77, 78, 80, 81
40	56, 76, 77, 80
46	62, 63, 79
48	62
49	65, 75, 76, 78, 80, 81
51	76, 77, 78
55	70, 79, 81
56	76, 77, 80
61	78
65	76, 78, 80, 81
75	80, 81

10.6. Root types of N^G

We give the type of the root sublattice of N^G, which is generated by vectors $v \in N^G$ with $\langle v, v\rangle = -2$, for each $(G, N) \in \mathcal{N}$ such that $[G] = \mathfrak{G}_n$ and $q(N_G) \cong q_n$ (see Table 10.2). In the list, elements in $\mathcal{N}'$ are enclosed by boxes (see Proposition 3.10), and the number of vectors $v \in N^G$ with

$\langle v,v\rangle = -4$ or -6 are given for the cases $n = 32, 41, 56, 63$. For Niemeier lattices $N = N_i$, see Table 10.1.

Table 10.6

$n = 1$:

i	3	6	7	8	9
Type	E_8	$A_1^{\oplus 9} \oplus E_7$	D_9	$A_1^{\oplus 8} \oplus D_8$	D_8

i	11	12	12	13	14
Type	$A_1^{\oplus 6} \oplus D_4 \oplus D_6$	$D_4^{\oplus 4}$	$D_4 \oplus E_6$	$A_1^{\oplus 10} \oplus D_6$	$D_5^{\oplus 2}$

i	15	16	16	16	18
Type	A_8	$A_1^{\oplus 8} \oplus D_4^{\oplus 2}$	$A_1^{\oplus 4} \oplus A_7$	$D_4 \oplus D_5$	$A_1^{\oplus 12} \oplus D_4$

i	18	19	19	20	21
Type	$A_1^{\oplus 3} \oplus A_3 \oplus A_5$	$A_3^{\oplus 4}$	$D_4^{\oplus 2}$	$A_4^{\oplus 2}$	$A_1^{\oplus 16}$

i	21	22	$\boxed{23}$
Type	$A_1^{\oplus 4} \oplus A_3^{\oplus 2}$	$A_2^{\oplus 4}$	$A_1^{\oplus 8}$

$n = 2$:

i	12	14	17	18	19	19	21	22	$\boxed{23}$
Type	E_6	D_6	A_6	$A_2 \oplus A_5$	$A_2^{\oplus 6}$	$D_4 \oplus A_2^{\oplus 2}$	$A_3^{\oplus 2}$	$A_2^{\oplus 3}$	$A_1^{\oplus 6}$

$n = 3$:

i	12	16	16	18	19	19	21
Type	$D_4^{\oplus 2}$	$A_1^{\oplus 8}$	$D_4^{\oplus 2}$	$A_1^{\oplus 6} \oplus A_3$	$A_3^{\oplus 2}$	$D_4^{\oplus 2}$	$A_1^{\oplus 4}$

i	21	21	21	$\boxed{22}$	23	23
Type	$A_1^{\oplus 8}$	$A_3 \oplus A_1^{\oplus 6}$	$A_3^{\oplus 2}$	$A_2^{\oplus 2}$	$A_1^{\oplus 4}$	$A_1^{\oplus 8}$

$n = 4$:

i	13	18	19	20	21	22	$\boxed{23}$
Type	D_5	D_4	$A_3^{\oplus 2}$	$A_1^{\oplus 2} \oplus A_4$	$A_1^{\oplus 2} \oplus A_3$	$A_1^{\oplus 2} \oplus A_2^{\oplus 2}$	$A_1^{\oplus 4}$

(*continued*)

$n=5,16$:

i	19	20	22	$\boxed{23}$
Type	D_4	A_4	$A_2^{\oplus 2}$	$A_1^{\oplus 4}$

$n=6$:

i	12	12	14	18	18	19
Type	D_4	E_6	D_5	$A_1^{\oplus 3}\oplus A_2$	$A_2\oplus A_5$	$A_2^{\oplus 4}$

i	19	19	21	22	22	$\boxed{23}$
Type	$A_2^{\oplus 2}\oplus A_3$	D_4	$A_1^{\oplus 2}\oplus A_3$	A_2	$A_2^{\oplus 3}$	$A_1^{\oplus 4}$

$n=7,18$:

i	12	18	19	19	21	22	$\boxed{23}$
Type	D_4	$A_1^{\oplus 3}\oplus A_2$	$A_2^{\oplus 2}$	A_3	$A_1^{\oplus 4}$	A_2	$A_1^{\oplus 2}$

$n=8,33$:

i	21	$\boxed{23}$
Type	A_3	$A_1^{\oplus 3}$

$n=9$:

i	21	21	$\boxed{23}$	23	23
Type	$A_1^{\oplus 4}$	$A_1^{\oplus 8}$	$A_1^{\oplus 2}$	$A_1^{\oplus 4}$	$A_1^{\oplus 8}$

$n=10$:

i	18	19	21	21	22	$\boxed{23}$	23
Type	A_3	$A_3^{\oplus 2}$	$A_1^{\oplus 4}$	$A_1^{\oplus 2}\oplus A_3$	$A_2^{\oplus 2}$	$A_1^{\oplus 2}$	$A_1^{\oplus 4}$

$n=11,22$:

i	21	$\boxed{23}$	23
Type	$A_1^{\oplus 4}$	$A_1^{\oplus 2}$	$A_1^{\oplus 4}$

$n=12$:

i	18	22	$\boxed{23}$
Type	D_4	$A_1^{\oplus 3}\oplus A_2$	$A_1^{\oplus 4}$

$n=13,24,28,29,37,40,43,44,45,59,60,67,69,71,77,80$:

i	$\boxed{23}$
Type	$A_1^{\oplus 2}$

$n = 14, 26$:

i	18	22	$\boxed{23}$
Type	A_3	$A_1 \oplus A_2$	$A_1^{\oplus 2}$

$n = 15, 30$:

i	19	22	$\boxed{23}$
Type	$A_2^{\oplus 3}$	$A_2^{\oplus 3}$	$A_1^{\oplus 3}$

$n = 17$:

i	19	19	21	21	22	23	23	$\boxed{23}$
Type	$A_2^{\oplus 2}$	$A_2 \oplus D_4$	A_3	$A_3^{\oplus 2}$	$A_2^{\oplus 2}$	$A_1^{\oplus 2}$	$A_1^{\oplus 4}$	$A_1^{\oplus 5}$

$n = 19, 20, 36, 47, 61$:

i	19	$\boxed{23}$
Type	$A_2^{\oplus 2}$	$A_1^{\oplus 2}$

$n = 21$:

i	23	$\boxed{23}$
Type	$A_1^{\oplus 4}$	$A_1^{\oplus 8}$

$n = 23, 39$:

i	$\boxed{23}$	23	23
Type	$A_1^{\oplus 2}$	$A_1^{\oplus 4}$	$A_1^{\oplus 4}$

$n = 25, 27, 42, 50, 57, 75$:

i	$\boxed{23}$
Type	$A_1^{\oplus 4}$

$n = 31$:

i	19	19	22	$\boxed{23}$
Type	A_2	A_2	A_2	A_1

$n = 32$:

i	19	20	20	22	$\boxed{23}$
Type	A_3	$A_1^{\oplus 2}$	A_4	$A_1 \oplus A_2$	$A_1^{\oplus 2}$
$\sharp\{v \in N^G \mid \langle v, v \rangle = -4\}$		14			22

(*continued*)

$n = 34$:

i	19	19	21	21	21	22	23	23	23	$\boxed{23}$
Type	$A_2^{\oplus 2}$	$A_2 \oplus A_3$	$A_1^{\oplus 2}$	$A_1^{\oplus 2} \oplus A_3$	A_3	$A_2^{\oplus 2}$	$A_1^{\oplus 2}$	$A_1^{\oplus 2}$	$A_1^{\oplus 3}$	$A_1^{\oplus 4}$

$n = 35, 51$:

i	21	21	$\boxed{23}$	23	23
Type	$A_1^{\oplus 2}$	$A_1^{\oplus 4}$	A_1	$A_1^{\oplus 2}$	$A_1^{\oplus 2}$

$n = 38, 54$:

i	18	22	$\boxed{23}$
Type	A_2	A_2	A_1

$n = 41$:

i	23	23	$\boxed{23}$
Type	$A_1^{\oplus 2}$	$A_1^{\oplus 2}$	$A_1^{\oplus 2}$
$\sharp\{v \in N^G \mid \langle v, v\rangle = -4\}$	26	26	42

$n = 46$:

i	22	22	$\boxed{23}$
Type	$A_1^{\oplus 2} \oplus A_2$	$A_1 \oplus A_2^{\oplus 2}$	$A_1^{\oplus 3}$

$n = 48$:

i	19	22	$\boxed{23}$
Type	A_2	A_2	A_1

$n = 49$:

i	23	23	$\boxed{23}$
Type	A_1	$A_1^{\oplus 4}$	$A_1^{\oplus 5}$

$n = 52, 53, 68, 72, 78$:

i	23	$\boxed{23}$
Type	A_1	$A_1^{\oplus 2}$

$n = 55$:

i	19	22	22	23	$\boxed{23}$
Type	D_4	A_2	$A_2^{\oplus 2}$	$A_1^{\oplus 3}$	$A_1^{\oplus 4}$

$n=56$:

i	23	$\boxed{23}$
Type	$A_1^{\oplus 2}$	$A_1^{\oplus 2}$
$\sharp\{v \in N^G \mid \langle v,v\rangle = -4\}$	26	42

$n=58$:

i	23	23
Type	$A_1^{\oplus 2}$	$A_1^{\oplus 2}$

$n=62$:

i	22	$\boxed{23}$
Type	A_2	A_1

$n=63$:

i	22	22	$\boxed{23}$
Type	$A_1^{\oplus 3}$	$A_1^{\oplus 2} \oplus A_2$	$A_1^{\oplus 3}$
$\sharp\{v \in N^G \mid \langle v,v\rangle = -6\}$	14		26

$n=64,73,81$:

i	23	$\boxed{23}$
Type	$A_1^{\oplus 3}$	$A_1^{\oplus 4}$

$n=65$:

i	23	23	23	$\boxed{23}$
Type	A_1	$A_1^{\oplus 2}$	$A_1^{\oplus 3}$	$A_1^{\oplus 4}$

$n=66,76$:

i	23	23	$\boxed{23}$
Type	A_1	A_1	$A_1^{\oplus 2}$

$n=70$:

i	19	22	23	$\boxed{23}$
Type	A_3	A_2	A_1	$A_1^{\oplus 2}$

$n=74$:

i	21	23	$\boxed{23}$
Type	A_3	$A_1^{\oplus 2}$	$A_1^{\oplus 3}$

(*continued*)

$n = 79$:

i	22	23	$\boxed{23}$
Type	$A_2^{\oplus 2}$	$A_1^{\oplus 2}$	$A_1^{\oplus 3}$

Appendix: computations using GAP

In this appendix, we briefly explain how to check Lemmas 8.2–8.4 using GAP [12].

For Lemma 8.2, consider the case $\mathfrak{H} = T_{192}$ $(= \mathfrak{G}_{77})$. By Table 10.4, it is sufficient to check that there exists only one conjugacy class of subgroups of T_{192} which are isomorphic to $T_{24} = \mathfrak{G}_{37}$ (resp., $(Q_8 * Q_8) \rtimes C_3 = \mathfrak{G}_{69}$). GAP has the catalog of all groups of small orders, and the command `SmallGroup`(k, i) returns the ith group of order k in the catalog (see [2]). For example, `SmallGroup`$(192, 1493)$ returns T_{192} by Table 10.2. The command `IsomorphicSubgroups`(G, H) enumerates all conjugacy classes of subgroups of G which are isomorphic to H. Hence, we can check the assertion as follows.*

```
gap> h:=SmallGroup(192,1493);;
gap> g1:=SmallGroup(24,3);;
gap> g2:=SmallGroup(96,204);;
gap> Size( IsomorphicSubgroups( h , g1 ) );
1
gap> Size( IsomorphicSubgroups( h , g2 ) );
1
```

Here the command `Size`(a) returns the size of the object a. The cases $\mathfrak{H} = T_{48}, H_{192}, M_{20}$ are similar.

For Lemma 8.3, we realize $G, \widetilde{G}$ as quotient groups of subgroups of $\mathfrak{S}_{36}$. For example, the linear transformations

$$(x, y, z, u, v, w) \mapsto (e^{2\pi i/6} x, y, z, u, v, w),$$
$$(x, y, z, u, v, w) \mapsto (y, x, z, u, v, w)$$

correspond to

$$(1\ 2\ 3\ 4\ 5\ 6),$$
$$(1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12),$$

respectively. We can check Lemma 8.3 as follows.

*A command terminated by two semicolons does not show the result.

```
a1:=(1,2,3,4,5,6);
a2:=(7,8,9,10,11,12);
a3:=(13,14,15,16,17,18);
a4:=(19,20,21,22,23,24);
a5:=(25,26,27,28,29,30);
a6:=(31,32,33,34,35,36);
a123456:=a1*a2*a3*a4*a5*a6;

b123:=(1,7,13)(2,8,14)(3,9,15)(4,10,16)(5,11,17)*
(6,12,18);
b456:=(19,25,31)(20,26,32)(21,27,33)(22,28,34)*
(23,29,35)(24,30,36);
b23:=(7,13)(8,14)(9,15)(10,16)(11,17)(12,18);
b56:=(25,31)(26,32)(27,33)(28,34)(29,35)(30,36);
b14:=(1,19)(2,20)(3,21)(4,22)(5,23)(6,24);
b2536:=(7,25,13,31)(8,26,14,32)(9,27,15,33)*
(10,28,16,34)(11,29,17,35)(12,30,18,36);

g0:=Group(
a1^3*a2^3,
a4^3*a5^3,
a5^2*a6^4*b123,
a2^4*a3^2*b456,
a1^3*a2^3*a3^3*b23*b56,
a123456
);
gg0:=ClosureGroup(gg0, b14*b2536 );
n:=Group(a123456);
f:=NaturalHomomorphismByNormalSubgroup(gg0,n);
g:=Image(f,g0);
gg:=Image(f);

list:=[[48,49],[48,30],[96,195],[144,184]];
   for nn in list do
   subgrps:=IsomorphicSubgroups(gg,SmallGroup(nn));
   subgrps:=Filtered(subgrps,x->IsSubgroup(g,Image(x)));
   Display(Size(subgrps));
od;
```

Here, for example, `a1^3*a2^3` corresponds to the transformation (8.6). The quotient groups `g`, `gg` by the group `n`, which corresponds to the subgroup of homothetic transformations, are $G, \widetilde{G}$, respectively. By Table 10.4, it is sufficient to check that there exists only one conjugacy class of subgroups of $\widetilde{G}$ which are isomorphic to $\mathfrak{G}_n$ and contained in G for $n = 52, 53, 68, 72$. This is done in the last paragraph of the above program. The result is the following:

```
1
1
1
1
```

Thus, Lemma 8.3 has been checked. Lemma 8.4 is similarly checked.

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