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A SUFFICIENT CONDITION FOR ASYMPTOTIC SUFFICIENCY OF INCOMPLETE OBSERVATIONS OF A DIFFUSION PROCESS

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Consider an m-dimensional diffusion process (X_t) with unknown drift and small known variance observed on a time interval $[0, T]$. We derive here a general condition ensuring the asymptotic sufficiency, in the sense of Le Cam, of incomplete observations of $(X_t)_{0 \le t \le T}$ with respect to the complete observation of the diffusion as the variance goes to 0. We then construct estimators based on these partial observations which are consistent, asymptotically Gaussian and asymptotically equivalent to the maximum likelihood estimator based on the observation of the complete sample path on $[0, T]$. Finally, we study when this condition is satisfied for various incomplete observations which often arise in practice: discrete observations, observation of a smoothed diffusion, observation of the first hitting times and positions of concentric spheres, complete or partial observation of the record process for one-dimensional diffusions.

1. Introduction. Drift estimation for diffusion processes continuously observed throughout a time interval $[0, T]$ has been largely investigated with asymptotic results as $T \to \infty$ or as the diffusion coefficient goes to 0 [see, for instance, Ibragimov and Has'minskii (1981), Liptser and Shiryaev (1977) and Kutoyants (1984)]. However, in practice, it may be difficult to observe the sample path of a diffusion process in every detail. We consider here an m-dimensional diffusion process X_t with an unknown parameter in the drift function and a small known variance. Our concern is to derive a general condition ensuring the asymptotic sufficiency, in the sense of Le Cam (1986), of incomplete observations of the sample path (X_t) on the time interval $[0, T]$, as the diffusion coefficient goes to 0. For the incomplete observations of $(X_t)_{0 \le t \le T}$ which meet this condition, we obtain estimators based on these observations which are optimal. We then study when this condition is satisfied for various incomplete observations which often arise in practice.

Let the diffusion model be defined by the stochastic differential equation:

$$dX_t = b(X_t, \theta)\,dt + \varepsilon\sigma(X_t)\,dW_t, \qquad X_0 = x, \tag{1}$$

where (W_t) is a standard m-dimensional Brownian motion, θ is an unknown parameter in the drift function $b(\cdot, \theta)$: $\mathbf{R}^m \to \mathbf{R}^m$ and the diffusion matrix σ: $\mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}^m$, $x \in \mathbf{R}^m$ and ε are known. Consider now that the sample

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path X_t is partially observed on $[0, T]$ and assume that it is possible to build on the basis of these incomplete observations a process (Y_t) defined on $[0, T]$ which is smooth enough and meets the condition

CONDITION 1. $\sup\{\varepsilon^{-1}\|X_t - Y_t\|, 0 \le t \le T\} \to 0$ in probability as ε goes to 0.

Under regularity assumptions on the drift and diffusion coefficients, the likelihood corresponding to the observations of $(X_t)_{0 \le t \le T}$ exists. In Section 1, we first obtain that, when suitably normalized, the net of experiments associated with (X_t) converges uniformly on the precompact subsets of Θ to the Gaussian shift experiment. Second, we prove that the logarithm of the likelihood ratio may be closely approximated by asymptotically normally distributed random vectors, which only depend on the incomplete observations (Y_t) [Theorem 1(i) and (ii)]. As a consequence of these two properties, we obtain that the incomplete observations $(Y_t)_{0 \le t \le T}$ are asymptotically sufficient in the sense of Le Cam (1986) with respect to the complete observation of $(X_t)_{0 \le t \le T}$ (Corollary 1). The key tool to these results is a stochastic Taylor expansion of (X_t) in powers of ε which may be applied under some regularity assumptions on $b(\cdot, \cdot)$ and $\sigma(\cdot)$. The main assumption, crucial for establishing Theorem 1, is that there exists a function $V(u, \theta)$ such that

$$\frac{\partial}{\partial u} V(u, \theta) = (\sigma(u)\sigma(u)')^{-1} b(u, \theta)$$

(we note A' the transposition of a matrix A and $\partial/\partial u$, the partial derivative with respect to u). We then prove in Proposition 1 the existence of estimators based on these incomplete observations which are consistent, asymptotically Gaussian and asymptotically equivalent to the maximum likelihood estimator based on the observation of the complete sample path (X_t) on the same time interval. In Section 2, we study which partial observations of (X_t) meet Condition 1. This is the case for discrete observations when the sampling interval $\Delta = \Delta(\varepsilon)$ is such that $\varepsilon^{-1/2}\Delta(\varepsilon) \to 0$ (Proposition 2). Sometimes, because of some recording device, one may observe, instead of (X_t), a smoothed path (Y_t) which is the result of the convolution of (X_t) with a kernel $\varphi_\eta(t) = \eta^{-1}\varphi(t/\eta)$, where φ is a smooth symmetric function with compact support. Then (Y_t) satisfies Condition 1 if $\varepsilon^{-1/2}\eta \to 0$ (Proposition 3). This condition is also satisfied when the observations consist of the first hitting times and positions of concentric spheres if $m \ge 2$ under the main assumption that the solution of (1) associated with $\varepsilon = 0$ leaves any sphere centered at x within a finite time (Proposition 4). For one-dimensional diffusions with positive drift function, Condition 1 is verified when the observation is the record process $M_t = \sup(X_s, 0 \le s \le t)$ between two prescribed levels $x = X_0$ and $A > x$ (Proposition 5). This observation may still be weakened without losing the sufficiency property. Indeed, consider the data consisting of the time intervals greater than a positive value η where M_t is constant, coupled with the value of M_t on each of these intervals. Then, if $\varepsilon^{-2}\eta$ converges to $C \ge 0$ as $\varepsilon \to 0$, Condition 1 is met by a process (Y_t) built on these observations.

2. Asymptotic sufficiency of the observations.

2.1. *Notation and regularity assumptions.* Let $\mathbf{R}^m$ be the m-dimensional Euclidean space and let $C = C([0, +\infty) \to \mathbf{R}^m)$ be the space of all continuou functions defined on $[0, +\infty)$ with values in $\mathbf{R}^m$. We shall denote by X_t the canonical coordinates of C, $\mathscr{C}_t$ the σ-field generated by $(X_s, 0 \le s \le t)$ and $\mathscr{C} = \bigvee_{t \ge 0} \mathscr{C}_t$. Consider now an m-dimensional Wiener process $W = (W^i)_{i \le m}$ (i.e., each W^i is a standard Wiener process and the W^i's are independent), a function $b(\cdot, \theta) = (b_i(\cdot, \theta))_{i \le m}$: $\mathbf{R}^m \to \mathbf{R}^m$ depending on a parameter θ and a function σ: $\mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}^m$. We shall denote by $\mathbf{P}_{\theta, \varepsilon}$ the probability measure on $(C, \mathscr{C})$ under which the canonical process $X = (X_t)$ is a time-homogeneous diffusion solution of the stochastic differential equation (1):

$$dX_t = b(X_t, \theta)\, dt + \varepsilon \sigma(X_t)\, dW_t, \qquad X_0 = x.$$

The diffusion matrix σ, the positive constant ε and the starting point $x \in \mathbf{R}^m$ are known, and θ is an unknown parameter in Θ, a subset of $\mathbf{R}^k$. Let $(D, \mathscr{D})$ denote the space of $\mathbf{R}^m$-valued functions defined on $[0, +\infty)$ which are left-continuous with right-hand limits, endowed with the σ algebra $\mathscr{D}$ generated by the Skorokhod topology on each compact set [see Billingsley (1968)]. We shall only consider in what follows incomplete observations $(Y_t)_{0 \le t \le T}$ of the sample path $(X_t)_{0 \le t \le T}$ which fulfill the condition

$$(2) \quad (Y_t)_{0 \le t \le T} = F((X_t)_{0 \le t \le T}) \quad \text{with } F\colon (C, \mathscr{C}) \to (D, \mathscr{D}) \text{ measurable.}$$

We shall use Condition 1 on the incomplete observations $(Y_t)_{0 \le t \le T}$:

CONDITION 1. For all $\theta \in \Theta$, $\sup_{0 \le t \le T} \|(Y_t - X_t)/\varepsilon\| \to 0$ in $\mathbf{P}_{\theta, \varepsilon}$-probability as $\varepsilon \to 0$.

We shall denote by $x_\theta(t)$ the solution of (1) associated with $\varepsilon = 0$:

$$(3) \qquad dx_\theta(t) = b(x_\theta(t), \theta)\, dt, \qquad x_\theta(0) = x.$$

The Fisher information matrix corresponding to the observation of $(X_t)_{0 \le t \le T}$ is

$$(4) \qquad I_T(\theta) = \int_0^T \left(\frac{\partial b(x_\theta(s), \theta)}{\partial \theta} \right)' e(x_\theta(s))^{-1} \left(\frac{\partial b(x_\theta(s), \theta)}{\partial \theta} \right) ds,$$

with

$$(5) \qquad e(u) = \sigma(u)\sigma(u)',$$

where $(\partial b(u, \theta))/\partial \theta$ denotes the $m \times k$ matrix containing the partial derivatives $(\partial b_i(u, \theta))/\partial \theta^j)_{i \le m;\, j \le k}$. Let θ_0 be the true value of the parameter.

ASSUMPTION 1. The functions $b(u, \theta)$ and $\sigma(u)$ have continuous partial derivatives in $\mathbf{R}^m \times \mathrm{Int}(\Theta)$ up to order 2 [$\mathrm{Int}(A)$ is the interior of set A].

ASSUMPTION 2. For all $\theta \in \Theta$, there exists a positive constant K_θ such that for all u in $\mathbf{R}^m$,

$$\|b(u,\theta)\|^2 + \|\sigma(u)\|^2 \le K_\theta(1 + \|u\|^2).$$

($\|\cdot\|$ denotes the Euclidean norm of $\mathbf{R}^m$.)

ASSUMPTION 3. There exists V: $\mathbf{R}^m \times \Theta \to \mathbf{R}$ such that $V(x,\theta) = 0$ and $(\partial V(u,\theta))/\partial u = e(u)^{-1}b(u,\theta)$.

ASSUMPTION 4. Θ is a compact subset of $\mathbf{R}^k$, $\theta_0 \in \operatorname{Int}(\Theta)$.

ASSUMPTION 5. For all $\theta \neq \theta'$ in Θ, $b(x_\theta(\cdot),\theta) \not\equiv b(x_\theta(\cdot),\theta')$.

ASSUMPTION 6. The $k \times k$ matrix $I_T(\theta)$ defined in (4) is positive definite on $\operatorname{Int}(\Theta)$.

Assumptions 1 and 2 ensure in particular that the solution of (1) is a strong Markov process with continuous sample paths uniquely determined on $[0, +\infty)$ and with infinite explosion time [see, for instance, Ikeda and Watanabe (1981)]. When the sample path (X_t) is continuously observed on $[0, T]$, Assumptions 1, 2 and 4–6 are standard and ensure that the statistical diffusion model is regular. Assumption 3 is specific to the situation of incomplete observations and is crucial for establishing Theorem 1.

Under Assumptions 1 and 2, the probability measures $\mathbf{P}_{\theta,\varepsilon}$ and $\mathbf{P}_{\theta_0,\varepsilon}$ are equivalent on $\mathscr{C}_T$ and the logarithm of the likelihood ratio is

$$\log\left(\frac{d\mathbf{P}_{\theta,\varepsilon}}{d\mathbf{P}_{\theta_0,\varepsilon}}\bigg|\mathscr{C}_T\right) = \lambda_\varepsilon(\theta) - \lambda_\varepsilon(\theta_0) \tag{6}$$

with

$$\lambda_\varepsilon(\theta) = \varepsilon^{-2}\left(\int_0^T \left(e(X_s)^{-1}b(X_s,\theta)\right)\cdot dX_s - \tfrac{1}{2}\int_0^T v(X_s,\theta)\,ds\right), \tag{7}$$

where $\cdot$ denotes the inner product of $\mathbf{R}^m$ and $v(u,\theta)$ is the real valued function defined by

$$v(u,\theta) = b(u,\theta)'e(u)^{-1}b(u,\theta). \tag{8}$$

In order to study the properties of the net of experiments $(\mathbf{P}_{\theta,\varepsilon}, \theta \in \Theta)_{\varepsilon>0}$ at point θ_0, we need the following notations. Let A be a ball $B(\theta_0, r) = \{z \in \mathbf{R}^k, \|z - \theta_0\| \le r\}$ such that A is included in Θ. Under Assumption 6, the matrix $I(\theta_0)$ is symmetric positive definite. Denote by $I(\theta_0)^{-1/2}$ the symmetric positive square root of $I(\theta_0)^{-1}$ and define the net of experiments:

$$\mathscr{E}_\varepsilon = \{(Q_{z,\varepsilon}), z \in A\}_{\varepsilon>0}, \quad \text{with } Q_{z,\varepsilon} = \mathbf{P}_{\theta_0+\varepsilon I(\theta_0)^{-1/2}z,\varepsilon}|\mathscr{C}_T. \tag{9}$$

The logarithm of the likelihood ratio associated with $\mathscr{E}_\varepsilon$ is

$$\log\left(\frac{d\mathbf{Q}_{z,\varepsilon}}{d\mathbf{Q}_{0,\varepsilon}}\right) = \Lambda_\varepsilon(z) = \lambda_\varepsilon\left(\theta_0 + \varepsilon I(\theta_0)^{-1/2} z\right) - \lambda_\varepsilon(\theta_0). \tag{10}$$

Now, let G_0 be the standard Gaussian distribution of $\mathbf{R}^k$ [$G_0 = \mathscr{N}(0, I_k)$ if I_k is the $k \times k$ identity matrix] and let G_z be G_0 translated by z. Then $\mathscr{G} = \{G_z, z \in \mathbf{R}^k\}$ is the standard Gaussian shift experiment of $\mathbf{R}^k$.

In what follows, we use a stochastic Taylor expansion of X_t in powers of ε due to Azencott (1982) [see also Freidlin and Wentzell (1984), with stronger assumptions on the drift and diffusion coefficients].

THEOREM A [Azencott (1982)]. *Under Assumptions* 1 *and* 2, *there exists a continuous centered Gaussian process* $(g_\theta(t))_{t \geq 0}$ *and a process* $R_\theta(t)$ *such that for all* $t \geq 0$,

$$X_t = x_\theta(t) + \varepsilon g_\theta(t) + \varepsilon^2 R_\theta(t) \quad \textit{with} \quad \lim_{\varepsilon \to 0, r \to +\infty} P_{\theta,\varepsilon}\left(\sup_{s \leq t} \| R_\theta(s) \| \geq r\right) = 0.$$

If $(\partial b(u, \theta))/\partial u$ denotes the $m \times m$ matrix composed of the partial derivatives $(\partial b_i(u, \theta))/\partial u_j$, the m-dimensional Gaussian process $g_\theta(t)$ is defined by

$$dg_\theta(t) = \frac{\partial b(x_\theta(t), \theta)}{\partial u} g_\theta(t)\, dt + \sigma(x_\theta(t))\, dW_t, \qquad g_\theta(0) = 0. \tag{11}$$

2.2. *Asymptotic sufficiency of the observations.* We can now study the statistical properties of the incomplete observations (Y_t). Under Assumption 3, we can define using (4) and (8) the net of centering variables with values in $\mathbf{R}^k$, $(\tilde{Z}_\varepsilon)_{\varepsilon > 0}$, which depend only on $(Y_t)_{0 \leq t \leq T}$:

$$\tilde{Z}_\varepsilon = \varepsilon^{-1} I(\theta_0)^{-1/2}\left(\frac{\partial V(Y_T, \theta_0)}{\partial \theta} - \frac{1}{2}\int_0^T \frac{\partial v(Y_s, \theta_0)}{\partial \theta}\, ds\right). \tag{12}$$

We are now in a position to state the main result of this section which is as follows:

THEOREM 1. *Consider Assumptions* 1–6. *Then,*

(i) *The net of experiments* $\mathscr{E}_\varepsilon = \{\mathbf{Q}_{z,\varepsilon}, z \in A\}_{\varepsilon > 0}$ *converges uniformly on the precompact subsets of A to the restriction to A of the standard Gaussian shift experiment* $\mathscr{G}$ *as* $\varepsilon \to 0$.

(ii) *Moreover, if* $(Y_t)_{0 \leq t \leq T}$ *meets Condition* 1, *one has, for all precompact subsets S of A,*

$$\forall z \in S, \qquad \Lambda_\varepsilon(z) = z'\tilde{Z}_\varepsilon - \|z\|^2/2 + \tilde{R}(\varepsilon, z, \theta_0), \tag{13}$$

with

$$\mathscr{L}\left(\tilde{Z}_\varepsilon \middle| \mathbf{P}_{\theta_0,\varepsilon}\right) \to \mathscr{N}(0, I_k) \quad \textit{as } \varepsilon \to 0, \tag{14}$$

$$\sup_{z \in S} \left| \tilde{R}(\varepsilon, z, \theta_0) \right| \to 0 \quad \textit{in } \mathbf{P}_{\theta_0,\varepsilon}\textit{-probability as } \varepsilon \to 0. \tag{15}$$

COROLLARY 1. *Consider Assumptions* 1–6 *and Condition* 1. *The observations* $(Y_t, 0 \le t \le T)$ *are then asymptotically sufficient in the sense of Le Cam* (1986): *If* $\mathscr{B}_\varepsilon$ *denotes the* σ *field generated by* $\tilde{Z}_\varepsilon$, *there exists a family of probability measures on* $\mathscr{C}_T$, $\{\mathbf{Q}'_{z,\varepsilon}, z \in A\}_{\varepsilon > 0}$ *such that*:

(a) *For the family* $(\mathbf{Q}'_{z,\varepsilon})$, *the* σ *field* $\mathscr{B}_\varepsilon$ *is sufficient.*

(b) *For all precompact sets* $S \subset A$, *one has* $\lim_{\varepsilon \to 0} \sup_{z \in S} \|\mathbf{Q}'_{z,\varepsilon} - \mathbf{Q}_{z,\varepsilon}\| = 0$, *where* $\|\cdot\|$ *is the* L^1 *distance between two probability measures.*

(c) *On the* σ *field* $\mathscr{B}_\varepsilon$, *measures* $\mathbf{Q}'_{z,\varepsilon}$ *and* $\mathbf{Q}_{z,\varepsilon}$ *coincide.*

(d) *The vectors* $(\tilde{Z}_\varepsilon)_{\varepsilon > 0}$ *behave as a distinguished sequence of statistics.*

PROOF OF THEOREM 1. (i) Clearly, property (i) implies that the family $(\mathbf{P}_{\theta,\varepsilon})$ is locally asymptotically normal in the usual sense. This is a known result [see, for instance, Kutoyants (1984), with the stronger assumption that the drift and diffusion coefficients have bounded continuous partial derivatives up to order 2]. However, in what follows, we need all the properties contained in (i), which are obtainable only under Assumption 1.

To avoid trivial complications, all the results proved in this section will be obtained in the case $m = 1$. The case $m > 1$ is just a repetition.

Let us define the random vectors:

$$(16)\qquad Z_\varepsilon = I(\theta_0)^{-1/2}\left(\int_0^T \frac{\partial b(X_s,\theta_0)}{\partial\theta^j}\left(\frac{dX_s - b(X_s,\theta_0)\,ds}{\varepsilon\sigma(X_s)^2}\right)\right)_{1 \le j \le k}$$

According to Le Cam [(1986), chapters 10 and 11], (i) will be proved if the following properties hold for the logarithm of the likelihood ratio: For all precompact subsets S of A, one has

$$(17)\qquad \forall z \in S, \qquad \Lambda_\varepsilon(z) = z'Z_\varepsilon - \|z\|^2/2 + R(\varepsilon, z, \theta_0),$$

with

$$(18)\qquad \mathscr{L}(Z_\varepsilon | \mathbf{P}_{\theta_0,\varepsilon}) \to \mathscr{N}(0, I_k)$$

and

$$(19)\qquad \sup_{z \in S} |R(\varepsilon, z, \theta_0)| \to 0 \quad \text{in } \mathbf{P}_{\theta_0,\varepsilon}\text{-probability as } \varepsilon \to 0.$$

Let us first prove (18). Since diffusion (X_t) satisfies (1), Z_ε verifies

$$(20)\quad Z_\varepsilon = I(\theta_0)^{-1/2}\int_0^T f(X_s,\theta_0)\,dW_s \quad \text{with } f(u,\theta) = \sigma(u)^{-1}\frac{\partial b(u,\theta)}{\partial\theta}.$$

By Theorem A, the quantity $(\int_0^T \|f(X_s,\theta_0) - f(x_{\theta_0}(s),\theta_0)\|^2\,ds)$ converges to 0 in $\mathbf{P}_{\theta_0,\varepsilon}$-probability and therefore Z_ε converges under $\mathbf{P}_{\theta_0,\varepsilon}$ to the variable $I(\theta_0)^{-1/2}\int_0^T f(x_{\theta_0}(s),\theta_0)\,dW_s$, which is Gaussian with mean 0 and covariance matrix I_k. Developing now $b(X_s,\theta)$ in Taylor series, the remainder term appearing in (17) may be expressed as

$$R(\varepsilon, z, \theta_0) = \tfrac{1}{2}z'(A_1 + A_2 + A_3)z,$$

where the matrices A_i are defined by

$$A_1 = \varepsilon \int_0^T \sigma(X_s)^{-1} \left(\int_0^1 \frac{\partial^2 b(X_s, \theta_0 + \varepsilon t z)}{\partial \theta^2} \, dt \right) dW_s,$$

$$A_2 = \int_0^T \frac{ds}{\sigma(X_s)^2} \int_0^1 (b(X_s, \theta_0) - b(X_5, \theta_0 + \varepsilon t z)) \frac{\partial^2 b(X_s, \theta_0 + \varepsilon t z)}{\partial \theta^2} \, dt,$$

$$A_3 = \int_0^T \left(H(x_{\theta_0}(s), \theta_0) - H(X_s, \theta_0 + \varepsilon t z) \right) ds$$

with

$$H(u, \theta) = \frac{1}{\sigma(u)^2} \frac{\partial b(u, \theta)}{\partial \theta} \left(\frac{\partial b(u, \theta)}{\partial \theta} \right)'.$$

The first term $A_1 = A_1(t)$ is a martingale with respect to $\mathscr{C}_t$. Therefore, by the Lenglart domination property [see, for instance, Jacod and Shiryaev (1988)], one has

$$\forall\, h, \eta > 0, \mathbf{P}_{\theta, \varepsilon}(\| A_1 \| \geq h) \leq \mathbf{P}_{\theta, \varepsilon}(\| B_1 \| \geq \eta) + \frac{\eta^2}{h} \text{ with}$$

$$B_1 = \varepsilon^2 \int_0^T \frac{ds}{\sigma(X_s)^2} \left(\int_0^1 \frac{\partial^2 b(X_s, \theta_0 + \varepsilon t z)}{\partial \theta^2} \, dt \right)^2.$$

Define the set, for $\delta > 0$,

$$C_\delta = \left| X_s - x_{\theta_0}(s) \right| \leq \delta. \tag{21}$$

By Theorem A, $\mathbf{P}_{\theta_0, \varepsilon}(C_\delta)$ converges to 1 as ε goes to 0. On C_δ, the norm of B_1 is bounded by $\varepsilon^2 T K_1$, where K_1 is a constant depending on T, θ_0, r and δ. Similarly, $\|A_2\|$ is bounded on C_δ by $\varepsilon T K_2(T, \theta_0, r, \delta)$ and $\|A_3\|$ by

$$\left| X_s - x_{\theta_0}(s) \right| \sup_{(u, \theta) \in B} \left| \frac{\partial H(u, \theta)}{\partial u} \right| + \varepsilon r \sup_{(u, \theta) \in B} \left\| \frac{\partial H(u, \theta)}{\partial \theta} \right\|,$$

where B is the compact set $\bigcup_{s \leq T} [x_{\theta_0}(s) - \delta, x_{\theta_0}(s) + \delta] \times (\|\theta - \theta_0\| \leq r)$. Combining these inequalities, one obtains that condition (19) is fulfilled by the remainder term. This completes the proof of (i).

(ii) Clearly, to obtain (ii), it suffices to show that the difference $\|\tilde{Z}_\varepsilon - Z_\varepsilon\| \to 0$ in $\mathbf{P}_{\theta_0, \varepsilon}$-probability as $\varepsilon \to 0$ uniformly for $z \in S$. By the Itô formula, one has

$$Z_\varepsilon - \tilde{Z}_\varepsilon = \frac{1}{\varepsilon} \left[\frac{\partial V(X_T, \theta_0)}{\partial \theta} - \frac{\partial V(Y_T, \theta_0)}{\partial \theta} \right]$$

$$- \frac{1}{\varepsilon} \int_0^T \left(\frac{\partial b(X_s, \theta_0)}{\partial \theta} \times \frac{b(X_s, \theta_0)}{\sigma(X_s)^2} - \frac{\partial b(Y_s, \theta_0)}{\partial \theta} \times \frac{b(Y_s, \theta_0)}{\sigma(Y_s)^2} \right) ds \tag{22}$$

$$- \frac{\varepsilon}{2} \int_0^T \frac{\partial}{\partial u} \left(\frac{1}{\sigma(X_s)^2} \times \frac{\partial b(X_s, \theta)}{\partial \theta} \right) \sigma(X_s)^2 \, ds.$$

Therefore, $\|Z_\varepsilon - \tilde{Z}_\varepsilon\|$ is bounded from above by

$$\left(\sup_{0 \le t \le T}\left|\frac{Y_t - X_t}{\varepsilon}\right|\right)(W_1 + W_2) + \varepsilon T W_3 \tag{23}$$

with

$$W_1 = \sup_{t \le T, \theta \in \Theta}\left\|\frac{\partial^2 V(X_s, \theta_0)}{\partial u\, \partial \theta}\right\|,$$

$$W_2 = \sup_{t \le T, \theta \in \Theta}\left\|\frac{\partial}{\partial u}\left(\frac{1}{\sigma(X_s)^2}\frac{\partial b(X_s, \theta_0)}{\partial \theta}\right)\right\|,$$

$$W_3 = \sup_{s \le T, \theta \in \Theta}\left\|\frac{\partial}{\partial u}\left(\frac{1}{\sigma(X_s)^2}\frac{\partial b(X_s, \theta_0)}{\partial \theta}\right)\sigma(X_s)^2\right\|.$$

It follows from Theorem A that the three random variables appearing above are bounded on the set C_δ defined in (21) by constants which are independent of z. Therefore, joining this with Condition 1 yields the uniform convergence to 0 of $\|Z_\varepsilon - \tilde{Z}_\varepsilon\|$. This completes the proof of Theorem 1. □

Proof of Corollary 1. Since the net of experiments $\mathscr{E}_\varepsilon$ verifies Theorem 1(i) and (ii), Theorem 10.1 of Le Cam (1986) applies. This leads to properties (a), (b) and (c): Here, the incomplete observations $(Y_t, 0 \le t \le T)$ are asymptotically sufficient in a very strong way. Let us now study (d). At first glance, the experiments obtained by observing only (Y_t) are weaker than the experiments $\mathscr{E}_\varepsilon$. However, we obtain here that the net of statistics $\tilde{Z}_\varepsilon$ form as good an experiment as $\mathscr{E}_\varepsilon$. According to Le Cam, this characterizes "distinguished statistics." For a precise definition, we refer to Le Cam [(1986), Chapter 7, Section 3]. Now, property (d) holds for the $(\tilde{Z}_\varepsilon)_{\varepsilon > 0}$, if the probability distributions $Q_{z,\varepsilon}$ and $Q'_{z,\varepsilon}$ asymptotically satisfy: For all precompact subset S of A, one has

$$\forall z \in S, \qquad \lim_{\varepsilon \to 0} \sup_{z \in S}\left\{k^2(Q_{z,\varepsilon}, Q_{0,\varepsilon}) - k^2\left(Q'_{z,\varepsilon}, Q'_{0,\varepsilon}\right)\right\} \le 0, \tag{24}$$

where $k^2(\cdot, \cdot)$ is the following χ^2-type distance between two probability measures P and Q:

$$k^2(P, Q) = \frac{1}{2}\int \frac{(dP - dQ)^2}{d(P + Q)}.$$

Now, one has

$$k^2(Q_{z,\varepsilon}, Q_{0,\varepsilon}) - k^2\left(Q'_{z,\varepsilon}, Q'_{0,\varepsilon}\right) \le k^2\left(Q_{z,\varepsilon}, Q'_{z,\varepsilon}\right) + k^2\left(Q_{0,\varepsilon}, Q'_{0,\varepsilon}\right),$$

$$k^2(P, Q) \le \|P - Q\|_{L^1}.$$

Therefore, (d) is merely here a consequence of (b).

An expected consequence of Corollary 1 is that all tests or estimates based on $(Y_t)_{0\le t\le T}$ will perform as well as those which could be obtained with $(X_t)_{0\le t\le T}$. Here, we shall only study estimators based on $(Y_t)_{0\le t\le T}$. For this, let us define the approximate likelihood:

$$L_\varepsilon(\theta) = l_\varepsilon(\theta) - l_\varepsilon(\theta_0),$$

$$\text{with}\quad l_\varepsilon(\theta) = \frac{1}{\varepsilon^2}\left[V(Y_T,\theta) - \frac{1}{2}\int_0^T \frac{b(Y_s,\theta)^2}{\sigma(Y_s)^2}\,ds\right]. \tag{25}$$

Using now (6), (7) and (25), we can successively define the maximum likelihood estimator $\hat{\theta}_\varepsilon$ of θ and a pseudo-maximum likelihood estimator $\tilde{\theta}_\varepsilon$ by the equations

$$\sup_{\theta\in\Theta}\Lambda_\varepsilon(\theta) = \Lambda_\varepsilon(\hat{\theta}_\varepsilon), \qquad \sup_{\theta\in\Theta} L_\varepsilon(\theta) = L_\varepsilon(\tilde{\theta}_\varepsilon). \tag{26}$$

Note that they are well defined under Assumption 4. □

The properties of $\tilde{\theta}_\varepsilon$ may be summarized as follows.

Proposition 1. *Under Assumptions* 1–6 *and Condition* 1, *the following properties hold*:

(a) *For all* $h > 0$, $\mathbf{P}_{\theta_0,\varepsilon}(\|\tilde{\theta}_\varepsilon - \theta_0\| > h) \to 0$ *as* $\varepsilon \to 0$.
(b) $\mathscr{L}(\varepsilon^{-1}(\tilde{\theta}_\varepsilon - \theta_0)|\mathbf{P}_{\theta_0,\varepsilon}) \to \mathscr{N}(0, I(\theta_0)^{-1})$ *as* $\varepsilon \to 0$.
(c) $\varepsilon^{-1}(\tilde{\theta}_\varepsilon - \hat{\theta}_\varepsilon) \to 0$ *in* $\mathbf{P}_{\theta_0,\varepsilon}$*-probability as* $\varepsilon \to 0$.

Proof of Proposition 1. (a) Let us first define

$$D_\delta = \left\{\left|Y_s - x_{\theta_0}(s)\right| \le \delta, 0 \le s \le T\right\} \quad\text{and}\quad U_\varepsilon(\theta) = -\varepsilon^2 L_\varepsilon(\theta). \tag{27}$$

By Theorem A and Condition 1, $\mathbf{P}_{\theta_0,\varepsilon}(D_\delta) \to 1$ as ε converges to 0. Hence, $U_\varepsilon(\theta)$ converges in $\mathbf{P}_{\theta_0,\varepsilon}$-probability to the function $K(\theta_0,\theta)$ defined by

$$\begin{aligned} K(\theta_0,\theta) &= V(x_{\theta_0}(T),\theta_0) - V(x_{\theta_0}(T),\theta) \\ &\quad - \frac{1}{2}\int_0^T \frac{b(x_{\theta_0}(s),\theta_0)^2 - b(x_{\theta_0}(s),\theta)^2}{\sigma(x_{\theta_0}(s))^2}\,ds, \\ &= \frac{1}{2}\int_0^T \frac{\left(b(x_{\theta_0}(s),\theta_0) - b(x_{\theta_0}(s),\theta)\right)^2}{\sigma(x_{\theta_0}(s))^2}\,ds. \end{aligned} \tag{28}$$

The identifiability assumption (Assumption 5) ensures that the function $K(\theta_0,\cdot)$ is positive for all $\theta \ne \theta_0$ and null at θ_0. Hence, the random function $U_\varepsilon(\theta)$ is a contrast function in the sense of Dacunha-Castelle and Duflo [(1983), Chapter 3]. The function $\theta \to K(\theta_0,\theta)$ is continuous. Therefore, according to Dacunha-Castelle and Duflo (1983), the consistency of $\tilde{\theta}_\varepsilon$ holds if the continuity modulus of $U_\varepsilon(\theta)$, $w(U_\varepsilon,\eta)=\sup_{\|\theta-\theta'\|\le\eta}|U_\varepsilon(\theta)-U_\varepsilon(\theta')|$, is such that

$$\forall\,\alpha>0,\ \exists\eta>0,\qquad \lim_{\varepsilon\to 0}\mathbf{P}_{\theta_0,\varepsilon}(w(U_\varepsilon,\eta)>\alpha)=0. \tag{29}$$

Clearly, U_ε verifies

$$|U_\varepsilon(\theta)-U_\varepsilon(\theta')|\le|V(Y_T,\theta)-V(Y_T,\theta')|+\int_0^T\frac{|b(Y_s,\theta)^2-b(Y_s,\theta')^2|}{2\sigma(Y_s)^2}\,ds.$$

Let $K=\{x_\theta(s),\ 0\le s\le T,\ \theta\in\Theta\}$ and $K_\delta=\{x\in\mathbf{R}^m,\ \exists\ y\in K\ \|y-x\|\le\delta\}$. On the compact set $K_\delta\times\Theta$, $b(u,\theta)$ is uniformly continuous, hence the two quantities

$$\beta(\eta)=\sup_{K_\delta}\ \sup_{\|\theta-\theta'\|\le\eta}|V(u,\theta)-V(u,\theta')|,$$

$$\gamma(\eta)=\sup_{K_\delta}\ \sup_{\|\theta-\theta'\|\le\eta}\left|\frac{b(u,\theta)^2-b(u,\theta')^2}{\sigma(u)^2}\right|$$

satisfy

$$\lim_{\eta\to 0}\beta(\eta)=\lim_{\eta\to 0}\gamma(\eta)=0.$$

Therefore, on the set D_δ defined in (27), one has $w(U_\varepsilon,\eta)\le\beta(\eta)+\frac{1}{2}T\gamma(\eta)$, which implies (29).

(b) To study $\tilde\theta_\varepsilon$, let us use the classical decomposition of U_ε for $j=1,\dots,k$:

$$0=\frac{1}{\varepsilon}\frac{\partial U_\varepsilon}{\partial\theta}(\theta_0)+\frac{1}{\varepsilon}\sum_{l=1}^{k}\left(\left(\tilde\theta_\varepsilon^l-\tilde\theta_0^l\right)\frac{\partial^2U_\varepsilon}{\partial\theta^l\,\partial\theta^j}(\theta_0)+R_{l,j}\left(\theta_0,\tilde\theta_\varepsilon-\theta_0\right)\right), \tag{30}$$

where the remainder term is

$$R_{l,j}(\theta,h)=\int_0^1\frac{\partial^2U_\varepsilon(\theta+sh)}{\partial\theta^l\,\partial\theta^j}\,ds.$$

Using this decomposition, the proof is straightforward since $\varepsilon^{-1}(\partial U_\varepsilon/\partial\theta)(\theta_0)=-I(\theta_0)^{1/2}\tilde Z_\varepsilon$, which converges to $\mathscr{N}(0,I(\theta_0))$ as $\varepsilon\to 0$, by Theorem 1(ii). Applying now Theorem A and using Condition 1, one easily obtains that $((\partial^2U_\varepsilon/\partial\theta^l\,\partial\theta^j)(\theta_0))_{1\le l,j\le k}$ converges to $I(\theta_0)$. Therefore, (30) yields

$$\varepsilon^{-1}\left(\tilde\theta_\varepsilon-\theta_0\right)=I(\theta_0)^{1/2}\tilde Z_\varepsilon+o_{P_{\theta_0,\varepsilon}}(1), \tag{31}$$

where $o_P(1)$ denotes a remainder converging to 0 in $\mathbf{P}$-probability.

(c) It follows from Theorem 1(i) that $\varepsilon^{-1}(\hat\theta_\varepsilon-\theta_0)=I(\theta_0)^{1/2}Z_\varepsilon+o_{P_{\theta_0,\varepsilon}}(1)$. Hence, one has $\varepsilon^{-1}(\tilde\theta_\varepsilon-\hat\theta_\varepsilon)=I(\theta_0)^{1/2}(\tilde Z_\varepsilon-\hat Z_\varepsilon)+o_{P_{\theta_0,\varepsilon}}(1)$. The proof of Proposition 1 is achieved using Theorem 1(ii). □

3. Incomplete observations of $(X_t)_{0\le t\le T}$ satisfying Condition 1. Let us now present various examples of incomplete observations satisfying Condition 1. For simplicity's sake, parameter θ will be omitted when possible.

3.1. *Discrete observations of the sample path.* The sampling interval is Δ; the observations therefore consist of $(X_{k\Delta})_{0\le k\le N}$, where $N = T/\Delta$. Let f be an arbitrary function defined at the points $k\Delta$, $k = 0,\dots, N$, with values in $\mathbf{R}^m$, and associate with f the interpolated function f^Δ: $[0,T]\to\mathbf{R}^m$ defined as

$$f^\Delta(t) = f(k\Delta) + \frac{(t-k\Delta)}{\Delta}\big(f((k+1)\Delta) - f(k\Delta)\big) \tag{32}$$

$$\text{for } k\Delta \le t < (k+1)\Delta.$$

From the discrete observations $X_{k\Delta}$, we can define the process

$$(Y_t)_{0\le t\le T} = \left(X_t^\Delta\right)_{0\le t\le T}. \tag{33}$$

Proposition 2. *Consider Assumptions* 1–3. *Then, if* $\Delta = \Delta(\varepsilon)$ *satisfies* $\varepsilon^{-1/2}\Delta(\varepsilon)\to 0$ *as* $\varepsilon\to 0$, *the interpolated process* $(Y_t)_{0\le t\le T}$ *defined in* (33) *meets Condition* 1.

Proof. The stochastic Taylor expansion of (X_t) given in Theorem A yields

$$Y_t - X_t = x^\Delta(t) - x(t) + \varepsilon\big(g^\Delta(t) - g(t)\big) + \varepsilon^2\big(R_2^\Delta(t) - R_2(t)\big),$$

where $x^\Delta(\cdot)$, $g^\Delta(\cdot)$ and $R_2^\Delta(\cdot)$ are, respectively, the interpolated functions on $[0,T]$ of $x(\cdot)$, $g(\cdot)$ and $R_2(\cdot)$. Set $w(g,\Delta) = \sup_{|t-s|\le\Delta}\|g(t) - g(s)\|$. One easily checks that

$$\frac{\|Y_t - X_t\|}{\varepsilon} \le \frac{\Delta^2}{\varepsilon}\sup_{[0,T]}\left\|\frac{d^2x}{d^2t}(s)\right\| + 2w(g,\Delta) + 4\sup_{[0,T]}\|\varepsilon R_2(t)\|. \tag{34}$$

By Assumption 1, the norms of the $m\times m$ matrices $d^2x/dt^2(s)$ are uniformly bounded on $[0,T]$. The process $g(t)$ is continuous and $\sup(\|\varepsilon R_2(t)\|, 0\le t\le T)$ converges to 0 by Theorem A. Therefore, if the sampling interval Δ satisfies $\varepsilon^{-1/2}\Delta\to 0$, inequality (34) leads to Proposition 2. □

Remark 1. From (34), one may observe that the order of magnitude of Δ with respect to ε is given by the goodness of the approximation of $x(\cdot)$ by $x^\Delta(\cdot)$, which clearly can be improved if the drift $b(u)$ is in $C^k(\mathbf{R}^m)$. Here, we only determine a sufficient condition for the asymptotic sufficiency of discrete observations. Using another method which takes into account the stochastic structure of these observations, Genon-Catalot (1990) has obtained the optimality of estimators based on discrete observations of $(X_t)_{0\le t\le T}$ for one-dimensional diffusions having ε for diffusion coefficient when the sampling interval satisfies $\Delta = \varepsilon^\alpha$, $0<\alpha\le 2$.

3.2. *Smoothed or filtered diffusion.* In practice, it often occurs that, because of some recording device, a smoothed path $(Y_t)_{0\le t\le T}$ is observed instead of $(X_t)_{0\le t\le T}$. This is not, strictly speaking, an incomplete observation of (X_t). Usually, the smoothing of the sample path occurs in the following way. Let φ

be a nonnegative function belonging to $C^l(\mathbf{R}, \mathbf{R})$, $l \geq 2$, with compact support included in $[-1, +1]$ and such that $\int_{-1}^{+1} \varphi(t)\, dt = 1$. Set $\varphi_\eta(t) = (1/\eta)\varphi(t/\eta)$. We assume that one observes on $[0, T]$ the process (Y_t) defined by

$$Y_t = X_t^\eta = (\varphi_\eta * X)_t = \int_0^T \varphi_\eta(t-s) X_s\, ds. \tag{35}$$

The process (Y_t) is also called a filtered diffusion.

PROPOSITION 3. *If φ is symmetric, and $\varepsilon^{-1/2}\eta \to 0$ as $\varepsilon \to 0$ [resp., $\int_{-1}^{+1} v\varphi(v)\, dv \neq 0$ and $\varepsilon^{-1}\eta \to 0$], then the process $(Y_t)_{0 \leq t \leq T}$, defined by* (35), *satisfies Condition* 1.

PROOF. It is similar to the proof of Proposition 2. □

3.3. *First hitting times and positions of concentric spheres.* Assume that $m \geq 2$ and recall that $X_0 = x$. Let $r > 0$. The first hitting time of the sphere $S(x, r)$ by the m-dimensional diffusion (X_t) is

$$T_r = \inf\{t \geq 0, \|X_t - x\| = r\}. \tag{36}$$

Under Assumptions 1 and 2, T_r is almost surely finite.

Let $R > 0$ be a prescribed positive number and consider now the incomplete observations of (X_t) consisting of (T_r, X_{T_r}) for $0 \leq r \leq R$. We shall compare these observations to the complete observation of (X_t) between 0 and T_R. For this, let us define

$$r_t = \sup_{0 \leq s \leq t} \|X_t - x\| \quad \text{and set} \quad Y_t = X_{T_{r_t}} \text{ for } 0 \leq t \leq T_R. \tag{37}$$

ASSUMPTION 7.

$$\forall\, \theta \in \Theta, \forall\, t > 0, \qquad (x_\theta(t) - x) \cdot b(x_\theta(t), \theta) > 0.$$

This ensures that $x_\theta(t)$ leaves any sphere centered at x within a finite time.

PROPOSITION 4. *Under Assumptions 1–3 and 7, the process $(Y_t)_{0 \leq t \leq T_R}$ defined in* (37) *meets Condition* 1 *on the time interval* $[0, T_R]$.

PROOF. The process $(T_r)_{r > 0}$ is left-continuous with right-hand limits and so belongs to D. The mapping: $(X_t)_{t>0} \to (T_r)_{r>0}$ is measurable from $(\mathbf{C}, \mathscr{C})$ into $(\mathbf{D}, \mathscr{D})$ since the mapping $(X_t) \to T_r$ is measurable and $\mathscr{D}$ is generated by its natural projections. Condition 1 is contained in Genon-Catalot (1989) in a different formulation and framework. Indeed, if φ is a smooth function,

$$D_T(\varphi) = \varepsilon^{-1}\left(\int_0^{T_R} \varphi(X_s)\, ds - \int_{[0,R)} \varphi(X_{T_r})\, dT_r\right) \quad \text{is an } o_P(1) \text{ as } \varepsilon \to 0.$$

Condition 1 is obtained using that $[0, T_R] = \bigcup_{0 \leq r < R} [T_r, T_{r^+}]$ and that $Y_t = X_{T_r}$ on $T_r \leq t \leq T_{r^+}$. □

REMARK 2. The statistical study is not modified by the fact that the observations are taken on a random time interval. Indeed, (6) and (7) are still valid if one substitutes T by T_R, since for all $\theta \in \Theta$, $\mathbf{P}_{\theta,\varepsilon}(T_R < \infty) = 1$. The Fisher information matrix is taken in this case on time interval $[0, t_{\theta_0}(R)]$, where $t_{\theta_0}(R) = \lim_{\varepsilon \to 0} T_R$ in $\mathbf{P}_{\theta_0,\varepsilon}$-probability.

3.4. *Record process for a one-dimensional diffusion.* In the case $m = 1$, an incomplete observation of (X_t) which occurs in practice is the record process of diffusion:

$$M_t = \sup_{0 \le s \le t} X_s. \tag{38}$$

Let $A > x = X_0$ and denote by T_a the first hitting time of a level $a > x$:

$$T_a = \inf\{t > 0, X_t = a\}. \tag{39}$$

We shall compare the observation of (X_t) on $[0, T_A]$ with the observation of (M_t) on the same time interval.

ASSUMPTION 8.

$$\forall\, u \in \mathbf{R}, \forall\, \theta \in \Theta,\ b(u, \theta) > 0.$$

PROPOSITION 5. *Consider Assumptions* 1, 2 *and* 8. *Then the process* (M_t) *defined in* (38) *fulfills Condition* 1 *on time interval* $[0, T_A]$.

PROOF. See Genon-Catalot and Laredo (1987). □

REMARK 3. Under Assumption 8, one has that for all $\theta \in \Theta$, T_A is $\mathbf{P}_{\theta,\varepsilon}$ almost surely finite. Therefore, the statistical analysis on the random time interval $[0, T_A]$ may be done as in Remark 2, the information matrix being taken between 0 and $t_{\theta_0}(A) = \lim_{\varepsilon \to 0} T_A$ in $\mathbf{P}_{\theta_0,\varepsilon}$-probability.

REMARK 4. Truncated observation of the record process. Like the original diffusion, the record process might not in practice be observed in every detail. The sample path M_t is nondecreasing and almost everywhere constant. Let η be a given positive constant. A natural incomplete observation of the record process is therefore composed of the data consisting of the time intervals greater than η, where M_t is constant, coupled with the value of M_t on each of these intervals. The following question then arises: How much is it possible to impoverish these observations (i.e, which are the greatest values of η) without losing the asymptotic sufficiency property on $[0, T_A]$? One can prove [see Laredo, 1989]:

(i) If $\eta = \eta(\varepsilon)$ satisfies $\varepsilon^{-2}\eta(\varepsilon) \to C \ge 0$ as $\varepsilon \to 0$, there exists a process (Y_t) built on these observations which fulfills Condition 1 on the time interval $[0, T_A]$.

(ii) If $\eta = \varepsilon^{\alpha}$, $\alpha < 2$, there exist examples where all information about θ is lost.

Here, using the method presented in the above section, one improves a previous result obtained in Genon-Catalot and Laredo (1990), where asymptotic sufficiency of these observations had been obtained for $\eta = \varepsilon^{\alpha}$, with $\alpha > 4$.

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