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A NOTE ON THE EXACT DISTRIBUTION OF A NONPARAMETRIC TEST STATISTIC FOR ORDERED ALTERNATIVES

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The exact distribution of a nonparametric test statistic for ordered alternatives, the so-called $\bar{\chi}^2_{\text{rank}}$ statistic, is discussed. Two tables are given.

1. Introduction and main result. Suppose that we have n populations and from population i a random sample $X_{i,1}, X_{i,2}, \cdots, X_{i,N_i}, i = 1, 2, \cdots, n$, is taken. Let $X_{i,j}$ have distribution function F_i for $j = 1, 2, \cdots, N_i, i = 1, 2, \cdots, n$, and let the samples from different populations be independent. We wish to test the hypothesis

$$H_0 : F_i = F \qquad i = 1, 2, \cdots, n$$

against the ordered alternative

$$H_1 : F_1 \geqslant F_2 \geqslant \cdots \geqslant F_n$$

where at least one inequality is strict.

If F_i is a normal distribution with mean μ_i and variance σ^2, the hypotheses to be tested become $H_0 : \mu_i = \mu$ and $H_1 : \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$. In this case the likelihood ratio test yields the so-called $\bar{\chi}^2$ test or $\bar{E}^2$ test, depending upon whether the common variance is known or not. The $\bar{\chi}^2$ test statistic is defined as

$$\bar{\chi}^2 = \Sigma_{i=1}^n N_i\left(\mu_i^* - \bar{X}\right)^2/\sigma^2,$$

where $\bar{X}$ is the grand mean of all the samples and μ_i^* is the maximum likelihood estimator of μ_i under the restriction $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$. It has been shown that the distribution function of the $\bar{\chi}^2$ statistic under the null hypothesis is a weighted sum of χ^2 distribution functions. The $\bar{\chi}^2$ and $\bar{E}^2$ tests are discussed in great detail in Chapter 3 of Barlow et al. (1972).

If F_i is known only to be continuous then the distribution-free version of the $\bar{\chi}^2$ statistic, the $\bar{\chi}^2_{\text{rank}}$ statistic, may be used; this statistic was introduced by Chacko (1963), extended by Shorack (1967), and is well summarized in Chapter 4, Section 4 of Barlow et al. (1972).

In order to define the $\bar{\chi}^2_{\text{rank}}$ statistic we need the following definitions. Let $R_{i,j}$ be the rank of $X_{i,j}$ among *all* ΣN_i observations. $\bar{R}_i = \Sigma_{j=1}^{N_i} R_{i,j}/N_i$, $N_. = \Sigma_{i=1}^n N_i$ and define $\mu_1^*, \mu_2^*, \cdots, \mu_n^*$ to be the solution to the problem of minimizing $\Sigma_{i=1}^n N_i(\bar{R}_i - \mu_i)^2$ subject to $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$. The μ_i^*'s are usually called the isotonic

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regression of the $\bar{R}_i$'s. A closed form expression for μ_i^* is

$$\mu_i^* \equiv \mu_i^*(\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_n) = \max_{r=1}^{i} \min_{t=i}^{n} \Sigma_{j=r}^{t} N_j \bar{R}_j / \Sigma_{j=r}^{t} N_j.$$

The definition of μ_i^* may also be expressed in terms of the greatest convex minorant of the set of points $(0, 0)$, $(\Sigma_{j=1}^{k} N_j, \Sigma_{j=1}^{k} N_j \bar{R}_j)$, $k = 1, 2, \cdots, n$. See Barlow et al. (1972), Chapter 1, Section 2 for a complete discussion on the computation of the isotonic regression.

The test statistic is then given by

$$(1.1) \qquad \bar{\chi}^2_{\text{rank}} = \frac{12}{N_.(N_. + 1)} \Sigma_{i=1}^{n} N_i \left(\mu_i^* - \frac{N_. + 1}{2} \right)^2.$$

The exact distribution of the $\bar{\chi}^2_{\text{rank}}$ statistic under H_0 is unknown; however, the asymptotic distribution of $\bar{\chi}^2_{\text{rank}}$ is that of $\bar{\chi}^2$ (see Theorem 4.5 of Barlow et al. (1972)).

In this note we wish to point out a special, but important, case when the exact null distribution of the $\bar{\chi}^2_{\text{rank}}$ statistic can be computed. If all the sample sizes are the same, say $N_i = N$ for all i, then we shall show that the distribution function of the $\bar{\chi}^2_{\text{rank}}$ statistic can be expressed as a weighted sum of standard Kruskal-Wallis statistic distributions.

We now state our result. The proof is essentially an application of Theorem 1.2 of Boswell and Brunk (1969); we give the proof in Section 2 of this note. We need the following definitions in order to state our result. For each $m = 1, 2, \cdots, n$ define

$$\mathcal{K}_m = \{k = (k_1, k_2, \cdots, k_n) : k_1, k_2, \cdots, k_n \text{ are nonnegative integers}, \Sigma_{i=1}^{n} i k_i = n, \Sigma_{i=1}^{n} k_i = m\}.$$

For k in $\mathcal{K}_m$ define

$$\mathcal{Q}^k = \{(\alpha_1, \alpha_2, \cdots, \alpha_m) : \alpha_1, \alpha_2, \cdots, \alpha_m \text{ are positive integers and exactly } k_i \text{ of the components are equal to } i, i = 1, 2, \cdots, n\}.$$

For $\alpha \in \mathcal{Q}^k$ define

$T_N(\alpha_1, \alpha_2, \cdots, \alpha_m)$ as the Kruskal-Wallis statistic based upon m populations with sample sizes $N\alpha_1, N\alpha_2, \cdots, N\alpha_m$ (if $m = 1$, $T_N(\alpha_1) \equiv 0$), that is,

$$T_N(\alpha_1, \cdots, \alpha_m) = \frac{12}{N_.(N_. + 1)} \Sigma_{i=1}^{m} N_i (\bar{R}_i - \frac{N_. + 1}{2})^2$$

with $N_i = N\alpha_i$, $i = 1, 2, \cdots, m$, and $N_. = \Sigma_{i=1}^{m} N_i = nN$.

THEOREM. *If equal sample sizes of size N are drawn from n populations then the distribution of the $\bar{\chi}^2_{\text{rank}}$ statistic under the null hypothesis can be expressed:*

$$P[\bar{\chi}^2_{\text{rank}} \leqslant y] = \Sigma_{m=1}^{n} \Sigma_{k \in \mathcal{K}_m} P[T_N(\alpha_1, \alpha_2, \cdots, \alpha_m) \leqslant y] / \Pi_{i=1}^{n} k_i! \, i^{k_i},$$

where α is selected arbitrarily in $\mathcal{Q}^k$ for each $k \in \mathcal{K}_m$.

If n is small this result is quite useful. We have for $0 < y$ and $n = 3$

$$P[\bar{\chi}^2_{\text{rank}} > y] = \tfrac{1}{6}P[T_N(1, 1, 1) > y] + \tfrac{1}{2}P[T_N(1, 2) > y], \tag{1.2}$$

and for $n = 4$ we have

$$P[\bar{\chi}^2_{\text{rank}} > y] = \tfrac{1}{24}P[T_N(1, 1, 1, 1) > y] + \tfrac{1}{4}P[T_N(1, 1, 2) > y] + \tfrac{1}{8}P[T_N(2, 2) > y] + \tfrac{1}{3}P[T_N(1, 3) > y].$$

In order to compute probabilities for the $\bar{\chi}^2_{\text{rank}}$ statistic we need the distribution of the Kruskal-Wallis statistic. The exact distribution of the Kruskal-Wallis statistic for three populations with equal sample sizes up to size eight has been compiled in Alexander and Quade (1968). The distribution of the Kruskal-Wallis statistic for two populations can be obtained from the distribution of the Mann-Whitney-Wilcoxon statistic; extensive tables have been computed in Buckle et al. (1969).

TABLE 1

$\bar{\chi}^2$ prescribed α-levels vs. true α – levels

3 populations					
prescribed α-level and critical value	.10 2.580	.05 3.820	.025 5.098	.01 6.822	.005 8.146
sample size			true α-level (percentage error)		
2	.1222 (22)	.0111 (−78)	0 (−100)	0 (−100)	0 (−100)
3	.0970 (−3.0)	.0518 (3.6)	.0238 (−4.8)	.0006 (−94)	0 (−100)
4	.1007 (.72)	.0494 (−1.1)	.0204 (−18)	.0053 (−47)	.0004 (−92)
5	.0986 (−1.4)	.0531 (6.3)	.0216 (−14)	.0063 (−37)	.0020 (−60)
6	.0968 (−1.4)	.0517 (3.4)	.0215 (−14)	.0078 (−22)	.0027 (−45)
7	.1025 (2.5)	.0476 (−4.8)	.0223 (−11)	.0073 (−27)	.0030 (−40)
8	.0998 (−.17)	.0510 (2.0)	.0241 (−3.7)	.0082 (−18)	.0034 (−32)

Using expression (1.2) and the tables in the above mentioned references the exact distribution of the $\bar{\chi}^2_{\text{rank}}$ statistic can be computed for three populations with equal sample sizes up to size eight. In Table 1 we have compared the $\bar{\chi}^2$ prescribed α-level with that of the true α-level for three populations. The $\bar{\chi}^2$ approximation appears reasonable for α-levels .05 and .1 and sample sizes $N = 3, 4, \cdots, 8$, but using the $\bar{\chi}^2$ approximation for α-levels .01 and .005 would perhaps yield too conservative a test.

In Table 2 some exact critical values which yield α-levels closest to .005, .01, .025 and .05 are given.

TABLE 2

Selected $\bar{\chi}^2_{\text{rank}}$ α-levels for three populations

Sample Size	2	3		4		5		6		7		8	
c.v.*	P	c.v.	P	c.v.	P	c.v.	P	c.v.	P	c.v.	P	c.v.	P
4.57	.0111	5.68	.00476	7.03	.00496	7.28	.00426	7.380	.00462	7.576	.00496	7.595	.00475
3.71	.0333	5.59	.00833	6.96	.00525	7.25	.00532	7.377	.00538	7.502	.00501	7.593	.00522
3.42	.1222	5.42	.0119	6.26	.00949	6.25	.00979	6.397	.00916	6.441	.00953	6.495	.00999
		5.40	.0238	6.03	.0101	6.17	.01006	6.394	.01056	6.434	.01069	6.485	.01004
		5.06	.0262	4.88	.0224	4.87	.0236	5.064	.02175	5.016	.02313	4.955	.02492
		4.26	.0470	4.76	.0297	4.85	.0282	5.052	.02502	5.009	.02569	4.940	.02517
		3.80	.0518	3.84	.0494	3.85	.0450	3.871	.04546	3.769	.04891	3.920	.04979
		2.75	.0970	3.73	.0512	3.84	.0531	3.868	.05117	3.762	.05418	3.885	.0502

*We define c.v. to be the critical value and $P \equiv P[\bar{\chi}^2_{\text{rank}} \geqslant \text{c.v.}]$.

2. Proof of theorem. We wish to apply Theorem 1.2 of Boswell and Brunk (1969). The statement of this theorem involves many special notations, so we refer the reader to page 372 of Boswell and Brunk (1969) for the special notation and Theorem 1.2 which we shall use below. (We have changed least concave majorant in Boswell and Brunk (1969) to greatest convex minorant.)

In the terminology of Boswell and Brunk (1969) let $I = R^1$ and

$$f_m(v, \xi) = \frac{12}{nN(nN+1)} \Sigma_{i=1}^m \alpha_i N\left(w_i - \frac{Nn+1}{2}\right)^2.$$

(Note that the right-hand side does not depend upon ξ.) We now verify the hypotheses of Theorem 1.2.

The function $f_m(v, \xi)$ is symmetric in the components of v in $\mathcal{V}_m$ and in ξ in R^n. Now,

$$f_m(y(\alpha, \xi), \xi) = \frac{12}{nN(nN+1)} \Sigma_{i=1}^m \alpha_i N\left(u_i(\alpha, \xi) - \frac{nN+1}{2}\right)^2,$$

and by definition of $u_i(\alpha, \xi)$ we will have $f_m(y(\alpha, \xi), \xi)$ continuous in ξ for all $\alpha \in \mathcal{Q}_m$, $m = 1, 2, \cdots, n$.

$$f_{m(\xi)}(z(\xi), \xi) = \frac{12}{nN(nN+1)} \Sigma_{i=1}^{m(\xi)} N\alpha_i(\xi)\left(u_i(\alpha(\xi), \xi) - \frac{nN+1}{2}\right)^2;$$

but we may also express

$$f_{m(\xi)}(z(\xi), \xi) = \frac{12}{nN(nN+1)} \Sigma_{i=1}^n N\left(\mu_i^*(\xi) - \frac{nN+1}{2}\right)^2 \tag{1.3}$$

with $\mu_i^*(\xi) = \max_{r=1}^i \min_{t=i}^n \Sigma_{j=r}^t \xi_j/(t-r+1)$ (see Barlow et al. (1972), Chapter 1, Section 2). The function $\mu_i^*(\xi)$ is continuous in ξ for ξ in R^n, and it follows that $f_{m(\xi)}(z(\xi), \xi)$ is continuous in ξ for ξ in R^n. Therefore, condition (1.4) of Theorem 1.2 is satisfied.

For our problem let $\Xi = (\Xi_1, \Xi_2, \cdots, \Xi_n) \equiv (\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_n)$. Under H_0, $\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_n$ are exchangeable and hence Theorem 1.2 may be applied. However, from expressions (1.1) and (1.3) it is clear that the random variable $f_M(Z, \Xi)$ in the left-hand side of equation (1.5) of Theorem 1.2 of Boswell and Brunk (1969) is just the $\bar{\chi}^2_{\text{rank}}$ statistic. Also, the random variable $f_m(y(\alpha, \Xi), \Xi)$ on the right-hand side of that same equation is just a Kruskal-Wallis statistic based on m populations with sample sizes $N\alpha_1, N\alpha_2, \cdots, N\alpha_m$. This concludes the proof.

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