The Annals of Probability
1975, Vol. 3, No. 1, 159-161

THE EXPECTATION AND VARIANCE OF THE NUMBER OF COMPONENTS IN RANDOM LINEAR GRAPHS

BY JOSEPH I. NAUS[1] AND LARRY RABINOWITZ

Rutgers-The State University of New Jersey and College of William and Mary

Formulas are given for the expectation and variance of the number of components for two definitions of random graphs. The results extend those of R. F. Ling (1973).

1. Introduction. A linear graph of order n consists of n labeled vertices together with some subset of the $\binom{n}{2}$ possible edges. Gilbert [1] considers random graphs, where each possible edge has probability $p = 1 - q$ of inclusion in the graph independently of other edges. Gilbert finds the probability, P_n, that a random graph of order n is connected.

Ling [2] considers the set $T_{n,r}$ of linear graphs of order n that have exactly r edges. He lets $\Gamma_{n,r}$ denote a graph picked at random from the $N(n, r) = \binom{\binom{n}{2}}{r}$ possible graphs in $T_{n,r}$. Ling notes that the probability that $\Gamma_{n,r}$ is connected is $C_{n,r}/N(n, r)$, where $C_{n,r}$ is a known quantity (Ling (1)).

Ling denotes the number of components (connected subgraphs) of $\Gamma_{n,r}$ of size j(j vertices) as $\Gamma_{n,r,j}$. He derives $E(\Gamma_{n,r,j})$ as a function of the $C_{j,l}$ terms. We provide a simple alternative derivation that readily yields the Var $(\Gamma_{n,r,j})$ and gives parallel results for the expectation and variance of the number of components for Gilbert's case.

2. The expectation and variance of $\Gamma_{n,r,j}$ (Ling's case). Ling has proved that

$$E(\Gamma_{n,r,j}) = \binom{n}{j} \textstyle\sum_l C_{j,l} N(n - j, r - l)/N(n, r)\,, \qquad l = \overline{j-1}(1)\binom{j}{2}\,. \tag{1}$$

We prove that

$$\operatorname{Var}(\Gamma_{n,r,j}) = E(\Gamma_{n,r,j}) - E^2(\Gamma_{n,r,j}) + n\binom{n-1}{j}\binom{n-j-1}{j-1}H(n, r, j)/j\,, \tag{2}$$

where

$$H(n, r, j) = \textstyle\sum \sum_{l_1,l_2} N(n - 2j, r - l_1 - l_2)C_{j,l_1}C_{j,l_2}/N(n, r)\,.$$

PROOF. Let $E_{i,j}$ denote the event that vertex i is in a component of size j. Note that

$$\Gamma_{n,r,j} = \textstyle\sum_{i=1}^n X_i\,,$$

where

$$\begin{aligned} X_i &= 1/j\,, && \text{if} \quad E_{i,j}\,, \\ &= 0\,, && \text{otherwise}\,. \end{aligned}$$

Received October 25, 1973; revised January 18, 1974.

[1] Supported in part by National Science Foundation Grant No. GP8210.

AMS 1970 *subject classifications.* Primary 60C05; Secondary 05C30.

Key words and phrases. Random linear graphs; connected subgraphs; number of components; expectations and variances; graphs.

Equation (1) follows immediately by noting that

$$E(\Gamma_{n,r,j}) = \sum_{i=1}^{n} E(X_i) = n \Pr(E_{i,j})/j$$

and

$$\Pr(E_{i,j}) = \binom{n-1}{j-1} \sum_l N(n-j, r-l)C_{j,l}/N(n, r) .$$

To find (2), write

$$\operatorname{Var}(\Gamma_{n,r,j}) = nE(X_i^2) + n(n-1)E(X_iX_k) - E^2(\Gamma_{n,r,j}) .$$

To find $E(X_iX_k) = \Pr(E_{i,j} \cap E_{k,j})/j^2$, note that vertex i and k may be in the same or different components:

$$\Pr(E_{i,j} \cap E_{k,j};\ \text{same component}) = \binom{n-2}{j-2} \sum_l N(n-j, r-l)C_{j,l}/N(n, r) ;$$
$$\Pr(E_{i,j} \cap E_{k,j};\ \text{different components}) = \binom{n-2}{j-1}\binom{n-j-1}{j-1}H(n, r, j) .$$

Asymptotic results. Ling (Corollary 1.1, Theorem 2, (4)—(5), Corollary 2.1) gives several approximations for $E(\Gamma_{n,r,j})$. To derive approximations for $\operatorname{Var}(\Gamma_{n,r,j})$, we can approximate $H(n, r, j)$ in our equation (2) as in the approach in Ling (Corollary 1.1):

$$H(n, r, j) \doteq \{N(n-2j, r-2j+2)C_{j,j-1}^2 + 2N(n-2j, r-2j+1)C_{j,j-1}C_{j,j} + N(n-2j, r-2j)C_{j,j}^2\}/N(n, r) ,$$

where $C_{j,j-1} = j^{j-2}$, and

$$C_{j,j} = \frac{(j-1)!}{2}\left(1 + j + \frac{j^2}{2!} + \cdots + \frac{j^{j-3}}{(j-3)!}\right) .$$

The other approximations can be carried out similarly.

3. The expectation and variance of the number of components in random graphs of order n (Gilbert's case). Let $Y_{n,j}$ denote the number of components of size j in a random graph of order n. Let $Y_n = \sum_{j=1}^{n} Y_{n,j}$ denote the total number of components. We prove that

$$E(Y_n) = \sum_{k=1}^{n} \binom{n}{k} P_k q^{k(n-k)} , \tag{3}$$

and

$$\operatorname{Var}(Y_n) = E(Y_n) - E^2(Y_n) + \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} \frac{n!}{r!\, s!\, (n-r-s)!} P_r P_s q^{n(r+s)-r^2-s^2-rs} , \tag{4}$$

where P_k is the probability that a random graph of order k is connected, and q is the probability that a given edge is excluded from the graph.

PROOF. Let $X_i^* = 1/j$, if $E_{i,j}$, for $j = 1, 2, \cdots, n$; note that $Y_n = \sum_{i=1}^{n} X_i^*$. The proof follows as before. Here we consider $\Pr(E_{i,r} \cap E_{k,s})$ where r and s may or may not be equal. If $r \neq s$, then vertex i and k must be in different components.

To use (3) and (4) first use the recurrence formula in Gilbert [1] to find values of P_k. Table 1 gives some values for the expectation and variance of the number of components in a random graph for Gilbert's case.

Asymptotic results. Gilbert ([1] page 1144) gives the result for large n:

$$P_n = 1 - nq^{n-1} + O(n^2q^{3n/2}) .$$

We substitute this together with the exact results $P_1 = 1$, $P_2 = 1 - q$ in equations (3) and (4) to find

$$E(Y_n) = 1 + nq^{n-1} + O(n^2q^{3n/2}) ,$$

and

$$\text{Var}\,(Y_n) = nq^{n-1} + O(n^2q^{3n/2}) .$$

TABLE 1

Expectation and variance of number of components in a random graph of order n

Expectation

n	q .1	.3	.5	.7	.9
2	1.10000	1.30000	1.50000	1.70000	1.90000
3	1.02900	1.24300	1.62500	2.12700	2.70100
4	1.00424	1.11918	1.53125	2.31918	3.40424
5	1.00051	1.04497	1.36524	2.32632	4.00996
6	1.00006	1.01538	1.21936	2.20655	4.52402

Variance

n	q .1	.3	.5	.7	.9
2	.09000	.21000	.25000	.21000	.09000
3	.03016	.23795	.48437	.54287	.26560
4	.00434	.12976	.53027	.88989	.52110
5	.00051	.04482	.33144	.98818	.89052
6	.00006	.01748	.27195	1.04536	1.38317

REFERENCES

[1] Gilbert, E. N. (1959). Random graphs. *Ann. Math. Statist.* **30** 1141–44.

[2] Ling, R. F. (1973). The expected number of components in random linear graphs. *Ann. Probability* **1** 876–81.

Joseph I. Naus
Statistics Center
Rutgers, the State University of New Jersey
New Brunswick, New Jersey 08903

Larry Rabinowitz
Department of Mathematics
College of William and Mary
Williamsburg, Virginia 23185