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BROWNIAN MOTION AND THE EQUILIBRIUM MEASURE ON THE JULIA SET OF A RATIONAL MAPPING[1]

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It is proved that if a rational mapping has ∞ as a fixed point in its Fatou set, then its Julia set has positive capacity and the equilibrium measure is invariant. If ∞ is attracting or superattracting, then the equilibrium measure is strongly mixing, whereas if ∞ is neutral, then the equilibrium measure is ergodic and has entropy zero. Lower bounds for the entropy are given in the attracting and superattracting cases. If the Julia set is totally disconnected, then the equilibrium measure is Gibbs and therefore Bernoulli. The proofs use an induced action by the rational mapping on the space of Brownian paths started at ∞.

1. Introduction. Let $Q(z) = P_1(z)/P_2(z)$ be a rational function of degree $d \geq 2$, and let $Q^n(z)$, $n \geq 0$, be its iterates:

$$Q^0(z) = z, \qquad Q^{n+1}(z) = Q(Q^n(z)).$$

The *Julia set* of $\mathscr{J}$ of Q is the set of points $z \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for which $\{Q^n\}_{n \geq 1}$ is not a normal family in any neighborhood of z. The *Fatou set* $\mathscr{F}$ is the complement of $\mathscr{J}$, that is, $\mathscr{F} = \overline{\mathbb{C}} \setminus \mathscr{J}$. The Julia set $\mathscr{J}$ is a nonempty, compact set satisfying $\mathscr{J} = Q(\mathscr{J}) = Q^{-1}(\mathscr{J})$ (Section 2).

The purpose of this paper is to investigate certain ergodic properties of the (normalized) *equilibrium measure* ν on $\mathscr{J}$ for rational mappings Q such that $Q(\infty) = \infty$ and $\infty \in \mathscr{F}$. See [12], Section 3.4 for the classical definition of ν. We shall adopt a probabilistic point of view, regarding ν as the distribution of the point of first entry into $\mathscr{J}$ by a Brownian motion started at ∞ (this may be taken as the definition of ν; see [12], Section 3.4). This will allow us to completely avoid methods and results of classical potential theory. Previous studies of the equilibrium measure on $\mathscr{J}$, for example, [3], [9], have not exploited its probabilistic interpretation.

It has been known since [3] that the equilibrium measure ν plays a distinguished role in the ergodic theory of *polynomial* mappings Q. Fix $x \in \mathbb{C}$ and consider the set $Q^{-n}(z) = \{\xi\colon Q^n(\xi) = z\}$. Observe that $Q^{-n}(z)$ has cardinality d^n, provided multiple roots are counted accordingly. Define μ_n^z to be the uniform distribution on $Q^{-n}(z)$, that is, μ_n^z is the probability measure which puts mass d^{-n} at each root of $Q^n(\xi) = z$.

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THEOREM (Brolin [3]). *If Q is a polynomial of degree $d \geq 2$, then $\mathscr{J}$ has positive (logarithmic) capacity, so ν is defined. For all but at most one $z \in \mathbb{C}$,*

$$\mu_n^z \to_{\mathscr{D}} \nu \quad as\ n \to \infty.$$

Furthermore, ν is an invariant measure for Q and the measure-preserving system $(\mathscr{J}, Q, \nu)$ is strongly mixing.

NOTE. $\to_{\mathscr{D}}$ indicates weak-convergence (convergence in distribution), that is, $\mu_n \to_{\mathscr{D}} \mu$ iff for every continuous function $f\colon \overline{\mathbb{C}} \to \mathbb{R}$, $\int f\, d\mu_n \to \int f\, d\mu$.

It is natural to wonder whether Brolin's theorem is true for an arbitrary rational mapping Q. This question has only recently been settled.

THEOREM (Ljubich [8]). *For all but at most two points $z \in \overline{\mathbb{C}}$, $\mu_n^z \to_{\mathscr{D}} \mu$, where μ is the unique maximum entropy invariant probability measure for $Q\colon \mathscr{J} \to \mathscr{J}$. Moreover, $(\mathscr{J}, Q, \mu)$ is strongly mixing and has entropy* $\log d$.

THEOREM (Lopes [9]). *If $Q(\infty) = \infty \notin \mathscr{J}$ and if $\nu = \mu$, then Q is a polynomial.*

One might now ask: (i) are there any rational mappings other than polynomials for which $\mathscr{J}$ has positive capacity; and (ii) if so, what can be said about the dynamical system $(\mathscr{J}, Q, \nu)$?

We shall assume henceforth that ∞ is a fixed point of Q (i.e., $Q(\infty) = \infty$) and that $\infty \notin \mathscr{J}$. Let $Q(z) = P_1(z)/P_2(z)$, where $P_1(z) = a_0 z^d + a_1 z^{d-1} + \cdots + a_d$ with $a_0 \neq 0$, and $P_2(z) = z^{d_*} + b_1 z^{d_* - 1} + \cdots + b_{d_*}$ with $d_* < d$, and $P_1(z)$ and $P_2(z)$ have no nontrivial common factors. If $d \geq d_* + 2$, say that ∞ is superattracting; if $d = d_* + 1$ and $|a_0| > 1$, say that ∞ is attracting; and if $d = d_* + 1$ and $|a_0| = 1$, say that ∞ is neutral. (The case $d = d_* + 1$ and $|a_0| < 1$ cannot occur, because in this case ∞ is a repelling fixed point and therefore $\infty \in \mathscr{J}$—see [1], Section 5.) Observe that if ∞ is attracting or superattracting, then there exists $C < \infty$ such that $\lim_{n\to\infty} |Q^n(z)| = \infty\ \forall\ |z| \geq C$. If $Q(z)$ is a polynomial, then ∞ is superattracting.

THEOREM 1. *If $Q(\infty) = \infty \in \mathscr{F}$, then the logarithmic capacity of $\mathscr{J}$ is positive, and hence the normalized equilibrium measure ν on $\mathscr{J}$ exists. Furthermore, ν is an invariant measure for Q. If ∞ is attracting or superattracting, then the measure-preserving system $(\mathscr{J}, Q, \nu)$ is strongly mixing, hence ergodic. If ∞ is neutral, then $(\mathscr{J}, Q, \nu)$ is a factor of an irrational rotation of the circle, hence is ergodic and has entropy zero. Consequently, ν and μ are mutually singular unless Q is a polynomial.*

A measure-preserving system (Ω_0, T_0, μ_0) is said to be a factor of another m.p.s. (Ω_1, T_1, μ_1) if there is a measurable map $\varphi\colon \Omega_1 \to \Omega_0$ onto $\Omega_0 \setminus N$ with $\mu_0(N) = 0$ such that $\mu_0 = \mu_1 \circ \varphi^{-1}$ and $\varphi \circ T_1 = T_0 \circ \varphi$. The entropy of (Ω_0, T_0, μ_0) is less than or equal to that of (Ω_1, T_1, μ_1), and if (Ω_1, T_1, μ_1) is ergodic, then so is (Ω_0, T_0, μ_0). Since irrational rotations of the circle are ergodic and have entropy zero, the same is true of their factors.

The fact that ν is strongly mixing in the attracting and superattracting cases implies that ν is ergodic. By Ljubich's theorem, μ is ergodic and by Lopes' theorem, $\mu \neq \nu$ unless Q is a polynomial. Since ergodic invariant measures are either equal or mutually singular, it follows that μ and ν are mutually singular unless Q is a polynomial.

Let $h(Q)$ be the entropy of the m.p.s. $(\mathscr{J}, Q, \nu)$. Ljubich's theorem and Theorem 1 imply that $h(Q) < \log d$ unless Q is a polynomial, in which case $h(Q) = \log d$.

THEOREM 2. *Assume that* ∞ *is attracting or superattracting.* (a) *Then* $h(Q) \geq \log(d - d_*)$. (b) *If all the branch points of* Q^{-1} *are contained in the connected component of* $\mathscr{F}$ *containing* ∞, *then* $h(Q) > \log(d - d_*)$, *provided* $d_* \geq 1$.

Let $\Sigma = \{1, 2, \dots, d\}^{\mathbb{N}}$ be the set of all sequences from the alphabet $\{1, 2, \dots, d\}$ and let $\sigma\colon \Sigma \to \Sigma$ be the forward shift.

THEOREM 3. *Assume that* ∞ *is attracting or superattracting and that all the branch points of* Q^{-1} *are contained in the connected component of* $\mathscr{F}$ *containing* ∞. *Then there is a homeomorphism* $\pi\colon \Sigma \to \mathscr{J}$ *such that* $\pi \circ \sigma = Q \circ \pi$ *and such that the induced measure* $\bar{\nu}$ *on* Σ *defined by* $\bar{\nu} \circ \pi^{-1} = \nu$ *is a Gibbs state. Consequently, the measure-preserving system* $(\mathscr{J}, Q, \nu)$ *is isomorphic to a Bernoulli shift.*

REMARK 1. The existence of the topological conjugacy π under the hypotheses of Theorem 3 is known, at least for polynomial mappings Q. (See [1], Section 9; however, the proof for the case degree $(Q) > 2$ has an error.) The main point (and by far the more difficult) is that $\bar{\nu}$ is a Gibbs state. See [2], Theorem 1.2, for the definition of a Gibbs state. See [2], Theorem 1.25, for the implication Gibbs $\Rightarrow$ Bernoulli.

REMARK 2. The situation described in the hypothesis of Theorem 3 is very common. If $Q_0(z)$ is any rational mapping for which ∞ is a superattracting fixed point, then $Q_a(z) \triangleq Q_0(z) + a$ satisfies the hypothesis of Theorem 3 for all $|a| \geq a_*$ (here a_* may depend on Q_0). See [1], Section 9, for the argument in the polynomial case (the rational case is essentially the same).

REMARK 3. That ν is a Gibbs state implies considerably more than the Bernoulli property; see [2], [6], [7]. For example, if $\mathscr{J} \to \mathbb{R}$ is a Hölder-continuous function *not* of the form $f = (\text{constant}) + g - g \circ Q$, then the sequence $S_n f = f + f \circ Q + f \circ Q^2 + \cdots + f \circ Q^{n-1}$ obeys the central limit theorem, law of the iterated logarithm, large deviations theorems, and so on, under ν.

REMARK 4. That the maximum entropy measure μ is a Gibbs state follows from the Gibbs variational principle ([2], Theorem 1.22). No such trivial proof can be given for ν.

REMARK 5. I conjecture that the main point of Theorem 3, that ν is a Gibbs state, remains true when the hypothesis concerning the branch points is weakened to expansivity of Q on $\mathscr{J}$, but may fail when $\mathscr{J}$ contains parabolic fixed points.

Our approach to all of the results concerning ν stated above is by way of a probabilistic characterization of the measure. Let Z_t be a standard Brownian motion process on $\overline{\mathbb{C}}$ started at ∞ (Section 3 below). Then Z_t enters $\mathscr{J}$ in finite time with probability 0 (if $\mathscr{J}$ has capacity 0) or 1 (if $\mathscr{J}$ has positive capacity) and in the latter case ν is the distribution of the first entrance point ([12], Chapter 3, Theorem 4.12). More important, $Q(Z_t)$ is also (after a reparametrization of time) a Brownian motion process started at ∞. Thus Q acts not only on $\overline{\mathbb{C}}$, but on the space of Brownian paths in $\overline{\mathbb{C}}$. This observation is the key to our results. To further emphasize the usefulness of Brownian paths, we shall give purely probabilistic proofs of (most of) Brolin's theorem (Section 5) and Lopes' theorem (Section 7); these are shorter, simpler and (we believe) more appealing to the intuition than the originals.

Some familiarity with the basic properties of Brownian motion—path continuity, the strong Markov property, rotational symmetry—is assumed; see [4] and [5], Section 1.1–1.7. The one deep property of Brownian motion that is needed, Lévy's conformal invariance theorem, is described in Section 3. For the convenience of the reader, some basic results of complex analytic dynamics are given in Section 2. Theorem 1 is proved in Sections 4 and 9, Theorem 2 in Section 6 and Theorem 3 in Section 8.

NOTE. Since writing this paper the author has learned that Theorem 3 has also been proved by Makarov and Volberg by a similar method, in an unpublished paper entitled "On the harmonic measure of discontinuous fractals."

2. Preliminaries: Complex analytic dynamics. The most interesting cases of Theorem 1 are when ∞ is an attractive or superattracting fixed point of Q. We shall assume in Sections 2–8 that ∞ is an attracting or superattracting fixed point. The alternative case, in which ∞ is a nonattractive (neutral) fixed point, will be considered separately in Section 9. If ∞ is attracting or superattracting, then there is a neighborhood $\mathscr{N}$ of ∞ in $\overline{\mathbb{C}}$ such that $Q^n(z) \to \infty$ as $n \to \infty$ uniformly for $z \in \mathscr{N}$.

A normal family in a domain $\mathscr{D}$ is a set $\{f_\lambda\}$ of functions meromorphic in $\mathscr{D}$ such that any sequence f_n has a subsequence that converges uniformly (with respect to the spherical metric) on compact subsets of $\mathscr{D}$. By the Arzela–Ascoli theorem, this is equivalent to the statement that $\{f_\lambda\}$ is equicontinuous in $\mathscr{D}$. If a set of analytic functions in $\mathscr{D}$ is uniformly bounded on every compact subset of $\mathscr{D}$, then it must be a normal family, because the Cauchy integral formula implies that the derivatives are uniformly bounded on compact subsets, and hence the set of functions is equicontinuous.

A set $\{f_\lambda\}$ of meromorphic functions is said to be normal at a point $z \in \overline{\mathbb{C}}$ if it is normal in some neighborhood of z. The *Fatou set* $\mathscr{F}$ of $Q(z)$ is defined [1] to be the set of $z \in \overline{\mathbb{C}}$ at which $\{Q^n\}_{n \geq 0}$ is normal. The Fatou set is clearly open and $\infty \in \mathscr{F}$ because $Q^n \to \infty$ uniformly in a neighborhood of ∞. The *Julia set* $\mathscr{J}$ is defined to be the complement of $\mathscr{F}$; it is evidently compact. Clearly, $Q(\mathscr{F}) = \mathscr{F}$ and $Q(\mathscr{J}) = \mathscr{J}$.

PROPOSITION 1. $\mathscr{J} \neq \varnothing$.

PROOF. If $\mathscr{J} = \varnothing$, then $\{Q_n\}_{n \geq 0}$ would be a normal family on $\overline{\mathbb{C}}$. Now $Q^n \to \infty$ uniformly in a neighborhood of ∞; consequently, if Q^{n_k} converges uniformly on $\overline{\mathbb{C}}$, then the limit function, being meromorphic, must be identically ∞. But it is impossible for $Q^{n_k} \to \infty$ uniformly on $\overline{\mathbb{C}}$, because each $Q^n\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is surjective. □

NOTE. See [1] for an argument that is valid even when ∞ is not an attracting or superattracting fixed point.

Define $\mathscr{F}_\infty$ to be the path-connected component of $\mathscr{F}$ that contains ∞, that is, the set of $z \in \mathscr{F}$ such that there is a continuous path from ∞ to z that lies entirely in $\mathscr{F}$.

PROPOSITION 2. *If $z \in \mathscr{F}_\infty$, then $Q(z) \in \mathscr{F}_\infty$ and $\lim_{n\to\infty} Q^n(z) = \infty$. Furthermore, this convergence is uniform on compact subsets of $\mathscr{F}_\infty$.*

PROOF. Let $\gamma(t)$, $0 \leq t \leq 1$, be a continuous path in $\mathscr{F}$ such that $\gamma(0) = \infty$ and $\gamma(1) = z$. Then $Q(\gamma(t))$ is a continuous path in $\mathscr{F}$ (because $\mathscr{F}$ and $\mathscr{J}$ are Q-invariant) such that $Q(\gamma(0)) = \infty$ and $Q(\gamma(t)) = Q(z)$; hence $Q(z) \in \mathscr{F}_\infty$. Since $\{Q^n\}_{n\geq 1}$ is normal in $\mathscr{F}$, every subsequence of Q^n has a subsequence which converges uniformly in a neighborhood of $\gamma([0,1])$. But $Q^n(\zeta) \to \infty$ uniformly for ζ in a neighborhood of ∞, hence for ζ in $\gamma([0,\varepsilon])$ for some $\varepsilon > 0$. Thus any subsequence Q^{n_k} which converges on $\gamma([0,1])$ must in fact converge to ∞, since the limit function must be meromorphic. It follows that $Q^n(z) \to \infty$.

For each $C < \infty$ sufficiently large, if $|z| > C$, then $|Q(z)| > C$. For each $z \in \mathscr{F}_\infty$, there is an integer $n \geq 1$ and a neighborhood $\mathscr{U}$ of z such that $Q^n(\mathscr{U}) \subset \{\zeta\colon |\zeta| > C\}$. It follows that $Q^n \to \infty$ uniformly on compact subsets of $\mathscr{F}_\infty$. □

For each $n \geq 1$, the inverse function of Q^n is multivalued, with branch points contained in $\mathscr{S}_n$, where

$$\mathscr{S}_0 = \{z \in \mathbb{C}\colon (dQ/dz) = 0\} \cup \{z \in \overline{\mathbb{C}}\colon Q(z) = \infty\},$$

$$\mathscr{S}_n = \bigcup_{m=0}^{n} Q^m(\mathscr{S}_0).$$

Let

$$\mathscr{S}_+ = \bigcup_{n=0}^{\infty} Q^n(\mathscr{S}_0).$$

The branches of the inverse function will be denoted by Q_i^{-n}, $i = 1, 2, \ldots, d^n$. Each Q_i^{-n} is a (single-valued) analytic function in any simply connected domain disjoint from $\mathscr{G}_n$.

Consider the set $\mathscr{G}_+ \cap \mathscr{F}_\infty$. If $\xi \in \mathscr{G}_0$ is such that $Q^m(\xi) \in \mathscr{F}_\infty$ for some $m \geq 0$, then $\lim_{n\to\infty} Q^n(\xi) = \infty$ by Proposition 2; consequently, the only possible accumulation point of $\mathscr{G}_+ \cap \mathscr{F}_\infty$ is ∞. It follows that each point of $\mathscr{F}_\infty \setminus \mathscr{G}_+$ has a simply connected neighborhood disjoint from $\mathscr{G}_+$.

PROPOSITION 3. *If Q is a polynomial, then $Q^{-1}(\mathscr{F}_\infty) = \mathscr{F}_\infty$.*

PROOF. By Proposition 2, $Q(\mathscr{F}_\infty) \subset \mathscr{F}_\infty$, so it suffices to show that $Q^{-1}(\mathscr{F}_\infty) \subset \mathscr{F}_\infty$. Let $z \in \mathscr{F}_\infty \setminus \{\infty\}$. There is a continuous path $\gamma(t)$, $0 \leq t \leq 1$, from ∞ to z such that $\gamma(t) \in \mathscr{F}_\infty \setminus \mathscr{G}_+$ for every $t \in (0, 1)$. This is because $\mathscr{F}_\infty \cap \mathscr{G}_+$ has no accumulation points in $\mathscr{F}_\infty$ except ∞.

If $\deg(Q) = d$, then ∞ is a d-fold root of $Q(\xi) = \xi$ and locally $Q(\xi)$ acts like (const) $\times\, \xi^d$. Thus $Q^{-1}(\gamma[0,1])$ consists of d distinct continuous paths, each beginning at ∞ and ending at one of the d points in $Q^{-1}(z)$. Each of these paths lies entirely in $\mathscr{F}$ since $\gamma[0,1] \subset \mathscr{F}$. By definition, each of the endpoints lies in $\mathscr{F}_\infty$. $\square$

PROPOSITION 4. *If $\mathscr{Q} = \{Q_i^{-n}\}_{n,i}$ is a collection of certain branches of Q^{-n} such that each $Q_i^{-n} \in \mathscr{Q}$ is single-valued and meromorphic in a domain $\mathscr{U}$ disjoint from a neighborhood of ∞, then $\mathscr{Q}$ is a normal family in $\mathscr{U}$.*

PROOF. Since ∞ is an attracting or superattracting fixed point, there exists $C < \infty$ such that if $|z| \geq C$, then $|Q(z)| > |z|$. Since $\mathscr{U}$ is disjoint from a neighborhood of ∞, $\bigcup_{n=0}^{\infty} Q^{-n}(\mathscr{U})$ is disjoint from a neighborhood of ∞. Hence, $\mathscr{Q}$ is uniformly bounded on $\mathscr{U}$. $\square$

Recall that if $z \in \mathscr{F}_\infty \setminus \mathscr{G}_+$, then z has a simply connected neighborhood containing no branch points of any Q^{-n}. Therefore, by Proposition 4, the collection $\{Q_i^{-n}, 1 \leq i \leq d^n, n \geq 1\}$ of *all* branches is a normal family at z.

PROPOSITION 5. *Let $Q_{i_k}^{-n_k}$, $k \geq 1$, be a sequence of branches of Q^{-n_k}, where $n_k \to \infty$, each of which is single-valued and meromorphic in $\mathscr{U}$, a connected open subset of $\mathscr{F}_\infty$. If $Q_{i_k}^{-n_k}$ converges uniformly on compact subsets of $\mathscr{U}$, then the limit is a constant function, and the constant is an element of the Julia set $\mathscr{J}$.*

PROOF. Let $f = \lim Q_{i_k}^{-n_k}$; then f is a meromorphic function in $\mathscr{U}$. Suppose that $\xi = f(z) \in \mathscr{F}$ for some $z \in \mathscr{U}$, $z \neq \infty$. By definition of $\mathscr{F}$, $\{Q^n\}$ would then be a normal family at ξ and consequently would be equicontinuous in a neighborhood of ξ. Since $Q_{i_k}^{-n_k}(z) \to \xi$ as $k \to \infty$, equicontinuity would imply that $Q^{n_k}(\xi) \to z$ as $k \to \infty$. But this is impossible, because by Proposition 2, $Q^n \to \infty$ uniformly on compact subsets of $\mathscr{F}_\infty$. This proves that $f(\mathscr{U} \setminus \{\infty\}) \cap \mathscr{F} = \varnothing$, so $f(\mathscr{U} \setminus \{\infty\}) \subset \mathscr{J}$.

Next, suppose that f is not constant on $\mathscr{U}$. Then $f(\mathscr{U} \setminus \{\infty\})$ is an open set, by the open mapping theorem for analytic functions. We will show that this is impossible by showing that every point of $\mathscr{J}$ is a boundary point of $\mathscr{F}$.

Let $\zeta \in \mathscr{J}$. Then for any open neighborhood $\mathscr{N}$ of ζ, $\{Q^n\}_{n \geq 0}$ is *not* normal in $\mathscr{N}$, by definition of $\mathscr{J}$. Thus $\{Q^n\}_{n \geq 0}$ is not uniformly bounded in $\mathscr{N}$. Since $\mathscr{F}$ contains a neighborhood of ∞ it follows that $Q^n(z) \in \mathscr{F}$ for some $z \in \mathscr{N}$ and some $n \geq 0$. But $Q^{-n}(\mathscr{F}) = \mathscr{F}$, so $z \in \mathscr{F}$. Thus every neighborhood of ζ intersects $\mathscr{F}$. □

PROPOSITION 6. *Let Γ be a simple closed curve in $\mathbb{C}$ that completely encloses $\mathscr{J}$. If γ is a continuous path from ∞ to a point of $\mathscr{J}$, then γ intersects $Q^{-n}(\Gamma)$ for each $n \geq 0$.*

PROOF. Consider the path $Q^n \circ \gamma$. This is a continuous path that starts at ∞ and terminates at a point of $\mathscr{J}$. Consequently, it must intersect Γ, since every continuous path from ∞ to $\mathscr{J}$ must cross Γ. It follows that γ intersects $Q^{-n}(\Gamma)$. □

PROPOSITION 7. *Let Γ be a simple closed curve in $\mathbb{C}$ that completely encloses $\mathscr{J}$. If $\gamma(t)$, $0 \leq t \leq t_*$, is a continuous path that starts at $\gamma(0) = \infty$ and intersects $Q^{-n}(\Gamma)$ for each $n \geq 0$, then $\gamma(t) \in \mathscr{J}$ for some $t \in [0, t_*]$.*

PROOF. Let $z_n \in Q^{-n}(\Gamma) \cap \mathscr{F}_\infty$ for $n \geq 0$. Then as $n \to \infty$, distance $(z_n, \mathscr{J}) \to 0$ because $\{z \in \mathscr{F}_\infty\colon \text{distance}(z, \mathscr{J}) \geq \varepsilon\}$ is a compact subset of $\mathscr{F}_\infty$ on which $Q^n \to \infty$ *uniformly*, by Proposition 2.

By hypothesis, $\gamma([0, t_*]) \cap Q^{-n}(\Gamma) \neq \varnothing \ \forall \ n \geq 0$, so we can choose $z_n \in \gamma([0, t_*]) \cap Q^{-n}(\Gamma)$. By the preceding paragraph, distance $(z_n, \mathscr{J}) \to 0$ as $n \to \infty$. Since $\mathscr{J}$ is compact, there is a subsequence z_k of z_n such that $z_k \to z \in \mathscr{J}$. But $\gamma[0, t_*]$ is closed, so $z \in \gamma([0, t_*])$. □

If ∞ is a fixed point of $Q = P_1/P_2$, then near ∞ the action of Q is close to that of a monomial with degree = degree(P_1) − degree(P_2). A useful way of formulating this statement is as follows.

PROPOSITION 8. *There is a neighborhood $\mathscr{U}$ of ∞ in $\overline{\mathbb{C}}$ and a conformal bijection $\varphi\colon \{z\colon |z| > r\} \to \mathscr{U}$ for some $r > 1$ such that $\varphi(\infty) = \infty$ and:*

(a) *if ∞ is superattracting, then $Q(\varphi(z)) = \varphi(\alpha z^{d-d_*})$ for every $z \in \mathscr{U}$, where $\alpha \neq 0$ is a constant*; *and*

(b) *if ∞ is attracting, then $Q(\varphi(z)) = \alpha\varphi(z)$ for every $z \in \mathscr{U}$, where α is a constant such that $|\alpha| > 1$.*

See [1], Section 3, Theorem 3.3–3.4.

3. Conformal invariance of Brownian motion. Let $\mathscr{D}$ be an open subset of $\mathbb{C}$ with smooth boundary $\partial\mathscr{D}$ and let f be an analytic function defined in a neighborhood of $\overline{\mathscr{D}}$. If Z_t is a Brownian motion in $\mathbb{C}$ started at

$z \in \mathscr{D}$ and if $T = \inf\{t \geq 0\colon Z_t \in \partial\mathscr{D}\}$, then $(f(Z_t))_{0 \leq t \leq T}$ is, after a time change, a Brownian motion started at $f(z)$ and run until it exits $f(\mathscr{D})$. This theorem is due to Lévy (cf. [4] or [5]). Lévy's theorem is clearly local in nature, and hence may be generalized to Brownian motion and analytic functions on an arbitrary Riemann surface.

The extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ may be identified with the unit sphere in $\mathbb{R}^3$ by the operation of stereographic projection. With this identification, $\overline{\mathbb{C}}$ inherits a (Riemannian) metric from the Euclidean metric on the unit sphere in $\mathbb{R}^3$, and thus also a Laplace–Beltrami operator Δ_{sphere}. Brownian motion on the Riemann sphere $\overline{\mathbb{C}}$ is the diffusion process with generator Δ_{sphere}; since Δ_{sphere} is a uniformly elliptic operator, the existence of this diffusion process follows from the results of [10], Sections 4.1–4.3. Thus we can talk about Brownian motion on $\overline{\mathbb{C}}$ started at ∞.

The relationship between planar Brownian motion and spherical Brownian motion is as follows. There is a C^∞ function $\rho > 0$ on $\mathbb{C}$ such that $\Delta_{\text{sphere}} = \rho\Delta$, where Δ is the usual Laplacian on $\mathbb{R}^2$ (this follows from the fact that stereographic projection is a conformal mapping). Consequently, spherical Brownian motion on $\overline{\mathbb{C}}$ started at any $z \in \mathbb{C}$ is just a time-changed planar Brownian motion started at z, the instantaneous time dilation factor being the current value of ρ.

Now let f be a (possibly multivalued) function that admits an analytic continuation along every continuous path in $\overline{\mathbb{C}} \setminus F$, where F is a finite set. If Z_t is a (spherical) Brownian motion started at $z \in \overline{\mathbb{C}} \setminus F$, then $f(Z_{\tau(t)})$ is a (spherical) Brownian motion started at $f(z)$, where

$$\tau(t) = \inf\left\{r\colon \int_0^r |\delta f(Z_s)|^2\, ds \geq t\right\} \tag{3.1}$$

and $|\delta f(\xi)|$ is the factor by which $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ expands (spherical) distances locally at ξ. This follows from the local form of Lévy's theorem together with the fact that spherical Brownian motion is time-changed planar Brownian motion.

These results carry over to arbitrary Riemann surfaces. Let M and N be compact Riemann surfaces and let $f\colon M \to N$ be analytic. Brownian motion on M (or N) is the strong Markov process whose infinitesimal generator is the Laplace–Beltrami operator on M (or N); its existence follows from [10], Sections 4.1–4.3. If Z_t is a Brownian motion on M started at z, then $f(Z_{\tau(t)})$ is a Brownian motion on N started at $f(z)$, where $\tau(t)$ is given by (3.1) and $|\delta f(\xi)|$ is the factor by which f expands distances locally at ξ.

4. Brownian motion in $\mathscr{F}_\infty$. Let Z_t, $t \geq 0$, be a Brownian motion process in $\overline{\mathbb{C}}$ started at $Z_0 = z$ under the probability measures P^z, $z \in \overline{\mathbb{C}}$. Then $\overline{Z}_t = Q(Z_{\tau(t)})$, with $\tau(t)$ given by (3.1) with $f = Q$, is a Brownian motion started at $Q(z)$. Recall that $Q(\infty) = \infty$.

Spherical Brownian motion Z_t is recurrent but does not hit individual points. In other words, (a) for any nonempty, open set $U \subset \overline{\mathbb{C}}$, any $z \in \overline{\mathbb{C}}$, and any $t_* < \infty$, $P^z\{Z_t \in U \text{ for some } t \geq t_*\} = 1$; and (b) for any $z, z' \in \overline{\mathbb{C}}$,

$P^z\{Z_t = z' \text{ for some } t > 0\} = 0$. These statements follow from the corresponding facts for planar Brownian motion ([5], Section 1.7).

Now consider the time change (3.1) with $f = Q$. Since Q is a rational function, there are only finitely many $z \in \overline{\mathbb{C}}$ where $|\delta Q(z)| = 0$. Also, $|\delta Q|$ is bounded since $\overline{\mathbb{C}}$ is compact. Since Brownian motion does not hit individual points, it follows that $\int_0^r |\delta Q(Z_s)|\, ds$ is strictly increasing in r and converges to ∞ as $r \to \infty$ a.s. (P^z). Thus, with P^z probability 1, $t \to \tau(t)$ is a homeomorphism of $[0, \infty)$. This proves:

PROPOSITION 9. *Q induces a measure-preserving transformation on the space of Brownian paths started at ∞, given by $Z_t \to Q(Z_{\tau(t)})$.*

In other words, if Ω_∞ is the set of continuous $\overline{\mathbb{C}}$-valued paths started at ∞, $\mathscr{G}$ the Borel σ-algebra on Ω_∞ and P^∞ the Wiener measure on $(\Omega_\infty, \mathscr{G})$, then the induced transformation Q: $(\Omega_\infty, \mathscr{G}) \to (\Omega_\infty, \mathscr{G})$ is measure-preserving.

Define a stopping time T by

$$T = \inf\{t \geq 0: Z_t \in \mathscr{J}\}.$$

On $\{T = \infty\}$ the path Z_t avoids $\mathscr{J}$ forever; on $\{T < \infty\}$ it enters $\mathscr{J}$ in finite time. Since $t \to \tau(t)$ is a homeomorphism of $[0, \infty)$ (with P^∞ probability 1) the events $\{T < \infty\}$ and $\{\tau^{-1}(T) < \infty\}$ coincide (a.s. (P^∞)) and

$$\tau^{-1}(T) = \inf\{t: Q(Z_{\tau(t)}) \in \mathscr{J}\},$$

because $\mathscr{F}$ and $\mathscr{J}$ are Q-invariant sets. Therefore, the distributions of Z_T and $Q(Z_T)$ are the same under P^∞ (we have not yet shown that $P^\infty\{T < \infty\} = 1$, so these distributions may be defective). Thus:

COROLLARY 1. *If $P^\infty\{T < \infty\} = 1$, then ν is a Q-invariant probability measure on $\mathscr{J}$.*

PROPOSITION 10. *If ∞ is an attracting or superattracting fixed point of Q, then $P^\infty\{T < \infty\} = 1$.*

The proof will use the existence of a local conjugacy with a monomial (Proposition 8), the recurrence of spherical Brownian motion and the following simple first-passage probability.

LEMMA 1. *Let Z_t be a Brownian motion in $\mathbb{R}^2$ started at $Z_0 = z$ under P^z, where $|z| = r > 0$. Let $\tau_R = \inf\{t: |Z_t| = R\}$. If $R_1 \leq r \leq R_2$, then*

$$P^z\{\tau_{R_2} < \tau_{R_1}\} = \frac{\log(r/R_1)}{\log(R_2/R_1)}.$$

See [4], Section 2, or [5], Section 1.7, for the proof.

PROOF OF PROPOSITION 10. Suppose first that $\mathscr{F} \neq \mathscr{F}_\infty$, that is, that $\mathscr{F}$ is not connected. Since $\mathscr{F}$ is open, there exists a nonempty open set $U \subset \mathscr{F} \setminus \mathscr{F}_\infty$.

Let $\tau_U = \inf\{t\colon Z_t \in U\}$. By the recurrence of spherical Brownian motion, $P^\infty\{\tau_U < \infty\} = 1$. The path Z_t, $0 \le t \le \tau_U$, is continuous, starts in $\mathscr{F}_\infty$, and ends in U, so it must pass through $\partial\mathscr{F}_\infty \subset \mathscr{J}$. Consequently $T \le \tau_U$ and so $P^\infty\{T < \infty\} = 1$.

Next, assume that ∞ *is a superattracting fixed point of* Q. By Proposition 8, there is a neighborhood U of ∞ in $\overline{\mathbb{C}}$ and a conformal homeomorphism $\varphi\colon \{|z| > r\} \to U$ such that $\varphi(\infty) = \infty$ and

$$Q(\varphi(z)) = \varphi(\alpha z^{d-d_*}) \qquad \forall\ |z| > r,$$

where α is a nonzero constant. Choose $R_1 < R_0 < R_{-1} < R_{-2} < \cdots$ satisfying $R_{i-1} = |\alpha| R_i^{d-d_*}$ for $i \le 1$ and $R_1 > r$; define

$$\begin{aligned} C_i &= \{z\colon |z| = R_i\}, \qquad i \le 1;\\ \Gamma_0 &= \varphi(C_0);\\ \Gamma_n &= Q^{-n}(\Gamma_0) \qquad \forall\ n \in \mathbb{Z}. \end{aligned}$$

Observe that $\Gamma_{-n} = \varphi(C_{-n})\ \forall\ n \ge 0$ and $\varphi(C_1) = \Gamma_1 \cap U$, but in general $\varphi(C_1) \ne \Gamma_1$. By Proposition 7, any continuous path $\gamma(t)$, $0 \le t \le t_*$, which starts at $\gamma(0) = \infty$ and intersects each Γ_n must intersect $\mathscr{J}$. Our objective will be to show that with probability 1, a Brownian path started at ∞ will hit all of the sets Γ_n, $n \ge 0$, in a finite time interval.

Let $\gamma(t)$, $0 \le t \le t_*$, be a continuous path with $\gamma(0) \in \Gamma_{n+1}$ and $\gamma(t_*) \in \Gamma_{n-k}$ for some $k \ge 1$. We will argue that γ must hit Γ_n. If $n \le 0$, this is because $\Gamma_m = \varphi(C_m)\ \forall\ m \le 0$, φ is a homeomorphism and C_m are concentric circles. If $n > 0$, then $Q^n(\gamma(t))$ is a continuous path from Γ_1 to Γ_{-k}. Since $\Gamma_1 \cap U = \varphi(C_1)$, the sets Γ_1 and Γ_{-k} are separated by $\Gamma_0 = \varphi(C_0)$; hence $Q^n(\gamma(t))$ must hit Γ_0, and so $\gamma(t)$ must hit Γ_n. Thus, for a Brownian path that reaches Γ_n, $n \ge 1$, to return to Γ_{-1}, it must hit Γ_{n-1}, then $\Gamma_{n-2}, \dots,$ then Γ_0 and finally Γ_{-1}.

Now let Z_t be a Brownian motion started at $z \in \overline{\mathbb{C}}$ under the probability measure P^z. Fix $z \in \Gamma_n$, $n \ge 0$; let $\xi = Q^n(z)$ and $\zeta = \varphi^{-1}(\xi)$; then by the conformal invariance of Brownian motion (since Q^n and φ^{-1} are analytic),

$$\begin{aligned} &P^z\{Z_t \text{ hits } \Gamma_{n+1} \text{ before } \Gamma_{n-1}\}\\ &\qquad \ge P^z\{Z_t \text{ hits } \Gamma_{n+1} \text{ before } Q^{-n}(\Gamma_{-1})\}\\ &\qquad = P^\xi\{Z_t \text{ hits } \Gamma_1 \text{ before } \Gamma_{-1}\}\\ &\qquad = P^\zeta\{Z_t \text{ hits } C_1 \text{ before } C_{-1}\}\\ &\qquad = \frac{\log(R_1/R_0)}{\log(R_2/R_0)} = \frac{d-d_*}{d-d_*+1} \ge \frac{2}{3}. \end{aligned}$$

[That $(d-d_*)/(d-d_*+1) \ge 2/3$ follows from the fact that ∞ is superattracting. In the attracting case, $(d-d_*)/(d-d_*+1) = 1/2$ and so the proof breaks down.]

Consider Brownian motion started at ∞. Since Γ_0 bounds two nonempty open disks in $\overline{\mathbb{C}}$, the recurrence of Brownian motion implies that it will reach

Γ_0 in finite time. The same argument shows that it will then return to Γ_{-1} in finite time. But there is positive probability that, after reaching Γ_0 for the first time, the path will visit *all* of the sets Γ_n, $n \geq 1$, before returning to Γ_{-1}. This is because upon reaching any Γ_n the path has chance at least $2/3$ of moving up to Γ_{n+1} before returning to Γ_{n-1}. For a formal proof, let X_m, $m \geq 0$, be the indices of successive sets Γ_n visited by the path after the first visit to Γ_0. Then 2^{-X_m}, $m \geq 0$ is a supermartingale with $2^{-X_0} = 2^0 = 1$, so by the maximal inequality for positive supermartingales,

$$\begin{aligned} P^{\infty}\{Z_t \text{ returns to } \Gamma_{-1} \text{ before visiting all } \Gamma_n, n \geq 0\} \\ = P^{\infty}\{2^{-X_m} = 2 \text{ for some } m \geq 0\} \\ \leq 1/2. \end{aligned}$$

But if Z_t visits all of the sets Γ_n, $n \geq 0$, before returning to Γ_{-1}, then it must do so in a finite time interval, because Z_t will return to Γ_{-1} in finite time. This proves that $P^{\infty}\{T < \infty\} > 0$. Now for any compact set $K \subset \overline{\mathbb{C}}$, it is either the case that $P^z\{\tau_K < \infty\} = 1 \ \forall\, z \in \overline{\mathbb{C}}$ or $P^z\{\tau_K < \infty\} = 0 \ \forall\, z \in \overline{\mathbb{C}} \setminus K$ ([12], Section 2.2), where $\tau_K = \inf\{t\colon Z_t \in K\}$. Therefore,

$$P^{\infty}\{T < \infty\} = 1.$$

Finally, assume that ∞ *is an attracting fixed point of* Q *and* $\mathscr{F} = \mathscr{F}_{\infty}$. By Proposition 8, there is a neighborhood U of ∞ in $\overline{\mathbb{C}}$ and a conformal homeomorphism $\varphi\colon \{|z| > r\} \to U$ such that $\varphi(\infty) = \infty$ and for some α, $|\alpha| > 1$,

$$Q(\varphi(z)) = \varphi(\alpha z) \qquad \forall\, |z| > r.$$

Choose $R > r$ and define

$$\begin{aligned} C_{-n} &= \{z\colon |z| = |\alpha|^{n+1}R\}, \qquad n \geq -1; \\ \Gamma_0 &= \varphi(C_0); \\ \Gamma_n &= Q^{-n}(\Gamma_0) \qquad \forall\, n \in \mathbb{Z}. \end{aligned}$$

As in the superattracting case, $\Gamma_{-n} = \varphi(C_{-n})\ \forall\ n \geq 0$ and $\varphi(C_1) = \Gamma_1 \cap U$. Also, $\varphi(C_1) \neq \Gamma_1$, because $z \to \alpha z$ is a 1-to-1 mapping of C_1 onto C_0 but $z \to Q(z)$ is a d-to-1 mapping of Γ_1 onto Γ_0 and we have assumed that $d \geq 2$. By Proposition 7 applied to Q^k for any $k \geq 1$, any continuous path $\gamma(t)$, $0 \leq t \leq t_*$, which starts at $\gamma(0) = \infty$ and intersects each Γ_{nk}, $n \geq 1$, must intersect $\mathscr{J}$.

We claim that there is an integer $k \geq 1$ and a constant $p > 1/2$ such that

$$P^z\{Z_t \text{ hits } \Gamma_{2k} \text{ before } \Gamma_0\} \geq p \qquad \forall\, z \in \Gamma_k.$$

Here is the proof. The function Q maps Γ_0 onto Γ_{-1} bijectively, but maps $Q^{-1}(\Gamma_{-1})$ d-to-1 onto Γ_1; hence $Q^{-1}(\Gamma_{-1}) \setminus \Gamma_0$ contains a closed curve Δ. Since $Q^{-1}(\Gamma_{-1}) \subset Q^{-k}(\Gamma_{-k})$ for all $k \geq 1$, $\Delta \subset Q^{-k}(\Gamma_{-k})$. Since $\mathscr{F} = \mathscr{F}_{\infty}$, there is a path β in $\mathscr{F}_{\infty}$ from Γ_0 to Δ. Since the sets Γ_k accumulate at $\mathscr{J}$ as $k \to \infty$, for all k sufficiently large, Γ_k will not intersect β. Now from any point $z \in \Gamma_k$, there is a continuous path from z to Γ_0 that does not intersect Γ_{2k}; consequently, for each $z \in \Gamma_k$, there is a continuous path from z to Δ that does not intersect $\Gamma_0 \cup \Gamma_{2k}$ (just follow a path from z *almost* to Γ_0, then move to β without hitting Γ_0 or Γ_{2k}, then follow β to Δ). It follows by routine arguments that

$$\begin{aligned} P^z\{Z_t \text{ hits } \Delta \text{ before } \Gamma_0 \cup \Gamma_{2k}\} &> 0 \qquad \forall\, z \in \Gamma_k;\\ P^z\{Z_t \text{ hits } \Gamma_k \text{ before } \Gamma_0\} &> 0 \qquad \forall\, z \in \Delta;\\ \Rightarrow P^z\{Z_t \text{ hits } \Gamma_{2k} \text{ before } \Gamma_0\} &> 0 \qquad \forall\, z \in \Delta. \end{aligned}$$

The conformal invariance of Brownian motion implies that for any $z \in \Gamma_k$, $\xi = Q^{2k}(z)$, $\zeta = \varphi^{-1}(\xi)$,

$$\begin{aligned} &P^z\{Z_t \text{ hits } \Gamma_{2k} \text{ before } Q^{-2k}(Q^{2k}(\Gamma_0))\}\\ &\qquad = P^{\xi}\{Z_t \text{ hits } \Gamma_0 \text{ before } \Gamma_{-2k}\}\\ &\qquad = P^{\zeta}\{Z_t \text{ hits } C_0 \text{ before } C_{-2k}\}\\ &\qquad = 1/2. \end{aligned}$$

Hence,

$$\begin{aligned} 1/2 &= P^z\{Z_t \text{ hits } Q^{-2k}(Q^{2k}(\Gamma_0)) \text{ before } \Gamma_{2k}\}\\ &\geq P^z\{Z_t \text{ hits } \Gamma_0 \text{ before } \Gamma_{2k}\} + P^z\{Z_t \text{ hits } \Delta \text{ before } \Gamma_{2k} \text{ and } \Gamma_{2k} \text{ before } \Gamma_0\}\\ &> P^z\{Z_t \text{ hits } \Gamma_0 \text{ before } \Gamma_{2k}\}. \end{aligned}$$

Since $P^z\{Z_t \text{ hits } \Gamma_0 \text{ before } \Gamma_{2k}\}$ is continuous in z (in fact, it is harmonic) and since Γ_k is compact, this proves the claim.

It now follows that for any $n \geq 1$, $z \in \Gamma_{nk}$, $\xi = Q^{(n-1)k}(z)$,

$$\begin{aligned} &P^z\{Z_t \text{ hits } \Gamma_{(n+1)k} \text{ before } \Gamma_{(n-1)k}\}\\ &\qquad \geq P^{\xi}\{Z_t \text{ hits } \Gamma_{2k} \text{ before } \Gamma_0\} \geq p > 1/2. \end{aligned}$$

The same argument that was used in the superattracting case now shows that $P^{\infty}\{T < \infty\} > 0$ and therefore

$$P^{\infty}\{T < \infty\} = 1. \qquad \square$$

COROLLARY 2. *If ∞ is an attracting or superattracting fixed point of Q, then*

$$P^z\{T < \infty\} = 1 \qquad \forall\, z \in \overline{\mathbb{C}}.$$

PROOF. If K is any compact subset of $\mathbb{C}$ and $\tau_K = \inf\{t\colon Z_t \in K\}$, then either $P^z\{\tau_K < \infty\} = 1\ \forall\, z \in \mathbb{C}$ or $P^z\{\tau_K < \infty\} = 0\ \forall\, z \notin K$ ([12], Chapter 2, Proposition 2.10). Since Brownian motion started at ∞ cannot reach $\mathcal{J}$ without going through some intermediate points of $\mathcal{F}$ and since $P^{\infty}\{T < \infty\} = 1$, it follows that $P^z\{T < \infty\} = 1$ for some, and therefore all, $z \in \mathbb{C}$. $\square$

PROPOSITION 11. *If ∞ is superattracting or attracting, then the measure-preserving system $(\mathcal{J}, Q, \nu)$ is strongly mixing.*

REMARK. If Q is a polynomial or if $\mathscr{J}$ is totally disconnected then $(\mathscr{J}, Q, \nu)$ is Bernoulli, which is considerably stronger than strong mixing. See Theorem 3 and Section 1, Remark 4.

PROOF. Let Z_t be a Brownian motion started at ∞ under P^∞. Define processes $Z_t^{(n)}$, $n \geq 0$, by

$$Z_t^{(n)} = Q^n\big(Z_{\tau_n(t)}\big), \qquad t \geq 0,$$

where $\tau_n(t)$ is given by (3.1) with $f = Q^n$. Let

$$T_n = \inf\{t\colon Z_t^{(n)} \in \mathscr{J}\}.$$

To prove the proposition it suffices to show that for all continuous functions $f, g\colon \mathscr{J} \to \mathbb{R}$,

$$\lim_{n\to\infty} E^\infty f\big(Z_{T_n}^{(n)}\big) g\big(Z_{T_0}^{(0)}\big) = E^\infty f(Z_T) E^\infty g(Z_T). \tag{4.1}$$

We may assume that $f, g\colon \overline{\mathbb{C}} \to \mathbb{R}$ are continuous on all of $\overline{\mathbb{C}}$.

Let Γ be a simple, closed curve in $\mathbb{C}$ that completely encloses $\mathscr{J}$ and define $\Gamma_n = Q^{-n}(\Gamma)$. If $\sigma_m = \inf\{t\colon Z_t \in \Gamma_m\}$, then $\lim_{m\to\infty} \sigma_m = T$ (see Proposition 6). Consequently, $\lim_{m\to\infty} g(Z_{\sigma_m}) = g(Z_T)$; since f and g are bounded on $\overline{\mathbb{C}}$, it follows that in order to prove (4.1) it suffices to show that for each $m \geq 1$,

$$\lim_{n\to\infty} E^\infty f\big(Z_{T_n}^{(n)}\big) g\big(Z_{\sigma_m}\big) = E^\infty f(Z_T) E^\infty g\big(Z_{\sigma_m}\big).$$

Now $T \geq \sigma_m$, so by the strong Markov property,

$$E^\infty\Big(f\big(Z_{T_n}^{(n)}\big) | Z_{\sigma_m} = z\Big) = E^\infty\Big(f(Q^n(Z_T)) | Z_{\sigma_m} = z\Big) = E^z f(Q^n(Z_T))$$

for all $z \in \Gamma_m$. By the conformal invariance of Brownian motion,

$$E^z\big(f(Q^n(Z_T))\big) = E^{Q^n(z)} f(Z_T).$$

But as $n \to \infty$, $Q^n(z) \to \infty$ uniformly for $z \in \Gamma_m$. Since $E^\xi f(Z_T)$ is a continuous function of $\xi \in \overline{\mathbb{C}}$, it follows that

$$\lim_{n\to\infty} E^\infty\Big(f\big(Z_{T_n}^{(n)}\big) | Z_{\sigma_m}\Big) = E^\infty f(Z_T).$$

The functions f and g are bounded, so by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{n\to\infty} E^\infty f\big(Z_{T_n}^{(n)}\big) g\big(Z_{\sigma_m}\big) \\
&\quad = \lim_{n\to\infty} E^\infty\Big(E^\infty\Big(f\big(Z_{T_n}^{(n)}\big) | Z_{\sigma_m}\Big)\Big) g\big(Z_{\sigma_m}\big) \\
&\quad = E^\infty\big(E^\infty f(Z_T)\big) g\big(Z_{\sigma_m}\big) \\
&\quad = E^\infty f(Z_T) E^\infty g\big(Z_{\sigma_m}\big).
\end{aligned}$$

□

NOTE. A similar argument shows that the stationary sequence $(Z_t^{(n)})_{n\geq 0}$ of random paths is strongly mixing.

In view of [12], Chapter 3, Theorem 4.12, Propositions 10 and 11 imply all of Theorem 1 except for the case where ∞ is a neutral fixed point. This case will be taken up in Section 9.

5. Polynomial mappings and Brolin's theorem. In this section we assume that $Q(z)$ is a polynomial of degree $d \geq 2$. Thus, ∞ is a superattracting fixed point, so Brownian motion started at ∞ reaches $\mathscr{J}$ in finite time and the hitting distribution ν is an invariant probability distribution (Propositions 8–9).

The property that distinguishes polynomials Q among the rational mappings that fix ∞ is that $z = \infty$ is the *only* solution of $Q(z) = \infty$. Therefore, all d branches Q_i^{-1} of Q^{-1} satisfy $Q_i^{-1}(\infty) = \infty$. Define $F_i = Q_i^{-1} \circ Q$ (for some indexing of the branches Q_i^{-1}); then each F_i is single-valued and analytic in a neighborhood of ∞. (In fact, by Proposition 2, the functions $F_1, F_2, \dots, F_d$ are analytically conjugate to the d rotations through angles $2\pi j/d$, $j = 0, 1, \dots, d$, in some neighborhood of ∞.) Moreover, each F_i has an analytic continuation along every path in $\mathscr{F}_\infty \setminus \{z \in \overline{\mathbb{C}}\colon Q'(z) = 0\}$ (but F_i may be multivalued).

Let $(Z_t)_{0 \leq t \leq T}$ be a Brownian motion started at ∞ and terminated at $\mathscr{J}$. Define the *trace* Z of the Brownian motion $(Z_t)_{0 \leq t \leq T}$ to be the equivalence class of all continuous paths that can be obtained from $(Z_t)_{0 \leq t \leq T}$ by a reparametrization of time. Observe that each of $F_i(Z)$, $i = 1, 2, \dots, d$, is a Brownian trace, as is $Q(Z)$, by conformal invariance. [Note: The original parametrization $(Z_t)_{0 \leq t \leq T}$ can be recovered from Z by a standard formula for the quadratic variation of a Brownian path.]

PROPOSITION 12. *Given the trace $Q(Z)$, the conditional distribution of Z is the uniform distribution on $F_1(Z), F_2(Z), \dots, F_d(Z)$.*

PROOF. Generate a trace $\tilde{Z}$ by choosing one of $F_1(Z), \dots, F_d(Z)$ at random. Since each of $F_i(Z)$ is a Brownian trace, so is $\tilde{Z}$; thus Z has the same distribution as $\tilde{Z}$. Furthermore, $Q(Z) = Q(\tilde{Z})$, since $F_i = Q_i^{-1} \circ Q$. Therefore, the joint distribution of $(Z, Q(Z))$ is the same as that of $(\tilde{Z}, Q(Z))$. But the conditional distribution of $\tilde{Z}$ given $Q(Z)$ is clearly uniform on $F_1(Z), \dots, F_d(Z)$, hence uniform on $F_1(\tilde{Z}), \dots, F_d(\tilde{Z})$. □

COROLLARY 3. *Given $Q(Z_T)$, the conditional distribution of Z_T is the uniform distribution on the d preimages of $Q(Z_T)$.*

This is an immediate consequence of Proposition 12, since the σ-algebra generated by $Q(Z_T)$ is contained in the σ-algebra generated by $Q(Z)$.

There is an easy, direct proof that $\mu_n^z \to \nu$ for each $z \in \mathscr{F}_\infty \setminus \{\infty\}$ based on Proposition 12. (Recall that Brolin's theorem states that $\mu_n^z \to \nu$ for all but at most one $z \in \mathbb{C}$.) Here is a sketch.

First, consider $z \in \mathscr{F}_\infty$, $z \neq \infty$, such that z is not a branch point of any Q^{-n}, $n \geq 1$. Let Γ be a simple closed curve in $\mathscr{F}_\infty$ which separates ∞ from $\mathscr{J}$, such that $z \in \Gamma$ and such that no point of Γ is a branch point of any Q^{-n}. Such a curve Γ exists because the branch points can only accumulate at ∞ in $\mathscr{F}_\infty$ (Section 2). Let Q_i^{-n}, $i = 1, \ldots, d^n$, be the distinct branches of Q^{-n} in a neighborhood of Γ; then $Q^{-n}(\Gamma) = \bigcup_{i=1}^{d^n} Q_i^{-n}(\Gamma)$. Observe that each $Q_i^{-n}(\Gamma)$ contains *exactly* one point of $Q^{-n}(z)$, so

$$(5.1) \qquad \mu_n^z(Q_i^{-n}(\Gamma)) = 1/d^n \qquad \forall\, i = 1, 2, \ldots, d^n.$$

The curve Γ may be covered by two simply connected neighborhoods contained in $\mathscr{F}_\infty$, neither containing branch points of any Q^{-n}. By Proposition 3, the collection of all branches of all Q^{-n}, $n \geq 1$, is a normal family in each of the two neighborhoods. Consequently, by Proposition 5,

$$(5.2) \qquad \lim_{n \to \infty} \max_{\xi \in Q^{-n}(\Gamma)} \operatorname{distance}(\xi, \mathscr{J}) = 0,$$

$$(5.3) \qquad \lim_{n \to \infty} \max_{1 \leq i \leq d^n} \operatorname{diameter}(Q_i^{-n}(\Gamma)) = 0.$$

Now consider Brownian motion $(Z_t)_{0 \leq t \leq T}$ started at ∞ and terminated at $\mathscr{J}$. By Proposition 7, the path Z_t must intersect each $Q^{-n}(\Gamma)$ before reaching $\mathscr{J}$. Let $\sigma_n = \inf\{t\colon Z_t \in Q^{-n}(\Gamma)\} < T$; since Z_t is continuous, (5.2) implies that $\sigma_n \to T$ and $Z_{\sigma_n} \to Z_T$ a.s. as $n \to \infty$. It follows that the distribution of Z_{σ_n} converges weakly to ν as $n \to \infty$. Now Proposition 12 implies that

$$(5.4) \qquad P^\infty\{Z_{\sigma_n} \in Q_i^{-n}(\Gamma)\} = 1/d^n \qquad \forall\, i = 1, 2, \ldots, d^n,$$

because for each i, exactly one of the d^n paths mapped into $Q^n(Z)$ by Q^n first hits $Q^{-n}(\Gamma)$ in $Q_i^{-n}(\Gamma)$. But (5.1) and (5.4) together with (5.3) imply that for large n, μ_n^z and the distribution of Z_{σ_n} are close in the weak topology. Therefore,

$$(5.5) \qquad \operatorname{weak} \lim_{n \to \infty} \mu_n^z = \nu.$$

Next, consider $z \in \mathscr{F}_\infty \setminus \{\infty\}$ such that z is a branch point of some Q^{-n}, $n \geq 1$. Recall (Proposition 3) that if Q is a polynomial, then $Q^{-1}(\mathscr{F}_\infty) = \mathscr{F}_\infty$; hence, for each $m \geq 1$, $Q^{-m}(z) \subset \mathscr{F}_\infty$. For large $m \geq 1$, all of the points of $Q^{-m}(z)$ must be near $\mathscr{J}$, by Proposition 2, so if m is sufficiently large, $Q^{-m}(z)$ contains no branch points of any Q^{-n}, $n \geq 1$, because the branch points can only accumulate at ∞ in $\mathscr{F}_\infty$. Consequently, for each $\xi \in Q^{-m}(z)$, $\lim_{n \to \infty} \mu_n^\xi = \nu$ by (5.5). But μ_{n+m}^z is a weighted average of μ_n^ξ, $\xi \in Q^{-m}(z)$. Therefore, $\lim_{n \to \infty} \mu_n^z = \nu$.

With just a little more work, one can show that for any nonexceptional $z_1, z_2 \in \mathbb{C}$ [an exceptional point being a d-fold root of $Q(z) = z$] the measures $\mu_n^{z_1}$ and $\mu_n^{z_2}$ becomes close in the weak topology as $n \to \infty$. Since this argument is carried out in [8], Section 4, we shall omit it. As there is at most one exceptional point of Q other than ∞, this proves Brolin's theorem.

6. Entropy of the equilibrium distribution. In this section we assume that $Q(z) = P_1(z)/P_2(z)$, where P_1 and P_2 are polynomials of degrees d and d_*, $d - d_* \geq 1$, and that ∞ is a superattracting or attracting fixed point.

Brownian motion started at ∞ reaches $\mathcal{J}$ in finite time and the hitting distribution ν (i.e., the equilibrium distribution) is an ergodic, invariant measure, by the results of Section 4.

Since ∞ is a $(d - d_*)$-fold root of $Q(z) = z$, there are $(d - d_*)$ distinct branches Q_i^{-1} of Q^{-1} that fix ∞. Define $F_i = Q_i^{-1} \circ Q$, $i = 1, 2, \dots, (d - d_*)$; each F_i is single-valued and analytic in a neighborhood of ∞, and $F_i(\infty) = \infty$. Also, each F_i has an analytic continuation along each path in $\mathcal{F}_\infty \setminus \{z \in \overline{\mathbb{C}}\colon Q'(z) = 0 \text{ or } Q(z) = \infty\}$.

Let $(Z_t)_{0 \le t \le T}$ be a Brownian motion started at ∞ and terminated at $\mathcal{J}$. Define the Brownian trace Z as in Section 5, and observe that each of $F_i(Z)$, $i = 1, 2, \dots, (d - d_*)$, is a Brownian trace.

PROPOSITION 13. *Given the trace $Q(Z)$, the conditional distribution of Z is the uniform distribution on $F_1(Z), F_2(Z), \dots, F_{d-d_*}(Z)$.*

PROOF. Same as for Proposition 12. □

Let $h(Q)$ be the entropy of the measure-preserving system $(\mathcal{J}, Q, \nu)$.

PROPOSITION 14. $h(Q) \ge \log(d - d_*)$.

PROOF. It suffices to prove that for any $\varepsilon > 0$, there exists a finite Borel partition $\mathcal{P}$ of $\mathcal{J}$ such that $h(\mathcal{P}, Q) \ge (1 - \varepsilon)\log(d - d_*)$. We will show that this inequality holds for any partition $\mathcal{P}$ of sufficiently small diameter. (Note: The notation for entropy is as in [11], Chapter 5.)

Choose $\varepsilon > 0$ small. There exists $\delta > 0$ so small that if $\operatorname{diam}(\mathcal{P}) < \delta$, then

$$\nu\{z\colon \text{cardinality}(Q^{-1}(z) \cap G) \ge 2, \text{ some } G \in \mathcal{P}\} < \epsilon \tag{6.1}$$

[multiple roots ξ of $Q(\xi) = z$ are counted according to multiplicity]. This follows from the fact that, with probability 1, Z_T has d distinct preimages under Q^{-1}.

According to a standard result ([11], Chapter 5, Proposition 2.12), $h(\mathcal{P}, Q) = H(\mathcal{P} | \bigvee_{n=1}^\infty Q^{-n}(\mathcal{P}))$. Now, conditioning on $\bigvee_{n=1}^\infty Q^{-n}(\mathcal{P})$ is the same as conditioning on the sequence of sets G_i in $\mathcal{P}$ containing $Q(Z_T), Q^2(Z_T), \dots$. Clearly, $Q(Z_T)$ determines this sequence, so the σ-algebra $\mathcal{G}$ generated by the Brownian trace $Q(Z)$ contains $\bigvee_{n=1}^\infty Q^{-n}(\mathcal{P})$. It follows ([11], Chapter 5, proposition 2.5 (2)) that

$$h(\mathcal{P}, Q) = H\left(\mathcal{P} \,\middle|\, \bigvee_{n=1}^{\infty} Q^{-n}(\mathcal{P})\right) \ge H(\mathcal{P} | \mathcal{G}).$$

By the result of the preceding paragraph, the probability that $Q(Z_T)$ has more than one preimage (under Q^{-1}) in any set of $\mathcal{P}$ is less than ε. Moreover, by Proposition 13, given $\mathcal{G}$ the conditional distribution of Z_T is the uniform distribution on $d - d_*$ of the d points in $Q^{-1}(Q(Z_T))$. Therefore,

$$H(\mathscr{P}|\mathscr{G}) = E \sum_{G \in \mathscr{P}} 1\{Z_T \in G\} \log P(Z_T \in G|\mathscr{G})^{-1}$$
$$\geq (1 - \varepsilon)\log(d - d_*). \qquad \square$$

A variant of this argument will be used to prove Theorem 2(b) in Section 8.

7. Rational mappings and Lopes' theorem. Let $Q(z) = P_1(z)/P_2(z)$, where P_1, P_2 have no nontrivial common factors, and assume that $Q(\infty) = \infty \notin \mathscr{J}$. Lopes' theorem states that if $\nu = \mu$, where μ is the maximum entropy invariant measure of Q, then Q is a polynomial. In this section we shall present a proof of Lopes' theorem under the additional hypothesis that ∞ is an attracting or superattracting fixed point of Q. The (less interesting) case in which ∞ is a neutral fixed point will be treated separately in Section 9 [by showing that $(\mathscr{J}; Q, \nu)$ is ergodic and has entropy zero].

The main step in the proof will be to show that if $\nu = \mu$, then $|P_2|$ is constant on $\mathscr{J}$. (Lopes [9] also does this, but his proof involves some laborious calculations.) Our argument will be based on a simple fact about harmonic measure which may be of some interest in its own right. Let K be a compact subset of $\overline{\mathbb{C}}$ such that $\infty \notin K$ and such that K has positive capacity, that is, Brownian motion started at ∞ will hit K with probability 1. Then Brownian motion started at any point of $\overline{\mathbb{C}}$ will hit K with probability 1. Let $\tau = \inf\{t: Z_t \in K\}$; for $\xi \in \mathbb{C}$, define $\nu_\xi(dz) = P^\xi\{Z_\tau \in dz\}$ (under P^ξ, Z_t is a Brownian motion started at ξ). Define $\nu = \nu_\infty$. Note: ν_ξ is the harmonic measure on K as seen from ξ.

PROPOSITION 15. *Let $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$ (the same point may be listed more than once). A necessary and sufficient condition for $\nu = n^{-1}\sum_{i=1}^n \nu_{\xi_i}$ is that $|\prod_{i=1}^n (z - \xi_i)|$ be a.e. constant for $z \in K$, relative to $\nu + \sum \nu_{\xi_i}$.*

PROOF. First we will show that if $K \subset L_a = \{z \in \mathbb{C}: |R(z)| = a\}$, where $R(z) = \prod_{i=1}^n (z - \xi_i)$ and $0 < a < \infty$, then $\nu = n^{-1}\sum_{i=1}^n \nu_{\xi_i}$. Let $(Z_t)_{0 \leq t \leq \sigma}$ be Brownian motion started at ∞ and stopped at $\sigma = \inf\{t: |Z_t| = a\}$, under P^∞. Define

$$\overline{Z}_t = a^2 Z_t / |Z_t|^2, \qquad t > 0,$$

that is, $\overline{Z}_t$ is the reflection of Z_t in the circle of radius a centered at 0. Since reflection in a circle is a conformal map (orientation-reversing), $\overline{Z}_t$ is a time-changed Brownian motion started at 0. With probability 1, neither $(Z_t)_{0 < t \leq \sigma}$ nor $(\overline{Z}_t)_{0 < t \leq \sigma}$ hits a branch point of R^{-1}.

The polynomial $R(z)$ has an inverse function R^{-1} with n distinct branches $R_1^{-1}, \dots, R_n^{-1}$ defined in a neighborhood of ∞. Choose one of the n paths $R_1^{-1}(Z_t), \dots, R_n^{-1}(Z_t)$, $t \geq 0$, at random (according to the uniform distribution on $\{1, 2, \dots, n\}$) and call it Y_t; if $Y_t = R_i^{-1}(Z_t)$, define $\overline{Y}_t = R_i^{-1}(\overline{Z}_t)$. Then $(Y_t)_{0 \leq t \leq \sigma}$ is a time-changed Brownian motion started at ∞ and stopped upon reaching L_a. This follows from virtually the same argument as that used in proving Proposition 12. Similarly, $(\overline{Y}_t)_{0 \le t \le \sigma}$ is a time-changed Brownian motion started at $\overline{Y}_0$ and stopped upon reaching L_a, where $P^\infty\{\overline{Y}_0 = \xi_i\} = n^{-1}$ for each $i = 1, 2, \dots, n$ (with multiple points ξ_i counted accordingly). By construction, $Y_\sigma = \overline{Y}_\sigma$; consequently, $\nu = n^{-1}\sum_{i=1}^n \nu_{\xi_i}$.

Next we will show that if $\nu = n^{-1}\sum_{i=1}^n \nu_{\xi_i}$, then $|R(z)|$ is constant a.s. on K w.r.t. ν. Suppose not; then there exists $a > 0$ such that

$$K_+ = K \cap \{z: |R(z)| > a\}$$

and

$$K_- = K \cap \{z: |R(z)| \le a\},$$

both have positive ν-measure. Let Y_t, $\overline{Y}_t$, σ, P^∞ be as in the previous paragraph, and define

$$\alpha = \inf\{t: Y_t \in K\},$$
$$\bar{\alpha} = \inf\{t: \overline{Y}_t \in K\}.$$

Observe that Y_α has distribution ν and $\overline{Y}_{\bar{\alpha}}$ has distribution $n^{-1}\sum_{i=1}^n \nu_{\xi_i}$. We will show that

$$P^\infty\{Y_\alpha \in K_+\} > P^\infty\{\overline{Y}_{\bar{\alpha}} \in K_+\},$$

contradicting the assumption $\nu = n^{-1}\sum_{i=1}^n \nu_{\xi_i}$.

Note first that

$$\alpha < \sigma \Rightarrow Y_\alpha \in K_+,$$
$$\bar{\alpha} < \sigma \Rightarrow \overline{Y}_{\bar{\alpha}} \in K_-.$$

Hence

$$\begin{aligned} P^\infty\{Y_\alpha \in K_+\} &= P^\infty\{\sigma > \alpha \vee \bar{\alpha}\} + P^\infty\{\bar{\alpha} \ge \sigma > \alpha\} \\ &\quad + P^\infty\{Y_\alpha \in K_+; \sigma \le \alpha \wedge \bar{\alpha}\} \\ &\quad + P^\infty\{Y_\alpha \in K_+; \bar{\alpha} < \sigma \le \alpha\} \end{aligned}$$

and

$$P^\infty\{\overline{Y}_{\bar{\alpha}} \in K_+\} = P^\infty\{\overline{Y}_{\bar{\alpha}} \in K_+; \bar{\alpha} \ge \sigma > \alpha\} + P^\infty\{\overline{Y}_{\bar{\alpha}} \in K_+; \sigma \le \alpha \wedge \bar{\alpha}\}.$$

On the event $\{\sigma \le \alpha \wedge \bar{\alpha}\}$ neither Y nor $\overline{Y}$ hits K before time σ. But $Y_\sigma = \overline{Y}_\sigma$, so beginning at time σ each of Y, $\overline{Y}$ is a Brownian motion started at the same point $T_\sigma = \overline{Y}_\sigma$ and hence by the strong Markov property,

$$P^\infty\{Y_\alpha \in K_+; \sigma \le \alpha \wedge \bar{\alpha}\} = P^\infty\{\overline{Y}_{\bar{\alpha}} \in K_+; \sigma \le \alpha \wedge \bar{\alpha}\}.$$

Consequently,

$$\begin{aligned} &P^\infty\{Y_\alpha \in K_+\} - P^\infty\{\overline{Y}_{\bar{\alpha}} \in K_+\} \\ &\quad = P^\infty\{\sigma > \alpha \vee \bar{\alpha}\} + P^\infty\{\bar{\alpha} \ge \sigma > \alpha; \overline{Y}_{\bar{\alpha}} \notin K_+\} + P^\infty\{Y_\alpha \in K_+; \bar{\alpha} < \sigma \le \alpha\}. \end{aligned}$$

We will show that the sum of the first two terms is strictly positive.

The region $\{z: |R(z)| > a\}$ is a connected open set, so there are continuous paths from ∞ to K_+ that do not hit $\{z: |R(z)| = a\}$. It follows that $P^\infty\{\sigma > \alpha\} > 0$. Suppose that $P^\infty\{\sigma > \alpha \vee \bar{\alpha}\} = 0$; then $P^\infty\{\bar{\alpha} \geq \sigma > \alpha\} > 0$. On the event $\{\sigma \leq \bar{\alpha}\}$, the path $\overline{Y}_t$ goes from $\overline{Y}_0$ to $\overline{Y}_\sigma$ without hitting K. Conditional on this event, there is *positive* probability that $\overline{Y}_t$ will *approximately* retrace its path from $\sigma \leq t \leq 2\sigma$, avoiding K and landing, at time 2σ, at a point near $\overline{Y}_0$. Since the unconditional probability of $\{\overline{Y}_{\bar{\alpha}} \in K_-\}$ is positive and since hitting probabilities are continuous functions of the initial point, it follows that

$$P^\infty\big(\overline{Y}_{\bar{\alpha}} \in K_- | \mathscr{G}_\sigma\big) 1\{\bar{\alpha} > \sigma\} > 0$$

(here $\mathscr{G}_\sigma$ is the σ-algebra generated by $\{Z_{t \wedge \sigma}, t \geq 0\}$, i.e., the stopping field). Thus,

$$P^\infty\{\bar{\alpha} \geq \sigma > \alpha\} > 0 \Rightarrow P^\infty\left\{\overline{Y}_{\bar{\alpha}} \in K_-; \bar{\alpha} \geq \sigma > \alpha\right\} > 0;$$

this proves that $P^\infty\{Y_\alpha \in K_+\} > P^\infty\{\overline{Y}_{\bar{\alpha}} \in K_+\}$. This completes the proof that if $\nu = n^{-1}\sum_{i=1}^n \nu_{\xi_i}$, then $|R(z)|$ is constant a.s. on K w.r.t. ν.

Finally, suppose that $|R(z)| = a$ a.e. $(\nu + \sum \nu_{\xi_i})$. Define $K' = K \cap L_a$. Then the hitting distribution of K' is the same as that of K for each of the processes Y_t and $\overline{Y}_t$, because $(\nu + \sum \nu_{\xi_i})(K \setminus K') = 0$. But $K' \subset L_a$, so the hitting distribution of K' is the same for each of the processes Y_t and $\overline{Y}_t$. Thus $\nu = n^{-1}\sum_{i=1}^n \nu_{\xi_i}$. □

Let $R(z) = \prod_{i=1}^n (z - \xi_i)$ and $L_a = \{z \in \mathbb{C}: |R(z)| = a\}$, where $0 < a < \infty$ and $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$. Then L_a is the union of a finite number of simple closed curves $L_a^{(1)}, \dots, L_a^{(k)}$, each of which surrounds a bounded region of $\mathbb{C}$ in which $|R| < a$.

LEMMA 2. *Let F be a rational function. If $|F(z)| = c > 0$ for infinitely many $z \in L_a^{(i)}$, then $|F(z)| = c$ for every $z \in L_a^{(i)}$.*

PROOF. Take $z_0 \in L_a^{(i)}$ such that $|F(z)| = c$ for infinitely many z in every neighborhood of z_0 in $L_a^{(i)}$. There is a 1-to-1 conformal map φ of the unit disk onto a neighborhood $\mathscr{U}$ of z_0 such that $\varphi(0) = z_0$ and $\varphi^{-1}(L_a^{(i)})$ consists of a finite number of line segments through 0, one of which is the real axis. Also, φ may be chosen so that $|F \circ \varphi(\xi)| = c$ for infinitely many $\xi \in \mathbb{R}$. A routine argument now shows that the power series of $i(\log F \circ \varphi - \log c)$ must have real coefficients. Consequently, $|F(z)| = c$ for every z in an open arc of $L_a^{(i)}$ containing z_0. But the same argument applies at the endpoints of this open arc; therefore, the arc of L_a on which $|F| = c$ may be extended indefinitely until it comes back on itself. □

The crucial fact about the maximum entropy measure μ for the argument below is that it is *balanced*, that is:

PROPOSITION 16. *Let X be a random variable with distribution μ, where μ is the maximum entropy measure for Q on $\mathscr{J}$. Let Y be a random variable such* *that, conditional on X, Y is uniformly distributed on the d points in $Q^{-1}(X)$. Then Y has distribution μ.*

PROOF. Choose $z \in \mathbb{C}$ such that $\mu_n^z \to \mu$ weakly as $n \to \infty$ (recall Ljubich's theorem). Let Y_n have distribution μ_n^z and let $X_n = Q(Y_n)$. Then X_n has distribution μ_{n-1}^z and, conditional on X_n, Y_n is uniformly distributed on the d points in $Q^{-1}(X_n)$ (this follows from the definition of μ_n^z and μ_{n-1}^z). Consequently, the random vector (X_n, Y_n) converges in distribution to (X, Y). Since Y_n has distribution μ_n^z and $\mu_n^z \to \mu$, it follows that Y has distribution μ. □

Assume for the remainder of this section that $Q(z) = P_1(z)/P_2(z)$, where P_1 and P_2 are relatively prime, with $d = \text{degree } P_1$ and $d_* = \text{degree } P_2$ satisfying $d \ge d_* + 1 \ge 2$, and assume that $\mu = \nu$. We will show that this leads to a contradiction. Take $P_2(z) = \prod_{i=1}^{d_*}(z - \xi_i)$ and let ν, ν_{ξ_i} be as in Proposition 15 for $K = \mathscr{J}$.

CLAIM 1. There exists $a > 0$ such that $|P_2| = a$ a.e. $(\nu + \sum_1^{d_*}\nu_{\xi_i})$.

PROOF. Let Z_t be Brownian motion started at ∞ (under P^∞) and run until the first time T it hits $\mathscr{J}$. Then Z_T has distribution $\nu = \mu$. Let $Q_1^{-1}(Z_t), Q_2^{-1}(Z_t), \dots, Q_d^{-1}(Z_t)$ be the d paths that map into Z_t by Q, listed so that $Q_i^{-1}(Z_0) = \xi_i$ for $i = 1, 2, \dots, d_*$ and $Q_i^{-1}(Z_0) = \infty$ for $i > d_*$. Choose one of the points $Q_1^{-1}(Z_T), \dots, Q_d^{-1}(Z_T)$ at random and call it Y; then Proposition 16 implies that Y has distribution $\mu = \nu$. Choose one of the points $Q_{d_*+1}^{-1}(Z_T), Q_{d_*+2}^{-1}(Z_T), \dots, Q_d^{-1}(Z_T)$ at random and call it W; then Proposition 13 implies that W has distribution $\nu = \mu$. Consequently, if one chooses one of the points $Q_1^{-1}(Z_T), Q_2^{-1}(Z_T), \dots, Q_{d_*}^{-1}(Z_T)$ at random and calls it X, then X has distribution ν.

If one chooses one of the *paths* $Q_1^{-1}(Z_t), \dots, Q_{d_*}^{-1}(Z_t)$ at random, the result is a Brownian motion started at a random point in $\{\xi_1, \xi_2, \dots, \xi_{d_*}\}$ and run until it hits $\mathscr{J}$. (This is because Z_t, $t > 0$, does not hit branch points of Q^{-1}, and each branch of Q^{-1} is conformal except at branch points.) Consequently, the distribution of X is $(d_*)^{-1}\sum_{i=1}^{d_*}\nu_{\xi_i}$. Thus $\nu = (d_*)^{-1}\sum_{i=1}^{d_*}\nu_{\xi_i}$, so the claim follows from Proposition 15. (The constant a cannot be zero, because $|P_2(z)| = 0$ only at $z = \xi_1, \xi_2, \dots, \xi_d$ and $\xi_i \notin \mathscr{J}$.) □

CLAIM 2. $|P_2(z)| = a$ for every $z \in \mathscr{J}$.

PROOF. Let $z \in \mathscr{J}$ and $\mathscr{U}$ be a neighborhood of z. There exists $n \ge 1$ such that $\nu(Q^n(\mathscr{U})) > 0$. [This follows from Montel's theorem (cf. [1], Section 5), which implies that $\bigcup_{n \ge 1} Q^n(\mathscr{U})$ excludes at most two points of $\overline{\mathbb{C}}$.] Since $\nu = \mu$, it follows from Proposition 16 that $\nu(\mathscr{U}) \ge d^{-n}\nu(Q^n(\mathscr{U})) > 0$. Consequently, by Claim 1, there exists $\xi \in \mathscr{U}$ such that $|P_2(\xi)| = a$. Therefore, since $\mathscr{U}$ is arbitrary, $|P_2(z)| = a$. □

Recall that $L_a = \{z \in \mathbb{C}: |P_2(z)| = a\}$ consists of a finite number of simple closed curves $L_a^{(1)}, L_a^{(2)}, \dots, L_a^{(k)}$, each of which surrounds a bounded region of $\mathbb{C}$ in which $|P_2| < a$.

CLAIM 3. $\mathscr{J} = \bigcup_{i=1}^m L_a^{(i)}$ for some $m \le k$, provided $L_a^{(1)}, \dots, L_a^{(k)}$ are labelled appropriately.

PROOF. By Claim 2, $\mathscr{J} \subset L_a$; consequently, if $|P_2(z)| > a$, then $z \in \mathscr{F}_\infty$ (because $\{z: |P_2(z)| > a\}$ is connected). Thus, if $z \in L_a$ and $z \notin \mathscr{J}$, then $z \in \mathscr{F}_\infty$ and so $Q^n(z) \to \infty$ as $n \to \infty$.

Since $Q^n(\mathscr{J}) = \mathscr{J}\ \forall\ n \ge 1$, it follows that $|P_2 \circ Q^n(z)| = a\ \forall\ z \in \mathscr{J}, n \ge 1$. Hence, by Lemma 2, if $\mathscr{J} \cap L_a^{(i)}$ is infinite for some i, then $|P_2 \circ Q^n(z)| = a$ $\forall\ z \in L_a^{(i)}$, $\forall\ n \ge 1$; consequently, $L_a^{(i)} \subset \mathscr{J}$, by the result of the previous paragraph.

To complete the proof it suffices to prove that $\mathscr{J}$ has no isolated points. But this follows from the argument in the proof of Claim 2. $\square$

It is now easy to obtain a contradiction. Consider $L_a^{(1)} \subset \mathscr{J}$ (note: $\mathscr{J} \neq \varnothing$ so there is at least one $L_a^{(i)}$ contained in $\mathscr{J}$, by Claim 3). The curve $L_a^{(1)}$ is a simple closed curve that surrounds a bounded region R_1 in which $|P_2| < a$; hence $R_1 \subset \mathscr{F}$ but $R_1 \cap \mathscr{F}_\infty = \varnothing$. It follows that $Q^{-1}(R_1) \subset \mathscr{F}$ but $Q^{-1}(R_1) \cap \mathscr{F}_\infty = \varnothing$, because $Q(\mathscr{F}_\infty) \subset \mathscr{F}_\infty$. However, if $Q^{-1}(R_1) \subset \mathscr{F}$ and $Q^{-1}(R_1) \cap \mathscr{F}_\infty = \varnothing$, then $Q^{-1}(R_1) \subset \bigcup_{i=1}^m R_i$, where R_i is the bounded region surrounded by $L_a^{(i)}$ and m is as in Claim 3. [Note: $Q^{-1}(R_1)$ cannot intersect $\bigcup_{i=m+1}^k R_i$, because $R_i \subset \mathscr{F}_\infty$ for $i \ge m+1$, since $L_a^{(i)} \not\subset \mathscr{J}$.]

Now each R_i, $i = 1, 2, \dots, m$, contains a zero of $P_2(z)$, by the argument principle (R_i is surrounded by $L_a^{(i)}$, on which $|P_2| \equiv a$ and $|P_2| < a$ in R_i). But Q maps the zeroes of P_2 to ∞; since R_i is a connected component of $\mathscr{F}$, it follows that $Q(R_i) \subset \mathscr{F}_\infty$. This is a contradiction, because $Q^{-1}(R_1) \subset \bigcup_{i=1}^m R_i$ and $R_1 \cap \mathscr{F}_\infty = \varnothing$. $\square$

8. Totally disconnected Julia sets.

Assume throughout this section that ∞ is a superattracting fixed point of Q and that the branch points of Q^{-1} are contained in $\mathscr{F}_\infty$. Our goal is to show that: (i) Q: $\mathscr{J} \to \mathscr{J}$ is topologically conjugate to the forward shift σ: $\Sigma \to \Sigma$ on the sequence space $\Sigma = \{1, 2, \dots, d\}^{\mathbb{N}}$; (ii) the equilibrium measure ν pulls back to a Gibbs state $\bar{\nu}$ on Σ; and (iii) the entropy $h(Q)$ of the system $(\mathscr{J}, Q, \nu)$ satisfies $h(Q) > \log(d - d_*)$.

8A. *Topological conjugacy.*

LEMMA 3. *There is a smooth Jordan curve Γ in $\mathbb{C}$ whose interior contains $\mathscr{J}$ and whose exterior contains $\bigcup_{n=1}^\infty Q^n(\mathscr{G}_0)$.*

PROOF. By hypothesis, $Q(\mathscr{S}_0) \subset \mathscr{F}_\infty$, because $Q(\mathscr{S}_0)$ consists of the branch points of Q^{-1}. It follows from Proposition 2 that $Q^n(\mathscr{S}_0) \subset \mathscr{F}_\infty\ \forall\ n \geq 1$ and that $Q^n(z) \to \infty\ \forall\ z \in \mathscr{S}_0$. Hence there is a large open disc D in $\mathbb{C}$ containing $\mathscr{J}$ and at most finitely many points of $\bigcup_{n=1}^{\infty} Q^n(\mathscr{S}_0)$. Label these points $\xi_1, \xi_2, \dots, \xi_r$. Since $\mathscr{F}_\infty$ is connected and $\xi_1, \dots, \xi_r \in \mathscr{F}_\infty$, there is a closed set $P \subset \mathscr{F}_\infty$ containing $\xi_1, \dots, \xi_r$ such that $D \setminus P$ is simply connected and contains $\mathscr{J}$. Let φ: $\{|z| < 1\} \to D \setminus P$ be a conformal homeomorphism of the unit disk onto $D \setminus P$ (such a mapping exists, by the Riemann mapping theorem). For r sufficiently close to 1, $\varphi(\{|z| < r\}) \supset \mathscr{J}$. Set $\Gamma = \varphi(\{|z| = r\})$. $\square$

Define

$$\mathscr{D} = \text{domain interior to } \Gamma.$$

Observe that $\mathscr{D}$ is simply connected, so by Lemma 3, all branches Q_i^{-n} of all Q^{-n}, $n \geq 1$, are single-valued and analytic in a neighborhood of $\overline{\mathscr{D}}$. Fix some definite labelling $Q_1^{-1}, Q_2^{-1}, \dots, Q_d^{-1}$ of the distinct branches of Q^{-1} in $\mathscr{D}$. For any finite sequence $i_1 i_2 \cdots i_d$ of symbols from $\{1, 2, \dots, d\}$, define

$$\mathscr{J}(i_1 i_2 \cdots i_n) = Q_{i_1}^{-1} \circ Q_{i_2}^{-1} \circ \cdots \circ Q_{i_n}^{-1}(\mathscr{J}),$$

$$\Gamma(i_1 i_2 \cdots i_n) = Q_{i_1}^{-1} \circ Q_{i_2}^{-1} \circ \cdots \circ Q_{i_n}^{-1}(\Gamma),$$

$$\mathscr{D}(i_1 i_2 \cdots i_n) = Q_{i_1}^{-1} \circ Q_{i_2}^{-1} \circ \cdots \circ Q_{i_n}^{-1}(\mathscr{D}).$$

[These are legitimate definitions, because (i) $Q^{-1}(\mathscr{J}) = \mathscr{J} \subset \mathscr{D}$ and (ii) all branches of Q^{-n} are single-valued and analytic on $\overline{\mathscr{D}}$, hence each must agree with some $Q_{i_1}^{-1} \circ \cdots \circ Q_{i_n}^{-1}$ on $\mathscr{J}$.] The definitions have some immediate but important consequences:

(a) $$\mathscr{J}(i_1 i_2 \cdots i_n) \subset \mathscr{J}(i_1 i_2 \cdots i_{n-1});$$

(b) $$\mathscr{J}(i_1 i_2 \cdots i_n) \cap \mathscr{J}(i_1' i_2' \cdots i_n') = \varnothing \quad \text{unless } i_j = i_j'\ \forall\, 1 \leq j \leq n;$$

(c) $$Q: \mathscr{J}(i_1 i_2 \cdots i_n) \to \mathscr{J}(i_2 i_3 \cdots i_n) \quad \text{is a surjective homeomorphism;}$$

(d) $$\lim_{n \to \infty} \max_{i_1 i_2 \cdots i_n} \operatorname{diameter}(\mathscr{J}(i_1 i_2 \cdots i_n)) = 0$$

[property (d) follows from Propositions 4–5]. Note that (a)–(d) imply that $\mathscr{J}$ is totally disconnected.

For each *infinite* sequence $i_1 i_2 \cdots \in \Sigma$ we may now define

$$\pi(i_1 i_2 \cdots) = \bigcap_{n=1}^{\infty} \mathscr{J}(i_1 i_2 \cdots i_n).$$

By (a), (c), this is the intersection of a nested sequence of nonempty, compact sets (see Proposition 1) and by (d) the intersection consists of a single point. Hence π: $\Sigma \to \mathscr{J}$. It follows from (a), (d) that this map is continuous and from (b) that it is 1-to-1; it is clearly onto, because for each n, $\mathscr{J} = \bigcup_{i_1 i_2 \cdots i_n} \mathscr{J}(i_1 i_2 \cdots i_n)$. Finally, by (c),

$$Q \circ \pi = \pi \circ \sigma.$$

Thus, we have exhibited a topological conjugacy between Q: $\mathscr{J} \to \mathscr{J}$ and σ: $\Sigma \to \Sigma$.

The curves $\Gamma(i_1 i_2 \cdots i_k)$ have played no role thus far. However, in studying Brownian paths started at ∞ and stopped at $\mathscr{J}$ they will be very useful, because each $\Gamma(i_1 i_2 \cdots i_k)$ surrounds the corresponding $\mathscr{J}(i_1 i_2 \cdots i_k)$. Unfortunately, the sets $\mathscr{D}(i_1 i_2 \cdots i_n)$ do not satisfy the nesting property (a) above. But Γ and the region exterior to Γ are contained in $\mathscr{F}_\infty$, so $Q^n(z) \to \infty$ as $n \to \infty$ uniformly for $z \notin \mathscr{D}$, by Proposition 2; consequently, there is an integer $r \geq 1$ large enough that

$$\text{(e)} \qquad Q^r(\Gamma) \subset (\overline{\mathscr{D}})^c,$$

$$\text{(f)} \qquad Q^{-n}(\overline{\mathscr{D}}) \subset \mathscr{D} \qquad \forall\, n \geq r.$$

Henceforth we shall assume that $r \geq 1$ is an integer large enough that both these statements hold. We now have

$$\text{(g)} \qquad \overline{\mathscr{D}(i_1 i_2 \cdots i_{(n+1)r})} \subset \mathscr{D}(i_1 i_2 \cdots i_{nr});$$

$$\text{(h)} \qquad \overline{\mathscr{D}(i_1 i_2 \cdots i_n)} \cap \overline{\mathscr{D}(i'_1 i'_2 \cdots i'_n)} = \varnothing \quad \text{unless } i_j = i'_j \ \forall\, 1 \leq j \leq n;$$

$$\text{(i)} \qquad Q\colon \overline{\mathscr{D}(i_1 i_2 \cdots i_n)} \to \overline{\mathscr{D}(i_2 i_3 \cdots i_n)} \quad \text{is a surjective homeomorphism;}$$

$$\text{(j)} \qquad \Gamma(i_1 i_2 \cdots i_n) = \partial \mathscr{D}(i_1 i_2 \cdots i_n);$$

$$\text{(k)} \qquad \mathscr{J}(i_1 i_2 \cdots i_n) \subset \mathscr{D}(i_1 i_2 \cdots i_n).$$

Finally, observe that $\mathscr{F} = \mathscr{F}_\infty$, so $\mathscr{F}$ is connected. Here is the proof. The region $(\overline{\mathscr{D}})^c$ exterior to Γ is contained in $\mathscr{F}_\infty$ by construction. For any $n \geq 1$, $Q^{-nr}((\overline{\mathscr{D}})^c) \subset \mathscr{F}_\infty$ by an easy induction argument using (g), (j), (k). But $\mathscr{F} = \bigcup_{n=1}^\infty Q^{-nr}((\overline{\mathscr{D}})^c)$, because $\mathscr{J} = \bigcap_{n=1}^\infty Q^{-nr}(\overline{\mathscr{D}})$.

8*B*. *Characterization of a Gibbs state.* We are to show that the pullback

$$\bar{\nu} = \nu \circ \pi$$

of the equilibrium measure ν is a Gibbs state on Σ. For that it suffices to show that there is a Hölder continuous function f: $\Sigma \to \mathbb{R}$ and constants $0 < c_1 < c_2 < \infty$ such that for every $i_1 i_2 \cdots \in \Sigma$ and $n \geq 0$,

$$\text{(8.1)} \qquad c_1 \leq \nu(\mathscr{J}(i_1 i_2 \cdots i_n)) / \exp\{S_n f(i_1 i_2 \cdots)\} \leq c_2,$$

where

$$S_n f = f + f \circ \sigma + f \circ \sigma^2 + \cdots + f \circ \sigma^{n-1}.$$

(see [2], Theorem 1.2. A function f: $\Sigma \to \mathbb{R}$ is Hölder continuous if there exist constants $C < \infty$, $0 < \beta < 1$ such that $|f(i_1 i_2 \cdots) - f(i'_1 i'_2 \cdots)| \leq C\beta^n$ whenever $i_j = i'_j\ \forall\, 1 \leq j \leq n$.)

LEMMA 4. *To prove* (8.1) *it suffices to prove that*

$$\text{(8.2)} \qquad \nu(\mathscr{J}(i_1 i_2 \cdots i_n)) > 0 \qquad \forall\, i_1 i_2 \cdots i_n,$$

and that there exist constants $C < \infty$, $0 < \beta < 1$ such that for any two sequences $i_1 i_1 \cdots i_n$ and $i'_1 i'_2 \cdots i'_{n'}$ satisfying $i_j = i'_j \ \forall\, 1 \le j \le k$, it is the case that

$$\left| \log\left\{ \frac{\nu(\mathscr{J}(i_1 i_2 \cdots i_n))/\nu(\mathscr{J}(i_2 i_3 \cdots i_n))}{\nu(\mathscr{J}(i'_1 i'_2 \cdots i'_{n'}))/\nu(\mathscr{J}(i'_2 i'_3 \cdots i'_{n'}))} \right\} \right| \le C\beta^k. \tag{8.3}$$

Proof. For any sequence $i_1 i_2 \cdots$, define

$$f(i_1 i_2 \cdots i_n) = \log\{\nu(\mathscr{J}(i_1 i_2 \cdots i_n))/\nu(\mathscr{J}(i_2 i_3 \cdots i_n))\},$$
$$f(i_1 i_2 \cdots) = \lim_{n\to\infty} f(i_1 i_2 \cdots i_n).$$

The hypotheses (8.2)–(8.3) imply that f is Hölder continuous on Σ and furthermore that

$$\frac{\nu(\mathscr{J}(i_1 i_2 \cdots i_n))}{\exp\{S_n f(i_1 i_2 \cdots)\}} = \frac{\exp\{\sum_{j=1}^n f(i_j i_{j+1} \cdots i_n)\}}{\exp\{S_n f(i_1 i_2 \cdots)\}}$$

is bounded above and below. □

To investigate the quantities in (8.2)–(8.3), we bring in once again the Brownian motion process started at ∞ and run until the time of first entry into $\mathscr{J}$ (since ∞ is attracting or superattracting, this time is finite with probability 1, by Proposition 10). Under the probability measure P^∞, let Z_t and $\tilde{Z}_t$ be Brownian motions satisfying $Z_0 = \tilde{Z}_0 = \infty$ and $Z = Q\tilde{Z}$ (as before, Z and $\tilde{Z}$ denote the traces of the paths Z_t and $\tilde{Z}_t$). Define $T = \inf\{t\colon Z_t \in \mathscr{J}\}$ and $\tilde{T} = \inf\{t\colon \tilde{Z}_t \in \mathscr{J}\}$. Then

$$\nu(\mathscr{J}(i_1 i_2 \cdots i_n)) = P^\infty\{\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i_1 i_2 \cdots i_n)\},$$
$$\nu(\mathscr{J}(i_2 i_3 \cdots i_n)) = P^\infty\{Z_T \in \mathscr{J}(i_2 i_3 \cdots i_n)\},$$

so

$$\frac{\nu(\mathscr{J}(i_1 i_2 \cdots i_n))}{\nu(\mathscr{J}(i_2 i_3 \cdots i_n))} = P^\infty\{\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i_1) | Z_T \in \mathscr{J}(i_2 i_3 \cdots i_n)\}.$$

This conditional probability may be rewritten in a form that eliminates the process $\tilde{Z}_t$. Consider the path Z_t, $0 \le t \le T$; it avoids the branch points of Q^{-1} (except for $Z_0 = \infty$) and terminates at $Z_T \in \mathscr{D}$. In the domain $\mathscr{D}$ the branches $Q_1^{-1}, \dots, Q_d^{-1}$ are single-valued and analytic, so $Q_i^{-1}(Z_T)$ is well-defined for each $i = 1, \dots, d$. By the monodromy theorem, Q_i^{-1} can be continued along Z_t from $t = T$ to $t = 0$ (t runs backwards), so we can define $Q_i^{-1}Z_t$ to be this path. Observe that $Q_i^{-1}Z_0 \in Q^{-1}(\infty)$; define the event

$$F_i = \{Q_i^{-1}Z_0 = \infty\}.$$

Lemma 5. *To prove* (8.3) *it suffices to show that there exist constants $C < \infty$, $0 < \beta < 1$ such that for each $i \in \{1, \dots, d\}$ and any two finite sequences*

$i_1 i_2 \cdots i_n$ and $i'_1 i'_2 \cdots i'_{n'}$ satisfying $i_j = i'_j$ $\forall\, 1 \le j \le k$, it is the case that

$$\left|\log\left\{\frac{P^\infty(F_i | Z_T \in \mathscr{J}(i_1 i_2 \cdots i_n))}{P^\infty(F_i | Z_T \in \mathscr{J}(i'_1 i'_2 \cdots i'_{n'}))}\right\}\right| \le C\beta^k. \tag{8.4}$$

PROOF. The event $\{\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i)\}$ is the same as the event $\{Q_i^{-1}Z = \tilde{Z}\}$. By Proposition 13,

$$P^\infty\big(Q_i^{-1}Z = \tilde{Z} | Z\big) = 1_{F_i}/(d - d_*).$$

Since the events $\{Z_T \in \mathscr{J}(i_1 i_2 \cdots i_n)\}$ and $\{Z_T \in \mathscr{J}(i'_1 i'_2 \cdots i'_{n'})\}$ are measurable with respect to the σ-algebra generated by the trace Z, it follows that

$$\begin{aligned} &P^\infty\big(\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i) | Z_T \in \mathscr{J}(i_1 i_2 \cdots i_n)\big) \\ &\qquad = (d - d_*)^{-1} P^\infty(F_i | Z_T \in \mathscr{J}(i_1 i_2 \cdots i_n)), \\ &P^\infty\big(\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i) | Z_t \in \mathscr{J}(i'_1 i'_2 \cdots i'_{n'})\big) \\ &\qquad = (d - d_*)^{-1} P^\infty(F_i | Z_T \in \mathscr{J}(i'_1 i'_2 \cdots i'_{n'})). \end{aligned}$$

□

NOTATIONAL CONVENTIONS. Let $(\Omega, \mathscr{B}, P)$ be a probability space, $A \in \mathscr{B}$, and $\mathscr{G}$ a σ-algebra contained in $\mathscr{B}$. Then $P(A|\mathscr{G})$ is the (essentially) unique $\mathscr{G}$-measurable random variable such that $P(A \cap G) = E(1_G P(A|\mathscr{G}))$ for all $G \in \mathscr{G}$. If $\mathscr{G}$ is generated by a random vector X, we will sometimes write $P(A|X)$ instead of $P(A|\mathscr{G})$; since this is a function of X we may sometimes let $P(A|X = x)$ denote the corresponding function of x. If $A, B \in \mathscr{B}$, $P(A|B) = P(A \cap B)/P(B)$. Similar conventions apply for conditional expectations.

8C. *Application of Harnack's inequality.* Verification of the inequalities (8.2) and (8.4) will require some auxiliary information about Brownian motion in $\mathscr{F}_\infty$. We begin with a version of Harnack's inequality. Assume that under P^ξ, Z_t is a Brownian motion process in $\overline{\mathbb{C}}$ with $P^\xi\{Z_0 = \xi\} = 1$. Let K be a compact subset of $\overline{\mathbb{C}}$; define $T_K = \inf\{t\colon Z_t \in K\}$. If $P^\xi\{T_K < \infty\} = 1$, define

$$\nu_K^\xi(dz) = P^\xi\{Z_{T_K} \in dz\}.$$

LEMMA 6. *Let D be a connected component of K^c, and assume that for some $\xi \in D$, $P^\xi\{T_K < \infty\} = 1$. Then $P^\zeta\{T_K < \infty\} = 1$ for every $\zeta \in D$ and for any two points ξ, $\zeta \in D$, the measures ν_K^ξ and ν_K^ζ are mutually absolutely continuous. For each compact $G \subset D$, there is a constant $c = c(G) < \infty$ such that for all ξ, $\zeta \in G$, $z \in K$,*

$$c^{-1} \le \frac{d\nu_K^\xi}{d\nu_K^\zeta}(z) \le c. \tag{8.5}$$

PROOF. It follows from [12], Chapter 2, Proposition 2.10 that either $P^\xi\{T_K < \infty\} = 0\ \forall\, \xi \in D$ or $P^\xi\{T_K < \infty\} = 1\ \forall\, \xi \in D$. Let A be any measurable subset of K; then $\nu_K^\xi(A)$ is a harmonic function of $\xi \in D$ (by the strong

Markov property, it satisfies the mean value property). Clearly $0 \le \nu_K^\xi(A) \le 1$, so by the maximum principle for harmonic functions ([12], Chapter 4, Proposition 1.4) either $\nu_K^\xi(A) = 0$ for all $\xi \in D$ or $\nu_K^\xi(A) > 0$ for all $\xi \in D$. Thus ν_K^ξ, ν_K^ζ are mutually a.c. By the Harnack inequality ([12], Chapter 4, Theorem 3.5), for any compact $G \subset D$ there is a constant $c = c(G)$ such that

$$c^{-1} \le \nu_K^\xi(A)/\nu_K^\zeta(A) \le c \qquad \forall\, \xi, \zeta \in G, \qquad \forall\, A \subset K;$$

the inequality (8.5) follows from this. □

Let $\mathscr{H}_T$ be the σ-algebra generated by the random variable Z_T (as usual, $T = \inf\{t\colon Z_t \in \mathscr{J}\}$).

LEMMA 7. *There exists a constant $\varepsilon > 0$ such that*

$$P^\infty(F_i|\mathscr{H}_T) \ge \varepsilon \quad a.s.\ \forall\, i = 1, 2, \dots, d. \tag{8.6}$$

PROOF. Let Γ_* be a smooth Jordan curve enclosing $\mathscr{J}$ such that $\Gamma_* \subset \mathscr{D}$. Any continuous path from ∞ to $\mathscr{J}$ must first intersect $\Gamma = \partial D$, then Γ_*, before reaching $\mathscr{J}$. Define

$$\tau = \inf\{t\colon Z_t \in \Gamma\},$$
$$\tau_* = \inf\{t\colon Z_t \in \Gamma_*\}.$$

For $\xi \in \mathbb{C}$, define measures ν^ξ, ν_*^ξ on $\mathscr{J}$ by

$$\nu^\xi(dz) = P^\xi\{Z_T \in dz\},$$
$$\nu_*^\xi(dz) = P^\xi\{Z_T \in dz \text{ and } T < \tau\}.$$

Note that for $\xi \notin \mathscr{D}$, $\nu_*^\xi = 0$; also, $(d\nu_*^\xi/d\nu^\xi) \le 1$. Using Lemma 6 we will show that there exists a constant $\varepsilon > 0$ such that

$$\frac{d\nu_*^\xi}{d\nu}(z) \ge \varepsilon \qquad \forall\, \xi \in \Gamma_*, \qquad \forall\, z \in \mathscr{J}.$$

For this it suffices to show that for some (possibly different) $\varepsilon > 0$,

$$\frac{d\nu_*^\xi}{d\nu^\xi}(z) \ge \varepsilon \qquad \forall\, \xi \in \Gamma_*, \qquad \forall\, z \in \mathscr{J},$$

because Lemma 6 (with $K = \mathscr{J}$, $D = \mathscr{F}_\infty$, $G = \Gamma_* \cup \{\infty\}$) implies that $d\nu^\xi/d\nu$ is bounded above and below. Consider a continuous path from Γ_* to $\mathscr{J}$. It may go directly to $\mathscr{J}$ (without hitting Γ); or it may hit Γ, return to Γ_*, then go directly to $\mathscr{J}$; or it may hit Γ and return to Γ_* n times, then go directly to $\mathscr{J}$. Thus, by the strong Markov property,

$$\nu^\xi = \nu_*^\xi + \int_{\Gamma_*} \nu_*^\zeta \, d\alpha^\xi(\zeta),$$

where the measure α^ξ satisfies $\alpha^\xi(\Gamma_*) \le \sum_{n=1}^\infty p^n$ with $p = \sup_{\zeta \in \Gamma_*} P^\zeta\{\tau < T\} < 1$, by Lemma 6 (with $K = \mathscr{J} \cup \Gamma$, $D = \mathscr{D} \setminus \mathscr{J}$ and $G = \Gamma_*$). Another application of Lemma 6 (again with $K = \mathscr{J} \cup \Gamma$, $D = \mathscr{D} \setminus \mathscr{J}$, $G = \Gamma_*$) shows that for some $c < \infty$,

$$c^{-1} \leq \frac{d\nu_*^\xi}{d\nu_*^\zeta}(z) \leq c \qquad \forall\, \xi, \zeta \in \Gamma_*, \qquad \forall\, z \in \mathscr{J},$$

so the integral representation above implies that

$$d\nu_*^\xi / d\nu^\xi \geq \left(1 + cp(1-p)^{-1}\right)^{-1}.$$

Now choose $\xi \in \Gamma_*$. For each $i = 1, 2, \dots, d$, there is a smooth path $\gamma_i(t)$, $0 \leq t \leq 1$, such that $\gamma_i(0) = \infty$, $\gamma_i(1) = \xi$, $\gamma_i(t) \in \mathscr{F}_\infty \setminus (\{\infty\} \cup \Gamma_* \cup Q(\mathscr{S}_0))$ for $0 < t < 1$, and such that if Q_i^{-1} is analytically continued backwards along γ_i from $\xi = \gamma_i(1)$, then $Q_i^{-1}\gamma_i(0) = \infty$. This follows from the fact that $Q(\mathscr{S}_0)$ (the set of branch points of Q^{-1}) lies outside Γ_* (Lemma 3), together with the fact that the Riemann surface of Q^{-1} is connected ([13], Section 3.2, problem 7). Observe that for small $\delta > 0$, if $\gamma(t)$, $0 \leq t \leq 1$, is a continuous path such that $\gamma(0) = \infty$ and

$$\text{distance}\big(\gamma(t), \gamma_i(t)\big) < \delta \qquad \forall\, 0 \leq t \leq 1$$

(here distance means spherical distance), then Q_i^{-1} continued analytically backwards along γ from $\gamma(1)$ will end at $Q_i^{-1}\gamma(0) = \infty$.

Consider Brownian motion Z_t started at ∞ and run until the first time τ_* that it reaches Γ_*. Let G_i be the event that *some* reparametrization of the path Z_t, $0 \leq t \leq \tau_*$, stays within distance δ of γ_i. Then

$$P^\infty(G_i) > 0 \qquad \forall\, i = 1, 2, \dots, d$$

(this may be proved by elementary arguments). By the strong Markov property,

$$P^\infty(F_i \cap \{Z_T \in dz\}) \geq E^\infty 1_{G_i} \nu_*^{Z_{\tau_*}}(dz);$$

since $d\nu_*^\xi / d\nu$ is bounded below, (8.6) follows. □

PROOF OF (8.2). This is by induction on n. For $n = 0$, the inequality (8.2) is trivial, because $\nu(\mathscr{J}) = 1$. Now

$$\begin{aligned}
&\big(\nu(\mathscr{J}(i_1 i_2 \cdots i_n))\big)/\big(\nu(\mathscr{J}(i_2 i_3 \cdots i_n))\big) \\
&\quad = P^\infty\big(\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i_1) | Z_T \in \mathscr{J}(i_2 i_3 \cdots i_n)\big) \\
&\quad = P^\infty\big(F_i | Z_T \in \mathscr{J}(i_2 i_3 \cdots i_n)\big)/(d - d_*) \\
&\quad \geq \varepsilon/(d - d_*)
\end{aligned}$$

by Lemma 7 (see the proof of Lemma 5). □

PROOF OF THEOREM 2(b). Let Z, $\tilde{Z}$, T, $\tilde{T}$ be as in the proof of Lemma 5. Recall that

$$\begin{aligned}
P^\infty\big(\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i) | Z\big) &= P^\infty\big(Q_i^{-1} Z = \tilde{Z} | Z\big) \\
&= 1_{F_i}/(d - d_*)
\end{aligned}$$

and

$$\sum_{i=1}^{d} 1_{F_i} = (d - d_*).$$

Consider the partition $\mathscr{P} = \{\{\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i)\}\}_{i=1,2,\dots,d}$; its entropy $h(\mathscr{P}, Q)$ satisfies

$$\begin{aligned} h(\mathscr{P}, Q) &\geq E^{\infty} \sum_{i=1}^{d} 1\{\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i)\} \log P^{\infty}\big(\tilde{Z}_{\tilde{T}} \in \mathscr{J}(i) | \mathscr{H}_T\big)^{-1} \\ &= E^{\infty} \sum_{i=1}^{d} \left\{\frac{P^{\infty}(F_i|\mathscr{H}_T)}{d - d_*}\right\} \log \left\{\frac{P^{\infty}(F_i|\mathscr{H}_T)}{d - d_*}\right\}^{-1} \\ &= \log(d - d_*) - E^{\infty} \sum_{i=1}^{d} \left\{\frac{P^{\infty}(F_i|\mathscr{H}_T)}{d - d_*}\right\} \log P^{\infty}(F_i|\mathscr{H}_T) \\ &> \log(d - d_*), \end{aligned}$$

because by Lemma 7, $0 < P^{\infty}(F_i|\mathscr{H}_T) < 1$. □

8*D*. *Exponential estimates.* It remains to prove the inequality (8.4). This will require some additional estimates.

Recall that Γ is the smooth Jordan curve bounding the domain $\mathscr{D}$ (Lemma 3). For $n \in \mathbb{Z}$, define

$$\Gamma_n = Q^{-n}(\Gamma).$$

By statements (e)–(f) of Section 8A, Γ_{-r} lies in the exterior of $\Gamma = \Gamma_0$, while for $n \geq r$, Γ_n lies in the interior of Γ. Observe that for $n \geq 1$,

$$\Gamma_n = \bigcup_{i_1 i_2 \cdots i_n} \Gamma(i_1 i_2 \cdots i_n).$$

LEMMA 8. *If r is sufficiently large, then there exist constants $C < \infty$, $0 < \beta < 1$ such that for all $n > 1$ and all $z \in \Gamma_{nr}$,*

$$P^z\{Z_t \text{ hits } \Gamma \text{ before } \mathscr{J}\} \leq C\beta^n. \tag{8.7}$$

PROOF. This is an extension of the argument used in proving Proposition 10. Consider first the case where ∞ is superattracting. In the proof of Proposition 10, we exhibited sets $\tilde{\Gamma}_n$, $n \in \mathbb{Z}$, with the following properties: (a) $\tilde{\Gamma}_n = Q^{-n}(\tilde{\Gamma}_0)$ $\forall$ $n \in \mathbb{Z}$. (b) $\tilde{\Gamma}_0$ is a smooth, Jordan curve in $\mathbb{C}$ containing $\mathscr{J}$ in its interior. (c) Any continuous path from $\tilde{\Gamma}_{n+1}$ to $\tilde{\Gamma}_{n-k}$, $k \geq 1$, must intersect $\tilde{\Gamma}_n$. (d) Any continuous path $\gamma(t)$, $0 \leq t \leq t_*$, that intersects each $\tilde{\Gamma}_n$, $n \geq 0$, must intersect $\mathscr{J}$. (e) For each $n \geq 0$ and each $z \in \tilde{\Gamma}_n$, $P^z\{Z_t$ hits $\tilde{\Gamma}_{n+1}$ before $\tilde{\Gamma}_{n-1}\} \geq 2/3$.

We claim that for all n, $k \geq 0$ and any $z \in \tilde{\Gamma}_{n+k}$,

$$P^z\{Z_t \text{ hits } \tilde{\Gamma}_k \text{ before } \mathscr{J}\} \leq 2^{-n}.$$

The proof is as follows. To get from z to $\tilde{\Gamma}_k$ before $\mathscr{J}$, the path Z_t must cross $\tilde{\Gamma}_{n+k+1}, \tilde{\Gamma}_{n+k-2}, \dots, \tilde{\Gamma}_k$ in that order before crossing *all* $\tilde{\Gamma}_{n+k+m}$, $m \geq 1$. Let

$X_0, X_1, \ldots$ be the indices of the successive sets $\tilde{\Gamma}_j$ hit by Z_t. Then under P^z the sequence 2^{-X_j} is a supermartingale with $2^{-X_0} = 2^{-n-k}$ (because the chance of moving up one before going down one is at least 2/3). Hence, $P^z\{2^{-X_j} = 2^{-k}$ for some $j \geq 0\} \leq 2^{-n}$, by the maximal inequality for supermartingales.

Now consider the sets Γ_n constructed in Section 8A. We may assume that $\tilde{\Gamma}_0$ lies in the exterior of $\Gamma = \Gamma_0$, because in the original construction of the sets $\tilde{\Gamma}_n$ (proof of Proposition 10) we could choose the radius R_0 of the circle C_0 as large as we like, forcing $\tilde{\Gamma}_0$ to be close to ∞. Choose $m \geq 1$ so large that $Q^m(\Gamma) = \Gamma_{-m}$ lies in the exterior of $\tilde{\Gamma}_0$; this is possible because $\Gamma \subset \mathscr{F}_\infty$ and $Q^m \to \infty$ uniformly on compact subsets of $\mathscr{F}_\infty$. Then $Q^n(\Gamma)$ lies in the exterior of $\tilde{\Gamma}_0$ for all $n \geq m$, because Q maps exterior $(\tilde{\Gamma}_0)$ into itself. Choose $r \geq m$ so large that $\tilde{\Gamma}_n$ is contained in $\mathscr{D}$ (= interior of Γ) for all $n \geq r$; this is possible because $\mathscr{J} \subset \mathscr{D}$ and the sets $\tilde{\Gamma}_n$ accumulate at $\mathscr{J}$ as $n \to \infty$.

Let γ be a continuous path from Γ_{nr} to Γ, where $n \geq 2$. Then $Q^{nr}(\gamma)$ is a continuous path from Γ to Γ_{-nr}, which must cross $\tilde{\Gamma}_0$ because Γ and Γ_{-nr} are on opposite sides of $\tilde{\Gamma}_0$. Thus γ must intersect $\tilde{\Gamma}_{nr} = Q^{-nr}(\tilde{\Gamma}_0)$. Let γ^* be the segment of γ that runs from $\tilde{\Gamma}_{nr}$ to Γ. Then $Q^m\gamma^*$ runs from $\tilde{\Gamma}_{nr-m}$ to Γ_{-m}. Since $n \geq 2$, $nr - m \geq m$, so $\tilde{\Gamma}_{nr-m} \subset \mathscr{D}$; hence $\tilde{\Gamma}_{nr-m}$ and Γ_{-m} lie on opposite sides of $\tilde{\Gamma}_0$. This implies that $Q^m\gamma^*$ intersects $\tilde{\Gamma}_0$, which in turn implies that γ^* intersects $\tilde{\Gamma}_m$. This proves that any continuous path γ from Γ_{nr} to Γ, $n \geq 2$, must first hit $\tilde{\Gamma}_{nr}$, then $\tilde{\Gamma}_m$, before reaching Γ. Consequently, for $n \geq 2$,

$$P^z\{Z_t \text{ hits } \Gamma \text{ before } \mathscr{J}\} \leq 2^{-nr+m} \qquad \forall\, z \in \Gamma_{nr}.$$

The inequality (8.7) follows $\forall\, n \geq 1$ by adjusting C, with $\beta = 2^{-r}$.

The case where ∞ is attracting rather than superattracting is similar, but requires modifications similar to those in the second half of the proof of Proposition 10. Since these modifications are routine, we omit them. □

Henceforth we shall assume that $r \geq 1$ has been chosen so large that the conclusion of Lemma 8 is valid. [Recall also that r should be large enough that statements (e)–(f) of Section 8A hold.]

Let $\tau < \infty$ be a stopping time for the process Z_t; define σ-algebras $\mathscr{F}_\tau$, $\mathscr{G}_\tau$, $\mathscr{H}_\tau$ by

$$\mathscr{F}_\tau = \sigma(\{Z_{t\wedge\tau}\}_{t\geq 0}),$$

$$\mathscr{G}_\tau = \sigma(\{Z_{t+\tau}\}_{t\geq 0}),$$

$$\mathscr{H}_\tau = \sigma(Z_\tau)$$

(i.e., $\mathscr{F}_\tau$, $\mathscr{G}_\tau$, $\mathscr{H}_\tau$ are the smallest σ-algebras making these collections of random variables measurable). Observe that $\mathscr{H}_\tau \subset \mathscr{G}_\tau$ and $\mathscr{H}_\tau \subset \mathscr{F}_\tau$; also, τ is not in general measurable w.r.t. $\mathscr{G}_\tau$. One should think of $\mathscr{F}_\tau$ as representing all information about the path Z_t up to time τ and $\mathscr{G}_\tau$ as representing all information about Z_t after time τ.

LEMMA 9. *For any event $F \in \mathscr{F}_\tau$, $P^\xi(F|\mathscr{G}_\tau) = P^\xi(F|\mathscr{H}_\tau)$.*

PROOF. Let $G \in \mathscr{G}_\tau$. By the strong Markov property and elementary properties of conditional expectation,

$$\begin{aligned} E^{\xi}&\left(1_G P^{\xi}(F|\mathscr{G}_\tau)\right) \\ &= E^{\xi} 1_G 1_F \\ &= E^{\xi}\left(1_F P^{\xi}(G|\mathscr{F}_\tau)\right) \\ &= E^{\xi}\left(1_F P^{\xi}(G|\mathscr{H}_\tau)\right) \\ &= E^{\xi}\left(P^{\xi}(F|\mathscr{H}_\tau) P^{\xi}(G|\mathscr{H}_\tau)\right) \\ &= E^{\xi}\left(1_G P^{\xi}(F|\mathscr{H}_\tau)\right); \end{aligned}$$

since this holds $\forall\ G \in \mathscr{G}_\tau$, it follows that $P^{\xi}(F|\mathscr{G}_\tau) = P^{\xi}(F|\mathscr{H}_\tau)$ a.s. $\square$

Define

$$\tau_n = \inf\{t: Z_t \in \Gamma_n\}, \qquad n \geq 0.$$

Statement (e), Section 8A implies that any continuous path from ∞ to $\mathscr{J}$ must intersect each Γ_n, $n \geq 1$, so $P^{\infty}\{\tau_n < T\} = 1$. Moreover, statements (f), (j) imply that for any $n \geq r$, $Z_{\tau_n} \in \mathscr{D}$. Let F_i^n, $n \geq r$, be the event that if Q_i^{-1} is continued analytically along Z_t, $0 \leq t \leq \tau_n$, backwards from Z_{τ_n}, then $Q_i^{-1} Z_0 = \infty$.

LEMMA 10. *There exist constants $C < \infty$, $0 < \beta < 1$ such that for each $k \geq 1$ and each $i = 1, 2, \dots, d$,*

$$|P^{\infty}(F_i|\mathscr{H}_T) - P^{\infty}(F_i^{kr}|\mathscr{H}_T)| \leq C\beta^k.$$

PROOF. The events F_i and F_i^{kr} differ only in paths Z_t which exit $\mathscr{D}$ after $t = \tau_{kr}$ and before $t = T$. (If Z_t does not make such an exit, then the analytic continuation of Q_i^{-1} along Z_t, $0 \leq t \leq T$, backwards from Z_T ends at the same value as the analytic continuation of Q^{-1} along Z_t, $0 \leq t \leq \tau_{kr}$, backwards from $Z_{\tau_{kr}}$.) To make such an exit, Z_t must hit $\Gamma = \partial D$. Thus it suffices to prove that

$$P^{\infty}(Z_t \in \Gamma, \text{some } \tau_{kr} \leq t \leq T|\mathscr{H}_T) \leq C\beta^k.$$

By the strong Markov property, for any Borel set $A \subset \mathscr{J}$,

$$\begin{aligned} P^{\infty}\{Z_T \in A &\text{ and } Z_t \in \Gamma \text{ for some } \tau_{kr} \leq t \leq T\} \\ &= E^{\infty}\left(P^{Z_{\tau_{kr}}}\{Z_T \in A \text{ and } \tau_0 < T\}\right) \\ &= E^{\infty} E^{Z_{\tau_{kr}}}\left(1\{\tau_0 < T\} P^{Z_{\tau_0}}\{Z_T \in A\}\right) \\ &\leq \left\{E^{\infty} E^{Z_{\tau_{kr}}} 1\{\tau_0 < T\}\right\}\left\{c P^{\infty}\{Z_T \in A\}\right\}. \end{aligned}$$

The last inequality follows from Lemma 6 (with $K = \mathscr{J}$, $D = \mathscr{F}_\infty$, $G = \Gamma \cup \{\infty\}$), since $Z_{\tau_0} \in \Gamma$; the constant $c < \infty$ does not depend on A or k. Lemma 8 implies that for some $C < \infty$, $0 < \beta < 1$,

$$E^{Z_{\tau_{kr}}} 1\{\tau_0 < T\} \le C\beta^k.$$

It now follows that $P^\infty(Z_t \in \Gamma, \text{ some } \tau_{kr} \le t \le T | \mathscr{H}_T) \le C'\beta^k)$. □

Let $i_1 i_2 \cdots i_{kr}$, $k \ge 1$, be a finite sequence of symbols from $\{1, 2, \ldots, d\}$. Any continuous path from ∞ to a point of $\Gamma(i_1 i_2 \cdots i_{kr})$ must intersect $\Gamma(i_1 i_2 \cdots i_{nr})$ for each $1 \le n \le k$, by statements (g), (j) of Section 8A. Therefore, for any $\xi \in \Gamma(i_1 i_2 \cdots i_{kr})$, we may define a probability measure $\mu^\xi_{i_1 i_2 \cdots i_{nr}}$ on $\Gamma(i_1 i_2 \cdots i_{nr})$, $n \le k$, by

$$\mu^\xi_{i_1 i_2 \cdots i_{nr}}(A) = \frac{P^\infty\big(Z_{\tau(i_1 i_2 \cdots i_{nr})} \in A; \tau(i_1 i_2 \cdots i_{kr}) < T; Z_{\tau(i_1 \cdots i_{kr})} \in d\xi\big)}{P^\infty\big(\tau(i_1 i_2 \cdots i_{kr}) < T; Z_{\tau(i_1 i_2 \cdots i_{kr})} \in d\xi\big)},$$

where

$$\tau(i_1 i_2 \cdots i_m) = \inf\{t \colon Z_t \in \Gamma(i_1 i_2 \cdots i_m)\}.$$

LEMMA 11. *There exist constants $C < \infty$, $0 < \beta < 1$ such that for all integers $1 \le n \le k$, all sequences $i_1 i_2 \cdots i_{kr}$, and any $\xi, \zeta \in \Gamma(i_1 i_2 \cdots i_{kr})$,*

$$\|\mu^\xi_{i_1 i_2 \cdots i_{nr}} - \mu^\zeta_{i_1 i_2 \cdots i_{nr}}\| \le C\beta^{k-n}. \tag{8.8}$$

NOTE. $\|\cdot\|$ denotes the total variation norm, which may be characterized as follows. For any two positive, finite measures μ_1, μ_2 on a measurable space $(\Omega, \mathscr{F})$ there exist unique, positive measures λ_0, λ_1, λ_2 with λ_1 and λ_2 mutually singular such that $\mu_1 = \lambda_0 + \lambda_1$ and $\mu_2 = \lambda_0 + \lambda_2$. The total variation distance between μ_1, μ_2 is then $\|\mu_1 - \mu_2\| = \lambda_1(\Omega) + \lambda_2(\Omega)$.

PROOF OF LEMMA 11. Since $\tau(i_1 i_2 \cdots i_{nr}) < \tau(i_1 i_2 \cdots i_{(n+1)r}) < \cdots < \tau(i_1 i_2 \cdots i_{kr})$ [see (g), (j), Section 8A], the strong Markov property implies that

$$\mu^\xi_{i_1 i_2 \cdots i_{nr}}(A) = \int \mu^{\xi'}_{i_1 i_2 \cdots i_{nr}}(A)\, d\mu^\xi_{i_1 i_2 \cdots i_{(n+1)r}}(\xi'),$$

$$\mu^\zeta_{i_1 i_2 \cdots i_{nr}}(A) = \int \mu^{\xi'}_{i_1 i_2 \cdots i_{nr}}(A)\, d\mu^\zeta_{i_1 i_2 \cdots i_{(n+1)r}}(\xi'),$$

where the integrals are over all $\xi' \in \Gamma(i_1 i_2 \cdots i_{(n+1)r})$. Let λ_0, λ_1, λ_2 be the unique, mutually singular, positive measures such that

$$\mu^\xi_{i_1 i_2 \cdots i_{(n+1)r}} = \lambda_0 + \lambda_1,$$

$$\mu^\zeta_{i_1 i_2 \cdots i_{(n+1)r}} = \lambda_0 + \lambda_2;$$

then

$$\mu^\xi_{i_1 i_2 \cdots i_{nr}} = \int \mu^{\xi'}_{i_1 i_2 \cdots i_{nr}}\, d\lambda_0(\xi') + \int \mu^{\xi'}_{i_1 i_2 \cdots i_{nr}}\, d\lambda_1(\xi'),$$

$$\mu^\zeta_{i_1 i_2 \cdots i_{nr}} = \int \mu^{\xi'}_{i_1 i_2 \cdots i_{nr}}\, d\lambda_0(\xi') + \int \mu^{\xi'}_{i_1 i_2 \cdots i_{nr}}\, d\lambda_2(\xi')$$

so

$$\|\mu^{\xi}_{i_1 i_2 \cdots i_{nr}} - \mu^{\zeta}_{i_1 i_2 \cdots i_{nr}}\| = \left\| \int \mu^{\xi'}_{i_1 i_2 \cdots i_{nr}} \, d(\lambda_1 - \lambda_2)(\xi') \right\|$$

$$\le \int \int \|\mu^{\xi'}_{i_1 i_2 \cdots i_{nr}} - \mu^{\xi''}_{i_1 i_2 \cdots i_{nr}}\| \, d\lambda_1(\xi') \, d\lambda_2(\xi'')/\|\lambda_1\|,$$

where the double integral ranges over all $\xi', \xi'' \in \Gamma(i_1 i_2 \cdots i_{(n+1)r})$. Note that $\|\lambda_1\| = \|\lambda_2\| = 1 - \|\lambda_0\|$ because λ_0, λ_1, λ_2 are mutually singular and $\lambda_0 + \lambda_1$, $\lambda_0 + \lambda_2$ are probability measures. Consequently,

$$\|\mu^{\xi}_{i_1 i_2 \cdots i_{nr}} - \mu^{\zeta}_{i_1 i_2 \cdots i_{nr}}\| / \|\mu_{i_1 i_2 \cdots i_{(n+1)r}} - \mu^{\zeta}_{i_1 i_2 \cdots i_{(n+1)r}}\|$$

$$\le \max_{\xi', \xi'' \in \Gamma(i_1 i_2 \cdots i_{(n+1)r})} \tfrac{1}{2} \|\mu^{\xi'}_{i_1 i_2 \cdots i_{nr}} - \mu^{\xi''}_{i_1 i_2 \cdots i_{nr}}\|.$$

Therefore, to prove (8.8), it suffices to show that there exists a constant $\beta < 1$ such that for all $n \ge 1$, all sequences $i_1 i_2 \cdots i_{(n+1)r}$ and all $\xi, \zeta \in \Gamma(i_1 i_2 \cdots i_{(n+1)r})$,

$$\|\mu^{\xi}_{i_1 i_2 \cdots i_{nr}} - \mu^{\zeta}_{i_1 i_2 \cdots i_{nr}}\| \le 2\beta. \tag{8.9}$$

For any sequence $i_1 i_2 \cdots i_m$ and any $z \notin \mathscr{J}$ define a subprobability measure $\lambda^z_{i_1 i_2 \cdots i_m}$ on $\Gamma(i_1 i_2 \cdots i_m)$ by

$$\lambda^z_{i_1 i_2 \cdots i_m}(A) = P^z\big(Z_{\tau(i_1 i_2 \cdots i_m)} \in A; \tau(i_1 i_2 \cdots i_m) < T\big).$$

Then for $\xi \in \Gamma(i_1 i_2 \cdots i_{(n+1)r})$ and $z \in \Gamma(i_1 i_2 \cdots i_{nr})$,

$$\mu^{\xi}_{i_1 i_2 \cdots i_{nr}}(dz) = \frac{\lambda^{\infty}_{i_1 i_2 \cdots i_{nr}}(dz)}{\|\lambda^{\infty}_{i_1 i_2 \cdots i_{nr}}\|} \cdot \frac{\lambda^z_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi)}{\big\{\lambda^{\infty}_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi)/\|\lambda^{\infty}_{i_1 i_2 \cdots i_{nr}}\|\big\}}$$

and

$$\frac{\lambda^{\infty}_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi)}{\|\lambda^{\infty}_{i_1 i_2 \cdots i_{nr}}\|} = \int_{z' \in \Gamma(i_1 i_2 \cdots i_{nr})} \lambda^{z'}_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi) \frac{\lambda^{\infty}_{i_1 i_2 \cdots i_{nr}}(dz')}{\|\lambda^{\infty}_{i_1 i_2 \cdots i_{nr}}\|}.$$

Consequently, to prove (8.9), it suffices to show that there exists $\varepsilon > 0$ such that for all $n \ge 1$, all sequences $i_1 i_2 \cdots i_{(n+1)r}$, all $\xi \in \Gamma(i_1 i_2 \cdots i_{(n+1)r})$ and all $z, z' \in \Gamma(i_1 i_2 \cdots i_{nr})$,

$$\varepsilon \le \frac{\lambda^z_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi)}{\lambda^{z'}_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi)} \le \varepsilon^{-1}. \tag{8.10}$$

Because of the symmetry in z and z', it is enough to establish only the upper bound.

For each $z \in \Gamma(i_1 i_2 \cdots i_{nr})$, define another subprobability measure on $\Gamma(i_1 i_2 \cdots i_{(n+1)r})$ by

$$\tilde{\lambda}_{i_1 i_2 \cdots i_{(n+r)}}(A) = P^z\big(Z_{\tau(i_1 i_2 \cdots i_{(n+1)r})} \in A; \tau(i_1 i_2 \cdots i_{(n+1)r}) < T \wedge \tau_{(n-1)r}\big)$$

(recall that $\tau_m = \inf\{t \colon Z_t \in \Gamma_m\}$). Observe that the event in this definition only involves that part of the path Z_t before its first exit from $\mathscr{D}(i_1 i_2 \cdots i_{(n-1)r})$—this is the point of stopping at $\tau_{(n-1)r}$. Recall that $Q^{(n-1)r}$ is an analytic homeomorphism of $\mathscr{D}(i_1 i_2 \cdots i_{(n-1)r})$ onto $\mathscr{D}$ [statement (i), Section 8A] that maps $\Gamma(i_1 i_2 \cdots i_{nr})$ onto $\Gamma(i_{(n-1)r+1} \cdots i_{nr})$ and $\Gamma(i_1 i_2 \cdots i_{(n+1)r})$ onto $\Gamma(i_{n-1)r+1} \cdots i_{(n+1)r})$. Therefore, by the conformal invariance of Brownian motion,

$$\tilde{\lambda}^{z}_{i_1 i_2 \cdots i_{(n+1)r}}(A) = \tilde{\lambda}^{Q^{(n-1)r}(z)}_{i_{(n-1)r+1} \cdots i_{(n+1)r}}(Q^{(n-1)r}(A)).$$

Since there are only finitely many sequences $i_{(n-1)r+1} \cdots i_{(n+1)r}$, it now follows from Lemma 6 that there are constants $0 < c_1 < c_2 < \infty$ such that for all sequences $i_1 i_2 \cdots i_{(n+1)r}$, all $\xi \in \Gamma(i_1 i_2 \cdots i_{(n+1)r})$ and all $z, z' \in \Gamma(i_1 i_2 \cdots i_{nr})$,

$$c_1 \le \frac{\tilde{\lambda}^{z}_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi)}{\tilde{\lambda}^{z'}_{i_1 i_2 \cdots i_{(n+1)r}}(d\xi)} \le c_2,$$

$$c_1 \le \|\tilde{\lambda}^{z}_{i_1 i_2 \cdots i_{(n+1)r}}\| = P^{z}\{\tau(i_1 i_2 \cdots i_{(n+1)r}) < T \wedge \tau_{(n-1)r}\}.$$

Consider a continuous path from $\Gamma(i_1 i_2 \cdots i_{nr})$ to $\Gamma(i_1 i_2 \cdots i_{(n+1)r})$ that does not intersect $\mathscr{J}$. It may go directly to $\Gamma(i_1 \cdots i_{(n+1)r})$ without hitting $\Gamma_{(n-1)r}$; or it may hit $\Gamma_{(n-1)r}$ first, then return to $\Gamma(i_1 i_2 \cdots i_{nr})$, then go directly to $\Gamma(i_1 \cdots i_{(n+1)r})$; or, in general, it may make $m \ge 0$ cycles between $\Gamma_{(n-1)r}$ and $\Gamma(i_1 i_2 \cdots i_{nr})$, then go directly to $\Gamma(i_1 \cdots i_{(n+1)r})$ [see (g), (h), (j), Section 8A]. Consequently, by the strong Markov property for $z \in \Gamma(i_1 i_2 \cdots i_{nr})$,

$$\lambda^{z}_{i_1 i_2 \cdots i_{(n+1)r}} = \tilde{\lambda}^{z}_{i_1 i_2 \cdots i_{(n+1)r}} + \sum_{m=1}^{\infty} \int_{z' \in \Gamma(i_1 \cdots i_{nr})} \tilde{\lambda}^{z}_{i_1 i_2 \cdots i_{(n+1)r}} \, d\alpha^{z}_{m}(z'),$$

where $\|\alpha^{z}_{m}\| \le (1 - c_1)^m$, by the last inequality of the preceding paragraph. The upper bound in (8.10) now follows directly from the second last inequality of the preceding paragraph, with $\varepsilon^{-1} = c_2 \sum_{m=0}^{\infty} (1 - c_1)^m$. □

Proof of (8.4). Let $i_1 i_2 \cdots i_n$ and $i'_1 i'_2 \cdots i'_{n'}$ be sequences of indices from $\{1, 2, \ldots, d\}$ satisfying $i_j = i'_j \ \forall\ 1 \le j \le 2kr$. [Note that the factor of $2r$ is irrelevant in establishing (8.4).] Let

$$A_1 = \{Z_T \in \mathscr{J}(i_1 i_2 \cdots i_n)\},$$

$$A_2 = \{Z_T \in \mathscr{J}(i'_1 i'_2 \cdots i'_{n'})\};$$

$$\tau = \tau(i_1 i_2 \cdots i_{kr}),$$

$$\tau_* = \tau(i_1 i_2 \cdots i_{2kr}).$$

On each of A_1, A_2, it is the case that $\tau_{kr} \le \tau < \tau_* < T$, by (g), (h), (j), (k) of

Section 8A. Since $\mathscr{H}_T \subset \mathscr{G}_{\tau_* \wedge T} \subset \mathscr{G}_{\tau \wedge T}$ and $F_i^{kr} \in \mathscr{F}_{\tau \wedge T}$, Lemma 9 implies

$$\begin{aligned}
&P^\infty(F_i^{kr}|\mathscr{H}_T)1_{A_j}\\
&\quad = E^\infty(P^\infty(F_i^{kr}1\{\tau < T\}|\mathscr{G}_{\tau \wedge T})|\mathscr{H}_T)1_{A_j}\\
&\quad = E^\infty(P^\infty(F_i^{kr}|\mathscr{H}_{\tau \wedge T})1\{\tau < T\}|\mathscr{H}_T)1_{A_j}\\
&\quad = E^\infty\Big(E^\infty(P^\infty(F_i^{kr}|\mathscr{H}_{\tau \wedge T})1\{\tau < T\}|\mathscr{G}_{\tau_* \wedge T})1\{\tau_* < T\}|\mathscr{H}_T\Big)1_{A_j}\\
&\quad = E^\infty\Big(E^\infty(P^\infty(F_i^{kr}|\mathscr{H}_{\tau \wedge T})1\{\tau < T\}|\mathscr{H}_{\tau_* \wedge T})1\{\tau_* < T\}|\mathscr{H}_T\Big)1_{A_j}.
\end{aligned}$$

On the event $\{\tau < T\}$, $P^\infty(F_i^{kr}|\mathscr{H}_{\tau \wedge T})$ is a function of Z_τ, which we will denote $f_i(Z_\tau)$. Thus

$$P^\infty(F_i^{kr}|\mathscr{H}_T)1_{A_j} = E^\infty\left(\int f_i(z)\, d\mu_{i_1 i_2 \cdots i_{kr}}^{Z_{\tau_*}}(z)1_{A_j}|\mathscr{H}_T\right),$$

where the integral is over all $z \in \Gamma(i_1 i_2 \cdots i_{kr})$. Lemmas 10–11 now imply that for any $\zeta \in \Gamma(i_1 i_2 \cdots i_{2kr})$,

$$\left|\left\{P^\infty(F_i|\mathscr{H}_T) - \int f_i\, d\mu_{i_1 i_2 \cdots i_{kr}}^{\zeta}\right\}1_{A_j}\right| \le C\beta^k, \qquad j = 1, 2$$

for suitable constants $C < \infty$, $0 < \beta < 1$. But

$$P^\infty(F_i|A_j) = E^\infty\big(P^\infty(F_i|\mathscr{H}_T)1_{A_j}\big)/P^\infty(A_j),$$

so by Lemma 7 and the preceding inequality,

$$\left|\log\left\{\frac{P^\infty(F_i|A_1)}{P^\infty(F_i|A_2)}\right\}\right| \le C\beta^k$$

for appropriate constants $C < \infty$, $0 < \beta < 1$. □

9. The neutral case. Assume now that ∞ is a neutral fixed point of Q and that $\infty \in \mathscr{F}$. Then the connected component $\mathscr{F}_\infty$ of $\mathscr{F}$ containing ∞ is a Siegel disk—see [1], Section 7 (the other four possibilities of [1], Section 7 are impossible). This means that there exists a surjective, analytic homeomorphism φ: $D_R \to \mathscr{F}_\infty$ (here $D_R = \{z \in \mathbb{C}: |z| < R\}$) and an irrational $\theta \in (0, 1)$ such that

$$\varphi(e^{2\pi i\theta}z) = Q(\varphi(z)) \qquad \forall\, z \in D_R.$$

This implies that $\varphi(0) = \infty$, that Q: $\mathscr{F}_\infty \to \mathscr{F}_\infty$ is 1-to-1 and that ∞ is the only periodic orbit in $\mathscr{F}_\infty$. Moreover, $\mathscr{F}_\infty \ne \mathscr{F}$, because Q: $\mathscr{F} \to \mathscr{F}$ is d-to-1, and we have assumed that $d \ge 2$. It is not necessarily the case that φ extends continuously to $\overline{D}_R$.

Let Z_t, $t \ge 0$, be a Brownian motion started at $Z_0 = \infty$ under P^∞ and let $T = \inf\{t\colon Z_t \in \mathscr{J}\}$. Since $\mathscr{F}_\infty \ne \mathscr{F}$, there is a nonempty, open disk $D \subset \mathscr{F} \setminus \mathscr{F}_\infty$. Brownian motion on $\overline{\mathbb{C}}$ is recurrent, so it must visit each open disk, with probability 1, hence $Z_t \in D$ for some $t < \infty$. But a continuous path from ∞ to

D must intersect $\mathscr{J}$, otherwise $D \subset \mathscr{F}_\infty$. This proves that

$$P^\infty\{T < \infty\} = 1,$$

so the logarithmic capacity of $\mathscr{J}$ is positive and ν is the distribution of Z_T under P^∞. The same arguments as used in the proof of Proposition 9 now show that $Z_T, Q(Z_T), Q^2(Z_T), \ldots$ is a stationary process, that is, that ν is an invariant measure. However, the arguments of Proposition 11 no longer apply.

Next, we must consider the boundary values of φ. As before, let D be a nonempty, open disk contained in $\mathscr{F} \setminus \mathscr{F}_\infty$, with center ζ. Let ψ be a linear fractional transformation such that $\psi(\zeta) = \infty$. Then $\psi \circ \varphi\colon D_R \to \mathbb{C}$ is a bounded analytic function, so by a well-known theorem of Fatou, $\psi \circ \varphi$ has radial limits a.e. Consequently, φ has radial limits a.e. Thus, φ extends to a (not necessarily continuous) function $\varphi\colon \overline{D}_R \to \overline{\mathbb{C}}$ such that $\lim_{r \uparrow R} \varphi(re^{i\alpha}) = \varphi(Re^{q\alpha})$ for a.e. $\alpha \in [0, 2\pi)$ (with respect to Lebesgue measure on $[0, 2\pi)$).

Let $\tilde{Z}_t$, $t \geq 0$, be a Brownian motion started at $\tilde{Z}_0 = 0$ under $\tilde{P}^0$ and let $\tilde{T}_r = \inf\{t\colon |\tilde{Z}_t| = r\}$. Then $\tilde{P}^0\{\tilde{T}_r < \infty\} = 1$, and under $\tilde{P}^0$, the distribution of $\tilde{Z}_{\tilde{T}_r}$ is the uniform distribution (Lebesgue measure) on $\{z\colon |z| = r\}$. The process $\psi \circ \varphi(\tilde{Z}_{\tilde{T}_{R-1/n}})_{n \geq 1}$ is a bounded, discrete time martingale under $\tilde{P}^0$, hence has a limit as $t \to \infty$ almost surely. An easy argument using the Poisson integral formula ([5], Section 5.2) shows that

$$\lim_{n \to \infty} \psi \circ \varphi\left(\tilde{Z}_{\tilde{T}_{R-1/n}}\right) = \psi \circ \varphi\left(\tilde{Z}_{\tilde{T}_R}\right) \text{ a.s. } (\tilde{P}^0)$$

$$\Rightarrow \lim_{n \to \infty} \varphi\left(\tilde{Z}_{\tilde{T}_{R-1/n}}\right) = \varphi\left(\tilde{Z}_{\tilde{T}_R}\right) \text{ a.s. } (\tilde{P}^0),$$

that is, the Brownian limit is the same as the radial limit.

Lévy's conformal invariance theorem implies that $\varphi(\tilde{Z}_t)$, $0 \leq t < \tilde{T}_R$, is a (time-changed) Brownian motion in $\mathscr{F}_\infty$ started at ∞. Clearly, $\lim_{t \uparrow \tilde{T}} \varphi(\tilde{Z}_t) \in \partial\mathscr{F}_\infty \subset \mathscr{J}$, so it follows that $\varphi(\tilde{Z}_{\tilde{T}_R})$ has the same distribution as Z_T under P^∞, namely ν. Now

$$\begin{aligned} Q\left(\varphi\left(\tilde{Z}_{\tilde{T}_R}\right)\right) &= \lim_{n \to \infty} Q\left(\varphi\left(\tilde{Z}_{\tilde{T}_{R-1/n}}\right)\right) \\ &= \lim_{n \to \infty} \varphi\left(e^{2\pi i\theta}\tilde{Z}_{\tilde{T}_{R-1/n}}\right) \\ &= \varphi\left(e^{2\pi i\theta}\tilde{Z}_{\tilde{T}_R}\right). \end{aligned}$$

Thus, $(\mathscr{J}, Q, \nu)$ is a factor of $(\partial D_R, M_{2\pi i\theta}, \lambda)$ where M_α is rotation by α and λ is (normalized) Lebesgue measure. (The conjugating map is φ restricted to ∂D_R.)

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