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SLOW POINTS IN THE SUPPORT OF HISTORICAL BROWNIAN MOTION[1]

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A slow point from the left for Brownian motion is a time during a given interval for which the oscillations of the path immediately to the left of this time are smaller than the typical ones, that is, those given by the local LIL. These slow points occur at random times during a given interval. For historical super-Brownian motion, the support at a fixed time contains an infinite collection of paths. This paper makes use of a branching process description of the support to investigate the slowness of these paths at the fixed time. The upper function found is the same as that found for slow points in the Brownian motion case.

1. Introduction. Consider the following branching Brownian motion, that comes from the study of superprocesses. Choose a random time θ_0 uniformly on $[0,1]$ and run a Brownian motion started at the origin up until θ_0. At this time, the process splits into two particles. Each of these moves as an independent Brownian motion started from the splitting point. These two particles split at independent random times distributed uniformly on $[\theta_0, 1]$. This splitting procedure is repeated ad infinitum. The separate paths and all the splitting times are independent of each other. In the limit, at time 1 there are uncountably many paths. This suggests that some of the paths may exhibit behavior that would not occur at a fixed time for Brownian motion. This paper studies the existence of slow paths, that is, paths that have unusually small oscillations from the left, at time 1. A typical path of a Brownian motion will have analogous slow points at random times in an interval. The results here indicate a relationship between exceptional times for Brownian motion and exceptional paths for this branching process.

To state the main results of this paper, we formalize the foregoing construction using the notion of a marked tree following Le Gall [7]. Let $\mathscr{B} = \{(0,\beta(1),\dots,\beta(n))\}$, with $n \geq 0$ and $\beta(i) \in \{0,1\}$, be the set of all finite sequences of zeroes and ones with first term 0, and let $\mathscr{B}_\infty = \{0, \beta(1), \beta(2), \cdots\}$ be the set of such infinite sequences. Define a marked tree to be an element of

$$\Omega = \{[0,1], \mathscr{C}([0,1],\mathbb{R})\}^{\mathscr{B}}.$$

A generic element of Ω will be denoted by

$$w = (\sigma(\beta), \psi(\beta)) \quad \text{or} \quad (\sigma_\beta, \psi_\beta).$$

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If $\beta = (0, \beta(1), \beta(2), \dots, \beta(m))$ [or $\beta = (0, \beta(1), \beta(2), \dots,)$], define the projection map for $n \le m$ ($n < \infty$) by

$$\beta_n = (0, \beta(1), \beta(2), \dots, \beta(n))$$

and set $\beta_0 = (0)$. For $\beta = (0, \beta(1), \dots, \beta(m))$, set $\delta(\beta) = \sigma_{\beta_0}\sigma_{\beta_1} \cdots \sigma_{\beta_m}$, and define W_t^β, $t \in [0,1]$, by

$$W_t^\beta = \begin{cases} \psi_{\beta_0}(t), & 0 \le t \le 1 - \delta(\beta_0), \\ \psi_{\beta_1}(t - (1 - \delta(\beta_0))) + W^\beta_{1-\delta(\beta_0)}, & 1 - \delta(\beta_0) < t \le 1 - \delta(\beta_1), \\ \cdots & \\ \psi_{\beta_m}(t - (1 - \delta(\beta_{m-1}))) + W^\beta_{1-\delta(\beta_{m-1})}, & 1 - \delta(\beta_{m-1}) < t \le 1 - \delta(\beta_m), \\ W^\beta_{1-\delta(\beta_m)}, & 1 - \delta(\beta_m) < t \le 1. \end{cases}$$

By [7] there exists a measure on Ω for which the following hold:

1. All the random elements $\{\sigma(\beta)\}_{\beta \in \mathscr{B}}$ and $\{\psi(\beta)\}_{\beta \in \mathscr{B}}$ are independent. The random variables $\{\sigma(\beta)\}$ are i.i.d. with uniform distribution on $[0,1]$ and $\{\psi(\beta)\}$ are i.i.d. with distribution like a Brownian motion started from 0.
2. Uniformly in $\beta \in \mathscr{B}_\infty$ and $t \in [0,1]$ the following limit exists almost surely:

$$W_t^\beta = \lim_{n \to \infty} W_t^{\beta_n}.$$

By construction we have the following properties almost surely.

1. For each $\beta \in \mathscr{B}_\infty$, W_t^β is distributed like a standard Brownian motion, independent of $\{\sigma(\gamma): \gamma \in \mathscr{B}\}$ (i.e., independent of the branching).
2. If $\beta_n = \gamma_n$ for an $n \ge 0$, then $W_t^\beta = W_t^\gamma$ for $0 \le t \le 1 - \delta(\beta_n)$.
3. If $\beta_n = \gamma_n$, but $\beta_{n+1} \ne \gamma_{n+1}$, then $W_t^\gamma - W^\gamma_{1-\delta(\gamma_n)}$ is conditionally independent of $W_t^\beta - W^\beta_{1-\delta(\beta_n)}$ for $t \in [1 - \delta(\beta_n), 1]$ given $\delta(\beta_n)$.

We prove the following theorem.

THEOREM 1.1. *For* $g(t) = t^{1/2}$,

$$\inf_\beta \limsup_{t \uparrow 1} \frac{|W_t^\beta - W_1^\beta|}{g(1-t)} = 1.$$

In [7] it is shown that the branching Brownian motion defined above gives a representation of the support of historical Brownian motion at time 1. If we set $W = \{W_\cdot^\beta: \beta \in \mathscr{B}_\infty\}$, then the support of historical Brownian motion at time 1 is found by taking a union of a finite, Poisson distributed number of independent copies of W. With this in mind, Theorem 1.1 says if we look at all the functions in the support of historical Brownian motion at time 1, then the function with the slowest point at time 1 has $t^{1/2}$ as an upper function.

A corresponding fast-point theorem may be formulated as follows.

THEOREM 1.2. *For* $h(t) = (2t \log 1/t)^{1/2}$,

$$1 \le \limsup_{t \uparrow 1} \sup_{\beta} \frac{W_t^\beta - W_1^\beta}{h(1-t)} \le \sup_{\beta} \limsup_{t \uparrow 1} \frac{W_t^\beta - W_1^\beta}{h(1-t)} \le 1.$$

This theorem has been established implicitly by Dawson and Perkins [6] improving upon the work in [5]. The theorem in [6] is for all time, not a fixed time. Their proof uses the familiar branching Brownian motion approximation to the superprocess. That is, a branching process with zero or two offspring at each branching time, and identically distributed interbranch times. They get an estimate on the variation of the paths in the branching process between two close time points in terms of the function $2^{1/2}h$, which is independent of the parameter controlling the branching rate. From here, to establish the characterization of the paths in the historical Brownian superprocess, nonstandard analysis is employed. With similar techniques, but applied to the branching picture described in the Introduction, one can prove Theorem 1.2 without the use of nonstandard analysis.

It is of interest here that the normalizing functions $g(t)$ and $h(t)$ correspond to the normalizing functions needed to answer the same questions for Brownian motion over an interval. That is, for X_t a standard Brownian motion, almost surely,

$$\sup_{r \in [0,1]} \limsup_{t \uparrow r} \frac{X_t - X_r}{h(r-t)} = 1,$$

$$\inf_{r \in [0,1]} \limsup_{t \uparrow r} \frac{|X_t - X_r|}{g(r-t)} = 1.$$

(cf. [9] and [4]).

It would be interesting to know how far this extends.

PROBLEM 1.1. Let X_t be a standard Brownian motion. Find a description of the set $\mathscr{G}$ for which $P(\exists t \in [0,1]$: the path $\{X_s : 0 \le s \le t\} \in \mathscr{G}) = 1$ if and only if $P(\exists \beta$: the path $\{W_s^\beta : 0 \le s \le 1\} \in \mathscr{G}) = 1$.

Outline. The paper is arranged as follows. The preliminary section contains a few general facts about this branching process are collected. The second section handles the slow-point problem. The key idea is to take an approach from [4] and modify it to fit the model at hand. The basic idea is to pick out a subtree of the tree at time t (those branches that are not too wiggly compared to how much their offspring will spread apart) and then to count the number of branches there are for this subtree. For the lower bound a calculation involving expectations suffices. To get the upper bound, an approach of Bramson [2] is used to get a handle on the dependencies of different branches. The problem reduces to estimating the variance of the number of paths with a certain property. The key formula is from Sawyer [10]; it allows one to reduce the calculation to one involving a branching picture with just a single branch time.

2. Preliminaries. Let N_t be the number of branches at time t. Notice, N_t is dependent only on the branching structure of W. The set $\mathscr{B}_\infty$ is naturally partitioned into those branches that have the same ancestor at time t. Let $\pi_t(\beta) = \inf\{n \geq 0: t \leq 1 - \delta(\beta_n)\}$. For each t, define the equivalence relation on $\mathscr{B}_\infty$ as follows $\beta \sim_t \gamma \Leftrightarrow \pi_t(\beta) = \pi_t(\gamma)$. Denote by $[\beta]_t$ the equivalence class of β, and write $\gamma \in [\beta]_t$ to indicate that $\gamma \sim_t \beta$. When a labeling of the branches at time t is necessary, we assign a random ordering independent of W to the N_t branches. Let $[\beta]_t(i)$ be the ith equivalence class corresponding to this ordering. For a set A, let $\#A$ denote its cardinality. Constants whose values are unimportant will be generically denoted by a k_i, for some i.

Recall that a Yule process is the name given the model of binary fission, where a particle splits into two after an exponentially distributed amount of time.

LEMMA 2.1. *Let $Z_t^\beta = W^\beta_{1-\exp(-t)}$ and $\hat{N}_t = N_{1-\exp(-t)}$. Then $\hat{N}_t$ is a Yule process and Z_t^β is a branching diffusion with diffusion part a time change of Brownian motion and branching structure of the Yule process.*

PROOF. It suffices to show that $\hat{N}_t$ is a Yule process, that is, the branch times are independent and identically distributed exponential random variables with parameter 1. The branch times for W have the structure $1 - \sigma_{\beta_0}\sigma_{\beta_1} \cdots \sigma_{\beta_m}$, where each σ_{β_i} is an i.i.d. uniform random variable on $[0,1]$. Thus the interbranch times for $\hat{N}$ have distribution

$$\log(\sigma_{\beta_0}\sigma_{\beta_1} \cdots \sigma_{\beta_m}) - \log(\sigma_{\beta_0}\sigma_{\beta_1} \cdots \sigma_{\beta_{m+1}}) = \log(1/\sigma_{\beta_{m+1}}),$$

but

$$P(\log 1/\sigma \geq t) = P(\sigma \leq e^{-t}) = e^{-t}.$$

Hence, the interbranch times are i.i.d. exponentials with the correct parameter. □

LEMMA 2.2. $E(s^{N_{1-t}}) = st/(1 - s(1 - t))$, *in particular* $E(N_{1-t}) = 1/t$.

PROOF. The easiest proof is that $\hat{N}_t$ is a Yule process, for which the moment generating function is well known (cf. [1]).

An alternative proof comes from the construction of the given branching process in terms of a Brownian excursion ζ_s from the origin in [7]. The branching structure arises from the binary tree structure constructed from a Brownian excursion by Neveu and Pitman [8].

To summarize, branch times for the branching process correspond to downward excursions from height 1, of a Brownian excursion from the origin. The maximum depth of an excursion determines a branch time. We have N_{1-t}, the number of branches for the branching Brownian motion at time $1 - t$, is equal in distribution to the number of excursions of ζ_s from 1 that reach a level $1 - t$. Let $p = 1 - t$ be the probability an excursion from 1 misses 0 given that it hits a level $1 - t$. Because all such excursions are independent, we have that the number of branches at time t for the branching Brownian motion, N_{1-t}, has

$$P(N_{1-t} = j) = p^{j-1}(1-p), \qquad j = 1, 2, \dots .$$

From this the moment generating function and the expectation are easy calculations. □

For the next two lemmas, let Y_t be a branching Brownian motion with branch times independent of the spatial motion. Let N_t be the number of particles at time t. The following useful lemma is well known in branching process theory.

LEMMA 2.3. *Write $Y_t = \{y_t^1, \dots, y_t^{N_t}\}$, where y_t^i is an element of $C([0,t], R)$ and let $\delta(x)$ designate point mass at x. Define the measure*

$$\nu_t = \sum_{i=1}^{N_t} \delta(y_t^i).$$

Then, for B a Borel subset of $C([0,t], R)$,

$$E\langle \nu_t, 1_B \rangle = EN_t P(y_t \in B).$$

PROOF. We have

$$\begin{aligned} E\langle \nu_t, 1_B \rangle &= E\left(\sum_{i=1}^{N_t} \langle \delta(y_t^i), 1_B \rangle \right) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{n} E(\langle \delta(cy_t^i), 1_B \rangle | N_t = n) P(N_t = n) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(y_t^i \in B) P(N_t = n) \\ &= \sum_{n=1}^{\infty} nP(N_t = n) P(y_t \in B) \\ &= EN_t P(y_t \in B). \end{aligned}$$

□

This next lemma says that n independent Brownian motions will spread more than a branching Brownian motion conditioned to have n branches.

LEMMA 2.4. *For X_t a Brownian motion independent of Y,*

$$E\left(P(|X_t - X_0| < c)^{N_t}\right) \le P\left(\sup_\beta |Y_t^\beta - Y_0^\beta| < c\right). \tag{2.1}$$

PROOF. The lemma will follow if it can be established that for all α,

$$P(|X_t - X_0 + \alpha| < c)^{N_t} \le P\left(\sup_\beta |Y_t^\beta - Y_0^\beta + \alpha| < c | N_t\right). \tag{2.2}$$

Proceed by induction on $\{N_t = n\}$. If $n = 1$, no branching occurs so this is a triviality because Y^β has distribution like a Brownian motion. Assume for all $k < n$ the lemma is true. Let $N_t^0(N_t^1)$ be the number of branches that have $\beta_1 = (0,0)$ [$\beta_1 = (0,1)$]. Then by looking at the two distinct family lines determined by the first branch, at time $1 - \delta_0$, [i.e., $\beta_1 = (0,0)$ or $\beta_1 = (0,1)$] we can employ their conditional independence to reduce the calculations to two branching processes with $N_t^j < n$ branches:

$$
\begin{aligned}
&P\Big(\sup_\beta |Y_t^\beta - Y_0^\beta + \alpha| < c | N_t = n\Big) \\
&= E\Big(P\Big(\sup_\beta |Y_t^\beta - Y_0^\beta + \alpha| < c | N_t = n, 1 - \delta_0\Big)\Big| N_t = n\Big) \\
&= E\Bigg(\sum_{i=1}^{n-1} P\big(N_t^1 = i | N_t = n, 1 - \delta_0\big) \\
&\qquad \times P\Big(\sup_{\beta_1 = (0,0)} |Y_t^\beta - Y_0^\beta + \alpha| < c, \\
&\qquad\qquad \sup_{\beta_1 = (0,1)} |Y_t^\beta - Y_0^\beta + \alpha| < c | N_t = n, 1 - \delta_0, N_t^1 = i\Big)\Big| N_t = n\Bigg) \\
&= E\Bigg(\sum_{i=1}^{n-1} P\big(N_t^1 = i | N_t = n, 1 - \delta_0\big) \\
&\qquad \times P\Big(\sup_{\beta_1 = (0,0)} |Y_t^\beta - Y_{1-\delta_0}^\beta + Y_{1-\delta_0}^\beta - Y_0^\beta + \alpha| < c, \\
&\qquad\qquad \sup_{\beta_1 = (0,1)} |Y_t^\beta - Y_{1-\delta_0}^\beta + Y_{1-\delta_0}^\beta - Y_0^\beta + \alpha| < c \\
&\qquad\qquad\qquad \times | N_t = n, 1 - \delta_0, N_t^1 = i\Big)\Big| N_t = n\Bigg) \\
&= E\Bigg(\sum_{i=1}^{n-1} P\big(N_t^1 = i | N_t = n, 1 - \delta_0\big) \\
&\qquad \times P\Big(\sup_{\beta_1 = (0,0)} |Y_t^\beta - Y_{1-\delta_0}^\beta + X_{1-\delta_0} - X_0 + \alpha| < c, \\
&\qquad\qquad \sup_{\beta_1 = (0,1)} |Y_t^\beta - Y_{1-\delta_0}^\beta + X_{1-\delta_0} - X_0 + \alpha| < c \\
&\qquad\qquad\qquad \times | N_t = n, 1 - \delta_0, N_t^1 = i\Big)\Big| N_t = n\Bigg).
\end{aligned}
$$

However, the two branching processes are conditionally independent, so we have that the above expression is equal to

$$
\begin{aligned}
&E\Bigg(\sum_{i=1}^{n-1} P(N_t^1 = i | N_t = n, 1-\delta_0) \\
&\qquad \times P\Big(\sup_{\beta_1=(0,0)} |Y_t^\beta - Y_{1-\delta_0}^\beta + X_{1-\delta_0} - X_0 + \alpha| < c | N_t^0 = n-i, 1-\delta_0\Big) \\
&\qquad \times P\Big(\sup_{\beta_1=(0,1)} |Y_t^\beta - Y_{1-\delta_0}^\beta + X_{1-\delta_0} - X_0 + \alpha| < c \\
&\qquad\qquad\qquad \times | N_t{}^1 = i, 1-\delta_0 | N_t = n\Big) \\
&\geq E\Bigg(\sum_{i=1}^{n-1} P(N_t^1 = i | N_t = n, 1-\delta_0) \qquad (2.3)\\
&\qquad \times P(|X_t - X_{1-\delta_0} + X_{1-\delta_0} - X_0 + \alpha| < c)^{n-i} \\
&\qquad \times P(|X_t - X_1 - \delta_0 + X_{1-\delta_0} - X_0 + \alpha| < c)^{i} | N_t = n\Bigg) \\
&= E\big(P(|X_t - X_0 + \alpha| < c)^n | N_t = n\big)
\end{aligned}
$$

Inequality (2.3) follows from the induction assumption, because given the time $1-\delta_0$, the two processes determined by this first splitting are both branching processes. □

3. Slow Points. Define $m_t = \sup_\beta \sup_{s\in[0,t]}(W_s^\beta - W_0^\beta)$.

LEMMA 3.1. *For all $k \geq 0$, $E(m_1^k) < \infty$.*

PROOF. Let $\theta = 1/2$, $\lambda > 1$, X_t be a Brownian motion independent of W, $x_n = ((n+1)\theta^n \log 2)^{1/2}$ and $s_n = P(\sup_{t\in[1-\theta^n, 1-\theta^{n+1}]}|X_t - X_{1-\theta^n}| \leq \lambda x_n)$. Set $L = \sum_{n=0}^\infty x_n < \infty$. Finally let $N_t^s(i)$ be the number of offspring the ith particle at time s has during $[s,t]$. Then

$$
\begin{aligned}
&P(m_1 > \lambda L) \\
&\leq P\Bigg(\bigcup_{n=0}^{\infty}\Big\{\sup_\beta \sup_{t\in[1-\theta^n, 1-\theta^{n+1}]} |W_t^\beta - W_{1-\theta^n}^\beta| > \lambda x_n\Big\}\Bigg) \\
&\leq \sum_{n=0}^\infty P\Big(\sup_\beta \sup_{t\in[1-\theta^n, 1-\theta^{n+1}]} |W_t^\beta - W_{1-\theta^n}^\beta| > \lambda x_n\Big) \\
&\leq \sum_{n=0}^\infty 1 - E\Bigg(\prod_{i=1}^{N_{1-\theta^n}} P\Big(\sup_{\beta\in[\beta]_{1-\theta^n}(i)} \sup_{t\in[1-\theta^n, 1-\theta^{n+1}]} |W_t^\beta - W_{1-\theta^n}^\beta| \leq \lambda x_n\Big)\Bigg) \qquad (3.1)\\
&\leq \sum_{n=0}^\infty 1 - E\Bigg(\prod_{i=1}^{N_{1-\theta^n}} P\Big(\sup_{t\in[1-\theta^n, 1-\theta^{n+1}]} |X_t - X_{1-\theta^n}| \leq \lambda x_n\Big)^{N_{1-\theta^{n+1}}^{1-\theta^n}(i)}\Bigg)
\end{aligned}
$$

(3.2)

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} 1 - E\left(s_n^{N_{1-\theta^{n+1}}}\right) \\
&= \sum_{n=0}^{\infty} 1 - \left(s_n \theta^{n+1}\right)/\left(1 - s_n(2 - \theta^{n+1})\right) \\
&\le \sum_{n=0}^{\infty} (1 - s_n)/\left(s_n \theta^{n+1}\right) \\
&\le \sum_{n=0}^{\infty} k_1 \exp\left(-(2\lambda^2(n+1)\log 2)/2\right)\theta^{-(n+1)} \\
&= \sum_{n=0}^{\infty} k_1 (1/2)^{(\lambda^2-1)(n+1)} \\
&= k_2 (1/2)^{\lambda^2 - 1}.
\end{aligned}
$$

Equation (3.1) comes from Lemma 2.4. From this it is clear that for $k < \infty$ we have $\int_0^\infty x^k P(m_1 > xL)\, dx < \infty$, and from this the lemma follows. □

REMARK 1. In fact even more can be said. Let

$$b_n = \left\{ \sup_{\beta} \sup_{y \in [1-\theta^n, 1-\theta^{n+1}]} |W_t^\beta - W_{1-\theta^n}^\beta| > 2x_n \right\}.$$

Then by letting $\lambda = 2$, we can see from the last calculation that

$$P(b_n \text{ i.o.}) = 0$$

by the Borel–Cantelli lemma. Because x_n sums, this has the consequence that

$$\lim_{t \uparrow 1} \sup_{\beta} \sup_{s \in [t,1]} |W_s^\beta - W_1^\beta| = 0. \tag{3.3}$$

For fixed β, s, define M_1 and M_2 by

$$
\begin{aligned}
M_1(\beta, s) &= \sup_{\gamma \in [\beta]_s} \sup_{t \in [s,1]} (W_t^\gamma - W_s^\gamma), \\
M_2(\beta, s) &= -\inf_{\gamma \in [\beta]_s} \inf_{t \in [s,1]} (W_t^\gamma - W_s^\gamma).
\end{aligned}
$$

By (3.3), $M_i(\beta, s) \downarrow 0$ as $s \uparrow 1$ uniformly in β. By scaling, $M_1(\beta, s)$ is distributed like $m_1(1-s)^{1/2}$. Notice for $i = 1$ or 2, $M_i(\beta, 0)$ does not depend on β; in this case we write $M_i = M_i(\beta, 0)$. M_i is distributed like m_1.

For $c > 0$, $n > 0$, define

$$
\begin{aligned}
\tau_c(\beta, s) = \sup\{t < s : W_t^\beta - W_s^\beta &= M_1(\beta, s) + c(1-t)^{1/2} \text{ or} \\
W_t^\beta - W_s^\beta &= -M_2(\beta, s) - c(1-t)^{1/2}\}.
\end{aligned}
$$

If the set above is empty, set $\tau_c(\beta, s) = 0$. The picture to keep in mind is a Brownian broom. Fix a β. Then the part of the branching process between s and 1 that corresponds to the offspring line for β is the fan of the broom, and the handle is the path W_t^β for $0 \le t \le s$. When $\tau_c(\beta, s) = 0$, the interpretation is that the handle is not too crooked compared to the size of the fan.

Let X_t be a standard Brownian motion independent of W and define

$$T_c = \inf\{t > 1: X_t - X_1 = M_1 + ct^{1/2} \text{ or } X_t - X_1 = -M_2 - ct^{1/2}\}.$$

By Brownian scaling and time reversal, we have for $r \in [0, t]$,

$$P(\tau_c(\beta, t) \ge r) = P\left(T_c \ge \frac{1-r}{1-t}\right). \tag{3.4}$$

LEMMA 3.2. *If $c < 1$, then $E(T_c) < \infty$; hence, $nP(T_c > n) \to 0$ as $n \to \infty$.*

PROOF. Let $M = \max(M_1, M_2)$. If

$$S_c = \inf\{t \ge 0: |X_{1+t} - X_1| = M + c(1+t)^{1/2}\},$$

then $T_c \le S_c + 1$. It suffices then to show S_c has a finite expectation. Let $c < 1$ be fixed and drop the dependence on c in the notation. Set $S_n = S \wedge n$. S_n is a bounded stopping time for the process $\{Y_t = X_{1+t} - X_1: t \ge 0\}$, a Brownian motion. Hence, by optional sampling and Hölder's inequality,

$$\begin{aligned} E(S_n) &= E\left(Y_{S_n}^2\right) \\ &\le E(M^2) + 2cE(M(S_n+1))^{1/2} + c^2E(S_n+1) \\ &\le E(M^2) + 2c(E(M^2))^{1/2}(E(S_n+1))^{1/2} + c^2E(S_n+1). \end{aligned}$$

So

$$(1-c^2)\frac{(E(S_n))}{(E(S_n+1))^{1/2}} \le \frac{E(M^2)+c^2}{(E(S_n+1))^{1/2}} + 2c(E(M^2))^{1/2}.$$

By Lemma 3.1, $E(M^2) < \infty$; hence, by monotone convergence the lemma follows by letting $n \to \infty$. □

Set

$$\begin{aligned} A_s(c) &= \{\beta: \forall r \in [0,s], -M_2(\beta, s) - c(1-r)^{1/2} \\ &\qquad < W_r^\beta - W_s^\beta < M_1(\beta, s) + c(1-r)^{1/2}\} \\ &= \{\beta: \tau_c(\beta, s) = 0\}, \\ B_s(c) &= \{[\beta]_s: \beta \in A_s\}. \end{aligned}$$

The Brownian-broom scheme is chosen to ensure monotonicity of the sets A_s. That is, for $s < t$, $A_t \subset A_s$.

LEMMA 3.3. *If* $c < 1$, *then* $P(\bigcap_{n \geq 1} A_{1-1/n}(c) \neq \varnothing) = 0$.

PROOF. Because $\{A_{1-1/n}(c)\}_{n=1}^{\infty}$ is a decreasing sequence, we have

$$
\begin{aligned}
P\Big(\bigcap_{n \geq 1} A_{1-1/n}(c) \neq \varnothing\Big) &= \lim_{n \to \infty} P\big(A_{1-1/n}(c) \neq \varnothing\big) \\
&\leq \lim_{n \to \infty} E(\# B_{1-1/n}) \\
&= \lim_{n \to \infty} E(N_{1-1/n}) P\big([\,\beta\,]_{1-1/n} \in B_{1-1/n}\big) \\
&= \lim_{n \to \infty} nP(\tau_c(\beta, 1 - 1/n) = 0) \\
&= \lim_{n \to \infty} nP(T_c > n) \qquad (3.5) \\
&= 0.
\end{aligned}
$$

T_c is from Lemma 3.2. Line (3.5) follows from (3.4) and Lemma 3.2. □

THEOREM 3.1. *Almost surely,*

$$\inf_{\beta} \limsup_{t \uparrow 1} \frac{|W_t^\beta - W_1^\beta|}{(1-t)^{1/2}} \geq 1.$$

PROOF. Suppose there exists a $d < 1$ for which there exists a β with positive probability such that

$$\limsup_{t \uparrow 1} \frac{|W_t^\beta - W_1^\beta|}{(1-t)^{1/2}} < d. \qquad (3.6)$$

Thus with positive probability there exists a random $\varepsilon > 0$ and β such that

$$|W_t^\beta - W_1^\beta| < d(1-t)^{1/2} \quad \text{for all } t \in [1-\varepsilon, 1].$$

Choose n so that $P(\varepsilon > 1/n) > 0$. With positive probability there exists β for which

$$|W_t^\beta - W_1^\beta| < d(1-t)^{1/2} \quad \text{for all } t \in [1 - 1/n, 1].$$

However, W^β is distributed like a Brownian motion; thus,

$$
\begin{aligned}
P^0\big(|W_t^\beta - W_1^\beta| < d(1-t)^{1/2} \quad &\text{for all } t \in [0, 1 - 1/n] \mid \\
|W_t^\beta - W_1^\beta| < d(1-t)^{1/2} \quad &\text{for all } t \in [1 - 1/n, 1]\big) > 0,
\end{aligned}
$$

because the part of a path between $[1 - 1/n, 1]$ is independent of the part between $[0, 1 - 1/n]$, given the value at time $1 - 1/n$. We conclude then that with positive probability there will be a β for which

$$|W_t^\beta - W_1^\beta| < d(1-t)^{1/2} \quad \text{for } 0 \leq t \leq 1.$$

However, by Lemma 3.3, with probability 1, for all β there exists a $t \in (0,1)$ for which

$$|W_t^\beta - W_1^\beta| = d(1-t)^{1/2}.$$

Thus no such $d < 1$ exists that satisfies (3.6). □

LEMMA 3.4. *For $c > 1$ there exists $\varepsilon(c) > 0$ such that $\varepsilon(c) \downarrow 0$ as $c \downarrow 1$ and for which as $t \to \infty$,*

$$P\big(|X_r - X_1| < cr^{1/2}, 1 \le r \le t\big) \sim t^{-(1-\varepsilon)}.$$

PROOF. This is a result of Breiman [3]. □

Recall the definition of $A_t(c), B_t(c)$:

$$A_t(c) = \big\{\beta : \forall r \in [0,t], -M_1(\beta,t) - c(1-r)^{1/2} < W_r^\beta - W_t^\beta < M_2(\beta,t) + c(1-r)^{1/2}\big\},$$

where M_i is defined as previously. $B_t(c)$ is the collection of equivalence classes corresponding to the N_t particles at time t. Let $C_t(c)$ be defined like $B_t(c)$, except with $M_1 = M_2 = 0$. It is clear that $C_t \subset B_t$. Let $H_t(c) = \#C_t(c)$.

LEMMA 3.5. *There exist constants k and K for which*

$$k(1-t)^{-\varepsilon} \le E(H_t) \le K(1-t)^{-\varepsilon}.$$

PROOF. This follows from the Lemmas 2.2 and 3.4 for

$$\begin{aligned} E(H_t) &= E(N_t)P([\beta]_t \in C_t) \\ &= E(N_t)P\big(|X_r - X_t| < c(1-r)^{1/2}, 0 \le r \le t\big) \\ &= E(N_t)P\big(|X_r - X_1| < c(1-r)^{1/2}, 1 \le r \le 1/(1-t)\big) \\ &\sim (1-t)^{-1}(1-t)^{1-\varepsilon}. \end{aligned}$$

□

LEMMA 3.6. *There exists a constant $k > 0$, independent of t, for which*

$$\lim_{t \uparrow 1} P(\#B_t(c) > 0) \ge k_4 > 0.$$

PROOF. For β, γ define $\beta \wedge \gamma = \sup\{n : \beta_n = \gamma_n\}$ and let $\alpha(\beta,\gamma) = 1 - \delta(\beta_{\beta \wedge \gamma})$. Then α corresponds to the time of splitting between the two paths indexed by γ and β. For ease of notation, let $\{\beta^i, 1 \le i \le N_t\}$ be a set of representatives for the different paths at time t and set $\alpha(i,j) = \alpha(\beta^i, \beta^j)$.

A consequence of Jensen's inequality gives

$$P(\#B_t(c) > 0) \ge P(H_t(c) > 0) \ge E(H_t)^2/E(H_t^2).$$

A count of the different branches at time t gives

$$E(H_t^2) = E(H_t) + E\left(\sum_i^{N_t} \sum_{j \ne i} 1(\beta^i \in C_t) 1(\beta^j \in C_t)\right).$$

The following formula is from Sawyer [10], except for a minor adjustment to accommodate the nonstationary of the process Z_t, from Lemma 2.1. At time t we denote Z_t by $\{z_t^1, z_t^2, \cdots, z_t^{N_t}\}$, where $z_t^i \in C([0,t],\mathbb{R})$. Label the branch times in chronological order with τ_n the time of the nth branch, and let A and B be Borel subsets of $C([0,t],\mathbb{R})$. Let $N_t^s(A)$ be the number of offspring from a branch at time s that are in a set A at time t. Define $\alpha(i,j)$ analogously.

We have the following decomposition:

$$
\begin{aligned}
&E\Big(\sum_i \sum_{j\neq i} 1_A(z_t^i)1_B(z_t^j)\Big)\\
&\quad= E\Bigg(\sum_n \sum_i \sum_{\substack{j\neq i\\ \alpha(i,j)=\tau_n}} 1_A(z_t^i)1_B(z_t^j)\Bigg)\\
&\quad= \sum_n \int_0^t P(\tau_n \in ds) E\Big(\sum_i \sum_{j\neq i} 1_A(z_t^i)1_B(z_t^j) | \alpha(i,j) = s\Big)\\
&\quad= \int_0^t \Big(\sum_n P(\tau_n \in ds) E\Big(\sum_i \sum_{j\neq i} 1_A(z_t^i)1_B(z_t^j) | \alpha(i,j) = s\Big) \qquad (3.7)\\
&\quad= 2\int_0^t e^s\, ds\, E\big(E^{U_s}(N_t^s(A))E^{U_s}(N_t^s(B))\big)\\
&\quad= 2\int_0^t e^s\, ds\, e^{2(t-s)} E\big(P^{U_s}(1_A(U_t))P^{U_s}(1_B(U_t))\big)\\
&\quad= 2\int_0^t e^{2t-s} P\big(1_A(U_t^1)1_B(U_t^2) | \alpha(U_t^1,U_t^2) = s\big)\, ds.
\end{aligned}
$$

Line (3.7) comes from Lemma 2.3. The appearance of 2 comes from the symmetry involved. At time s the branch splits in two, and we count the expected number of paths from each branch. The symmetry in the expected number of those from the first branch that end up in A and those from the second branch that end up in B and those from the second branch ending in A. That $(\Sigma_n P(\tau_n \in ds)) = e^s\, ds$ is a simple calculation that follows by conditioning on the location of the first branch.

Time changing back to get W, we have the formula

$$
\begin{aligned}
&E\Bigg(\sum_{i=1}^{N_t} \sum_{j\neq i}^{N_t} 1(\beta^i \in C_t, \beta^j \in C_t)\Bigg) \qquad (3.8)\\
&\quad= 2(1-t)^{-2} \int_0^t (1-s) P\big(X_t^1 \in C_t, X_t^2 \in C_t | \alpha(1,2) = s\big)\, ds,
\end{aligned}
$$

where X^i is distributed like a Brownian motion and X^1 and X^2 fork apart at time s. That is, $X^1(r) = X^2(r)$ for $r \le s$ and $X_r^1 - X_s^1$ is independent of $X_r^2 - X_s^2$ for $r \ge s$. Simplify notation by setting $r' = 1 - r$ for any r. To analyze this, it suffices to ignore the part common to the two branches:

$$\begin{aligned}
&P\left(X_t^1 \in C_t, X_t^2 \in C_t \mid \alpha(1,2) = s\right)\\
&\quad = P\Big(\forall r \in [0,t], |X_r^2 - X_t^1| < c(r')^{1/2},\\
&\qquad\qquad |X_r^2 - X_t^2| < c(r')^{1/2} | X_r^1 = X_r^2, r \le s\Big)\\
&\quad \le P\Big(\forall r \in [s,t], |X_r^1 - X_t^1| < c(r')^{1/2}, |X_r^2 - X_t^2| < c(r')^{1/2}\Big)\\
&\quad = P\Big(\forall r \in [s,t], |X_r - X_t| < c(r')^{1/2}\Big)^2\\
&\quad = P\Big(\forall r \in [t',s'], |X_r - X_{t'}| < cr^{1/2}\Big)^2\\
&\quad \le k_1(t'/s')^{2(1-\varepsilon)}.
\end{aligned}$$

This follows from the time reversal properties of Brownian motion, Brownian scaling and Lemma 3.4. With this, (3.8) becomes

$$\begin{aligned}
&2(t')^2 \int_0^t s' P\left(X_t^1 \in C_t, X_t^2 \in C_t \mid \alpha(1,2) = s\right) ds\\
&\quad \le k_2(t')^{-2} \int_0^t s'(t'/s')^{2(1-\varepsilon)}\, ds\\
&\quad = k_2(t')^{-2\varepsilon} \int_0^t (s')^{-(1-2\varepsilon)}\, ds\\
&\quad \le k_2(t')^{-2\varepsilon}/(2\varepsilon)\\
&\quad \le k_3 E(H_t)^2.
\end{aligned}$$

Here $k_3 < \infty$ is independent of t, but $E(H_t) \uparrow \infty$ as $t \uparrow 1$ by Lemma 3.5; thus, we get

$$\begin{aligned}
P(\#A_t(c) > 0) &\ge E(H_t)^2 / E(H_t^2)\\
&\ge E(H_t)^2 / \left(E(H_t) + k_3 E(H_t)^2\right)\\
&\ge k_4 > 0.
\end{aligned}$$

Here k_4 is a positive constant independent of t. □

THEOREM 3.2. *Almost surely*,

$$\inf_\beta \limsup_{t \uparrow 1} \frac{|W_t^\beta - W_1^\beta|}{(1-t)^{1/2}} \le 1.$$

PROOF. Fix $c > 1$. The sets $A_t(c)$ are monotone decreasing in t as $t \uparrow 1$, so the last lemma assures us that with positive probability

$$\bigcap_{n=1}^{\infty} A_{1/n}(c) \neq \varnothing.$$

Thus, there exists with positive probability γ such that for all $n > 0$ and for $t \in [0, 1 - 1/n]$,

$$-M_1(\gamma, 1/n) - c(1-t)^{1/2} < |W_t^\gamma - W_{1-1/n}^\gamma| < M_2(\gamma, 1/n) + c(1-t)^{1/2}.$$

Set $M(t) = M_1(\gamma, t) \vee M_2(\gamma, t)$, $M(t) \downarrow 0$ uniformly from Remark 1. Fix $t_0 < 1$ and $d > c$. Then we can find n such that $t_0 < 1 - 1/n$ and

$$2M(1 - 1/n) + c(1-t_0)^{1/2} < d(1-t_0)^{1/2}.$$

Thus we must have that

$$\begin{aligned}|W_{t_0}^\gamma - W_1^\gamma| &\le |W_{t_0}^\gamma - W_{1-1/n}^\gamma| + |W_{1-1/n}^\gamma - W_1^\gamma| \\ &< M(1-1/n) + c(1-t_0)^{1/2} + M(1-1/n) \\ &< d(1-t_0)^{1/2}.\end{aligned}$$

Thus we have that for each $d > c$ there exists a γ for which

$$\limsup_{t \uparrow 1} \frac{|W_t^\gamma - W_1^\gamma|}{(1-t)^{1/2}} \le d.$$

So with positive probability,

$$\inf_\beta \limsup_{t \uparrow 1} \frac{|W_t^\beta - W_1^\beta|}{(1-t)^{1/2}} \le c. \tag{3.9}$$

To establish that (3.9) holds with probability 1, we now show that there is a 0–1 law for this problem. Let τ be the time of the first branch of W and let W^1, W^2 be the two separate branching processes determined by the branch at τ. Set

$$\Gamma(W) = \left\{\beta\colon \limsup_{t \uparrow 1} \frac{|W_t^\beta - W_1^\beta|}{(1-t)^{1/2}} \le c\right\}.$$

Then

$$\begin{aligned}&P(\Gamma(W) \ne \varnothing) \\ &\quad = P(\Gamma(W^1) \ne \varnothing \text{ or } \Gamma(W^2) \ne \varnothing) \\ &\quad = \int_0^1 P(\tau \in dt)\big[P(\Gamma(W^1) \ne \varnothing | \tau = t) + P(\Gamma(W^2) \ne \varnothing | \tau = t) \\ &\qquad\qquad - P(\Gamma(W^1) \ne \varnothing | \tau = t) P(\Gamma(W^2) \ne \varnothing | \tau = t)\big] \\ &\quad = \int_0^1 P(\tau \in dt)\left[2P(\Gamma(W^1) \ne \varnothing | \tau = t) - P(\Gamma(W^1) \ne \varnothing | \tau = t)^2\right] \\ &\quad = 2P(\Gamma(W) \ne \varnothing) - P(\Gamma(W) \ne \varnothing)^2.\end{aligned}$$

Solving for $P(\Gamma(W) \ne \varnothing)$ gives, for $c > 1$ almost surely,

$$\inf_\beta \limsup_{t \uparrow 1} \frac{|W_t^\beta - W_1^\beta|}{(1-t)^{1/2}} \le c.$$

The theorem follows by letting $c \downarrow 1$ along a countable sequence. □

Combining Theorems 3.1 and 3.2 gives Theorem 1.1.

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