CONTINUOUS SAMPLING PROCEDURES WITHOUT CONTROL[1]

By C. Derman, M. V. Johns, Jr., and G. J. Lieberman

Columbia University, Stanford University, and Stanford University

1. Summary. Several modifications of the Dodge CSP-1 procedure [1] are presented. Changes are made in the rule of action when a defective item is observed while on sampling. The Average Outgoing Quality Limit (AOQL) for these new procedures are derived without the assumption of control. These results are compared with the AOQL assuming control. A production process is said to be in statistical control if there is a constant probability p that an item is defective, and if the states of all the items (defective or nondefective) are stochastically independent. Further, the AOQL for the CSP-1 procedure using probability sampling (looking at every item with probability $1/k$ when on sampling) is derived without the assumption of control.

2. Introduction and results. Two continuous sampling procedures are considered. The first procedure is denoted by CSP-4[2] and is as follows:

a) At the outset, inspect 100 per cent of the units consecutively as produced and continue such inspection until i units in succession are found clear of defects.

b) When i units in succession are found clear of defects, discontinue 100 per cent inspection, and inspect only a fraction $1/k$ of the units, choosing the item to be observed at random from a segment of size k (this type of sampling will be called random sampling).

c) If a sample unit is found defective revert immediately to 100 per cent inspection, eliminating from the production process the remaining $(k - 1)$ items in the segment, and commencing 100 per cent inspection with the next item following the eliminated segment. Continue 100 per cent inspection until again i units in succession are found clear of defects, as in paragraph (a).

d) Correct or replace with good units all defective units found.

It is important to discuss the implications of (c). These eliminated units can be considered as a source of good items for (d). Furthermore, under certain mathematical models for the production process such as "a state of statistical control" condition c is equivalent to the following:

> If a sample unit is found defective revert immediately to 100 per cent inspection, commencing such inspection with the segment in which the defective item is observed. Continue 100 per cent inspection until again i units in succession are found clear of defects, as in paragraph (a).

The second continuous sampling procedure considered will be denoted by CSP-5 and is the same as CSP-4 except for condition (c) which is as follows:

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[2] CSP-2 and CSP-3 have already been used to denote other continuous sampling procedures.

c′) If a sample unit is found defective screen the remaining $k - 1$ items in the segment. Upon completion of this screening, commence 100 per cent inspection with the next item produced. Continue 100 per cent inspection until again i units in succession, not including the $k - 1$ screened items, are found clear of defects, as in paragraph (a).

These procedures differ from the Dodge CSP-1 procedure in paragraphs (b) and (c). Dodge's statements [1] analogous to (b) and (c) are as follows:

When i units in succession are found clear of defects, discontinue 100% inspection and inspect only a fraction $1/k$ of the units, selecting individual sample units one at a time from the flow of product, in such a manner as to assure an unbiased sample.

If a sample unit is found defective, revert immediately to a 100% inspection of succeeding units and continue until again i units in succession are found clear of defects, as in paragraph (a).

It is not immediately evident what Dodge meant by the phrase, "$\cdots$, *selecting individual sample units one at a time from the flow of product, in such a manner as to assure an unbiased sample.*" However, Dodge did study properties of his procedure and presented equations and charts for determining the Average Outgoing Quality Limit (AOQL) as functions of the parameters k and i, under the assumption that the process is in a state of statistical control. There are several interpretations of the sampling procedure while on partial inspection which lead to Dodge's operating characteristics under the assumption of control. These are as follows: (1) look at every kth item. This type of sampling is denoted as systematic sampling and has the practical disadvantage that the particular item to be chosen is known in advance. (2) sample every item with probability $1/k$. This type of sampling is denoted as probability sampling and has the disadvantage that the number of uninspected items is a random variable. The result showing the coincidence of the operating characteristic using this type of sampling with CSP-1 is due to Resnikoff [2]. (3) sample only a fraction $1/k$ of the units, choosing the item to be observed at random from a segment of size k (random sampling). If the sample unit is found defective begin 100% inspection with the item *following the segment* in which the defective item was observed, allowing the $k - 1$ uninspected items to enter into the production stream.

The CSP-4 and CSP-5 procedures are variations of this last type of sampling, i.e., random sampling. These procedures are investigated under the assumption of the existence of a state of statistical control and the AOQL's so obtained do not coincide exactly with the values given by Dodge for CSP-1. More important, however, the CSP-4 and CSP-5 procedures are analyzed without the assumption of the existence of a state of statistical control.

The problem of determining an AOQL for a Dodge type procedure without the assumption that the process is in a state of statistical control was first considered by Lieberman in [3], where it was shown that the CSP-1 procedure guarantees an AOQL whether or not the process is in a state of statistical control. In fact, for this case the AOQL equals $(k - 1)/(k + i)$. This result was obtained under the hypothesis of random sampling while on partial inspection. The same result is obtained in this paper under the hypothesis of probability sampling while on partial inspection. For a given k and i, the above value of the AOQL is always higher than that obtained using Dodge's equations. This is to be expected since the AOQL, without the assumption of control, is the least upper bound of the average quality level that a production process is able to achieve. This is not to imply that this is the average outgoing quality of a typical production process, but rather, that the average outgoing quality of the process can never exceed this AOQL value. The production process that actually achieves this level is one which alternates between producing all defective items during partial inspection and producing all non-defective items during 100% inspection.

It is the authors' contention that the assumption of control is not always justified. Whereas a production process which achieves the AOQL found by Lieberman seems unlikely, it should be emphasized that deviations from control can produce values of the average outgoing quality ranging up to the AOQL found by Lieberman.

It is intuitively clear that under CSP-4 and CSP-5 a production process which alternates between producing all defective items during partial inspection and producing all non-defective items during 100% inspection, will *not* represent the least favorable case. It is shown in this paper that both of these procedures guarantee a non-trivial AOQL whether or not the process is in a state of statistical control. In fact, for CSP-4

$$\text{AOQL} = \begin{cases} \dfrac{(c_4 + 2) - 2\sqrt{c_4 + 1}}{c_4^2}, & c_4 \neq 0 \\ \frac{1}{4}, & c_4 = 0 \end{cases} \qquad \text{where } c_4 = (i - k + 1)/k$$

The AOQL is actually achieved when the process alternates between producing

$$d_4 = \begin{cases} \dfrac{k^2\sqrt{(i + 1)/k} - k^2}{i - k + 1}, & i \neq k - 1 \\ k/2, & i = k - 1 \end{cases}$$

defective items in a block of size k during partial inspection and producing all non-defective items during 100 per cent inspection. Similarly, for CSP-5

$$\text{AOQL} = \frac{(c_5 + 2) - 2\sqrt{c_5 + 1}}{c_5^2}, \qquad \text{where } c_5 = i/k.$$

Note that the AOQL depends only on the ratio i/k, and not on the individual values. This AOQL is achieved when the process alternates between producing

$$d_5 = \frac{k^2\sqrt{i/k + 1} - k^2}{i}$$

defective items in a block of size k during partial inspection and producing all non-defective items during 100 per cent inspection.

Naturally, these results are always higher than those obtained assuming control. However, the values of d given are not so high as to be unrealistic. For example, if an operator knows that only 1 in k items is to be chosen at random and observed, he may be careless enough to produce d defective items in this block, whereas if he knows every item is to be looked at (100 per cent inspection) he will be very careful and produce all good items. Hence, the AOQL values given above may not be unreasonably large.

Finally, the CSP-4 or CSP-5 procedures are used in practice because of a reluctance to pass a segment in which a defective item has already been observed. Usually, the equations for the AOQL of CSP-1 under the assumption of control are used to find the necessary parameters i and k for the CSP-4 or CSP-5 procedures since this is a "conservative" approximation. However, its conservatism depends upon the realism of the assumption of control. It is interesting to point out that the CSP-5 procedure guarantees that the AOQL will never exceed 25% regardless of the choice of i and k.

3. Theorems and proof for the AOQL without the assumption of control for CSP-4 and CSP-5. Define

D_{st} = number of defects produced in the sth block of the tth cycle,

$$D_{st} = 0, 1, \cdots, k \text{ for all } s, t.$$

A cycle is the period where partial inspection begins to the time a defective is observed. A block is a segment of k items produced while on partial inspection from which a single item is chosen at random for inspection.

N_t = number of blocks (of k items) sampled in the tth cycle. It is pointed out that the cycle terminates when a defective is found and that for the procedures considered the block in which the defective is drawn is not put directly into the production stream. However, it will still be considered as part of the tth cycle. Under CSP-4, the block is eliminated and under CSP-5, the block is screened replacing all defective items by good ones.

X_t = total number of defects being passed in the tth cycle. $X_t = \sum_{s=1}^{N_t-1} D_{st}$.

δ_{st} are zero-one random variables and indicate whether the sth item in the 100% inspection sequence preceding the tth cycle of partial inspection are non-defective or defective.

M_t = number of items inspected in the 100% inspection sequence preceding the tth cycle of partial inspection. This is a sure function of δ_{st}.

A strategy of nature is characterized by a pair of doubly infinite sequences of possibly dependent random variables

$$\{\{D_{st}\}, \{\delta_{st}\}\}$$

Define the number L_j, $(j = 4, 5)$, as the smallest numbers with the property that for every process the probability is zero that

$$\limsup_{m\to\infty} \frac{\sum_{t=1}^{m} X_t}{k\sum_{t=1}^{m} N_t - m\alpha_j + \sum_{t=1}^{m} M_t} > L_j\,; \qquad (j = 4, 5) \tag{1}$$

where

$$\alpha_j = \begin{cases} k-1, & j = 4 \\ 0, & j = 5 \end{cases}$$

The numbers L_4 and L_5 are called the AOQL for CSP-4 and CSP-5 respectively. It is evident that the ratio whose lim sup is taken in (1) is just the total number of defectives contributed to the outgoing product in the first m cycles divided by the total number of items contributed to the outgoing product in the m cycles.

It is clear that in order to determine L we may confine ourselves to consideration of strategies of nature for which the number of cycles is infinite with probability 1. Furthermore, if we choose $\{\delta_{st}\} = \{0, 0, \cdots, 0, \cdots\}$ with probability 1, independent of the past, we are assured that $M_t = i$, $(t = 1, 2, \cdots)$, with probability 1. Hence, any strategy of nature for which the δ_{st} are not of this form is dominated by a corresponding strategy for which they are. Similarly it is sufficient to consider the special class of strategies for which the number of defectives in every block on partial inspection is $\geqq 1$. Hence, by confining ourselves to such strategies we may characterize nature's strategy by the single infinite sequence $\{D_{st}\}$, where the random variables D_{st} take on the values $1, 2, \cdots, k$, with probability 1. It then follows that

$$\limsup_{m\to\infty} \frac{\sum_{t=1}^{m} X_t}{k\sum_{t=1}^{m} N_t - m\alpha_j + \sum_{t=1}^{m} M_t} \leqq \limsup_{m\to\infty} \frac{\sum_{t=1}^{m} X_t}{k\sum_{t=1}^{m} N_t - m\alpha_j + mi}\,; \tag{2}$$

$$(j = 4, 5).$$

THEOREM 1:[3] *For every strategy $\{D_{st}\}$ of nature and for all m*

$$\frac{\sum_{t=1}^{m} E(X_t \mid \underline{D}_t)}{k\sum_{t=1}^{m} E(N_t \mid \underline{D}_t) + m(i - \alpha_j)} \leqq L(c_j) \qquad (j = 4, 5); \tag{3}$$

[3] The authors are indebted to Professor S. Karlin for suggesting the method of proof used in this theorem.

where

$$L(c_j) = \begin{cases} \dfrac{(c_j + 2) - 2\sqrt{c_j + 1}}{c_j^2}, & c_j \neq 0 \\ \frac{1}{4}, & c_j = 0 \end{cases} \qquad (j = 4, 5); \tag{4}$$

$$c_j = \frac{i - \alpha_j}{k}; \tag{5}$$

and

$$\underline{D}_t = \{D_{1t}, D_{2t}, \cdots\}. \tag{6}$$

PROOF: We may write

$$X_t = \sum_{s=1}^{\infty} D_{st} U_{st}, \qquad \text{where} \quad U_{st} = \begin{cases} 1, N_t > s \\ 0, \text{otherwise} \end{cases} \tag{7}$$

Hence,

$$\begin{aligned} E(X_t \mid \underline{D}_t) &= \sum_{s=1}^{\infty} D_{st} E(U_{st} \mid \underline{D}_t) \\ &= D_{1t}\left(1 - \frac{D_{1t}}{k}\right) + D_{2t}\left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right) \\ &\quad + D_{3t}\left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right)\left(1 - \frac{D_{3t}}{k}\right) + \cdots . \end{aligned} \tag{8}$$

This is a geometric series that is bounded uniformly by the convergent series $k\sum_{s=1}^{\infty} (1 - 1/k)^s$.

Similarly,

$$N_t = 1 + \sum_{s=1}^{\infty} U_{st}, \tag{9}$$

so that

$$\begin{aligned} E(N_t \mid \underline{D}_t) &= 1 + \left(1 - \frac{D_{1t}}{k}\right) \\ &+ \left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right) + \left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right)\left(1 - \frac{D_{3t}}{k}\right) + \cdots . \end{aligned} \tag{10}$$

Again, this is uniformly bounded by a convergent geometric series.

From (8) it follows that

$$\sum_{t=1}^{m} E(X_t \mid \underline{D}_t) = \sum_{t=1}^{m} \left[D_{1t}\left(1 - \frac{D_{1t}}{k}\right) + D_{2t}\left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right) + \cdots \right]$$

$$\begin{aligned} (11) \qquad &= \sum_{t=1}^{m} \left[\left(\frac{D_{1t}\left(1 - \frac{D_{1t}}{k}\right)}{k + (i - \alpha_j)\frac{D_{1t}}{k}} \right)\left(k + (i - \alpha_j)\frac{D_{1t}}{k}\right) \right. \\ &\quad + \left(\frac{D_{2t}\left(1 - \frac{D_{2t}}{k}\right)}{k + (i - \alpha_j)\frac{D_{2t}}{k}} \right)\left(k + (i - \alpha_j)\frac{D_{2t}}{k}\right)\left(1 - \frac{D_{1t}}{k}\right) \\ &\quad + \left(\frac{D_{3t}\left(1 - \frac{D_{3t}}{k}\right)}{k + (i - \alpha_j)\frac{D_{3t}}{k}} \right)\left(k + (i - \alpha_j)\frac{D_{3t}}{k}\right) \\ &\quad \left. \cdot \left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right) + \cdots \right]. \end{aligned}$$

From (10) it follows that

$$\begin{aligned} (12) \qquad & k \sum_{t=1}^{m} E(N_t \mid \underline{D}_t) + m(i - \alpha_j) \\ &= \sum_{t=1}^{m} \left[k + k\left(1 - \frac{D_{1t}}{k}\right) + k\left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right) + \cdots + (i - \alpha_j) \right]. \end{aligned}$$

Noting that

$$\frac{D_{1t}}{k} + \frac{D_{2t}}{k}\left(1 - \frac{D_{1t}}{k}\right) + \frac{D_{3t}}{k}\left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right) + \cdots = 1$$

since the left hand side is just the probability of ultimately achieving a success when performing successive Bernoulli trials with success probabilities bounded away from zero, we see that expression (12) can be written as

$$\begin{aligned} & k \sum_{t=1}^{m} E(N_t \mid \underline{D}_t) + m(i - \alpha_j) \\ (13) \qquad &= \sum_{t=1}^{m} \left[\left(k + (i - \alpha_j)\frac{D_{1t}}{k}\right) + \left(k + (i - \alpha_j)\frac{D_{2t}}{k}\right)\left(1 - \frac{D_{1t}}{k}\right) \right. \\ &\quad \left. + \left(k + (i - \alpha_j)\frac{D_{3t}}{k}\right)\left(1 - \frac{D_{1t}}{k}\right)\left(1 - \frac{D_{2t}}{k}\right) + \cdots \right]. \end{aligned}$$

Hence,

$$\frac{\sum_{t=1}^{m} E(X_t \mid \underline{D}_t)}{k \sum_{t=1}^{m} E(N_t \mid \underline{D}_t) + m(i - \alpha_j)}$$

is merely a non-negatively weighted average of quantities of the form

$$f(D_{st}; i, k) = \frac{D_{st}\left(1 - \frac{D_{st}}{k}\right)}{k + (i - \alpha_j)\frac{D_{st}}{k}}, \qquad (j = 4, 5; s = 1, 2, \cdots) \tag{14}$$

and has an upper bound obtained by maximizing each of these expressions independently. Taking the derivative of (14) with respect to the value d_{st} of D_{st} (treated as a continuous variable) we obtain

$$f'(d_{st}, i, k) = \begin{cases} \dfrac{k - 2d_{st}}{k^2 + (i - \alpha_j)d_{st}} - \dfrac{(kd_{st} - d_{st}^2)(i - \alpha_j)}{[k^2 + (i - \alpha_j)d_{st}]^2}, & i \neq \alpha_j \\ \dfrac{1}{k} - 2\dfrac{d_{st}}{k^2}, & i = \alpha_j \end{cases} \tag{15}$$

The quantity $f(d_{st}, i, k)$ is clearly maximized by setting (15) equal to zero. Denoting the maximizing value of d_{st} by d_j since it is independent of s and t we obtain

$$d_j = \begin{cases} \dfrac{k^2\sqrt{(i - \alpha_j)/k + 1} - k^2}{(i - \alpha_j)}, & i \neq \alpha_j \\ k/2, & i = \alpha_j \end{cases} \qquad (j = 4, 5); \tag{16}$$

It then follows that

$$\begin{aligned} \frac{\sum_{t=1}^{m} E(X_t \mid \underline{D}_t)}{k \sum_{t=1}^{m} E(N_t \mid \underline{D}_t) + m(i - \alpha_j)} &\leqq \frac{d_j\left(1 - \frac{d_j}{k}\right)}{k + (i - \alpha_j)\frac{d_j}{k}} \\ &= \left.\begin{cases} \dfrac{(c_j + 2) - 2\sqrt{c_j + 1}}{c_j^2}, & c_j \neq 0 \\ 1/4, & c_j = 0 \end{cases}\right\} = L(c_j); \qquad (j = 4, 5); \end{aligned} \tag{17}$$

where $c_j = (i - \alpha_j)/k$.

THEOREM 2: *For any strategy* $\{D_{st}\}$ *of nature, for either* CSP-4 *or* CSP-5

$$\lim_{m \to \infty} \left[\frac{1}{m}\sum_{t=1}^{m} X_t - \frac{1}{m}\sum_{t=1}^{m} E(X_t \mid \underline{D}_t)\right] = 0, \tag{18}$$

with probability 1, *and*

$$\lim_{m\to\infty}\left[\frac{1}{m}\sum_{t=1}^{m} N_t - \frac{1}{m}\sum_{t=1}^{m} E(N_t \mid \underline{D}_t)\right] = 0, \tag{19}$$

with probability 1.

PROOF: For $t = 1, 2, \cdots$, let

$$Z_t = X_t - E(X_t \mid \underline{D}_t). \tag{20}$$

Then

$$E(Z_t \mid \underline{D}_t) = E[X_t - E(X_t \mid \underline{D}_t) \mid \underline{D}_t] = 0 \tag{21}$$

so that $E(Z_t) = 0$. Furthermore, for $t > s$, Z_t and Z_s are conditionally independent given $\underline{D}_t$ so that

$$E(Z_s Z_t) = E[E(Z_s Z_t \mid \underline{D}_t)] = E[E(Z_t \mid \underline{D}_t)\, E(Z_s \mid \underline{D}_t)] = 0. \tag{22}$$

Now

$$\begin{aligned} E(Z_t^2) &= E(X_t^2) - E[E^2(X_t \mid \underline{D}_t)] \leqq E(X_t^2) < k^2 E(N_t^2) \\ &= k^2 E[E(N_t^2 \mid \underline{D}_t)] \leqq k^2 \sum_{s=1}^{\infty} s^2 \left(1 - \frac{1}{k}\right)^{s-1} < \infty, \end{aligned} \tag{23}$$

since $D_{st} \geqq 1$ with probability 1. Now by a well known Law of Large Numbers for sums of orthogonal random variables ([4] Chapter IV, Theorem 5.2) equation (22) together with the uniform boundedness of $E(Z_t^2)$ shown by (23) implies that

$$\lim_{m\to\infty} \frac{1}{m}\sum_{t=1}^{\infty} Z_t = 0, \tag{24}$$

with probability 1,
so that (18) is established. Letting $Z_t^* = N_t - E(N_t \mid \underline{D}_t)$, the proof of (19) is similar.

THEOREM 3. *For any strategy* $\{D_{st}\}$ *of nature*

$$L_j \leqq L(c_j) \qquad (j = 4, 5). \tag{25}$$

PROOF. By Theorem 1 we have

$$\frac{1}{m}\sum_{t=1}^{m} E(X_t \mid \underline{D}_t) - L(c_j)\left[\frac{k}{m}\sum_{t=1}^{m} E(N_t \mid \underline{D}_t) + (i - \alpha_j)\right] \leqq 0, \tag{26}$$

for all m. If for each m we let

$$V_m = \frac{1}{m}\sum_{t=1}^{m} X_t - \frac{1}{m}\sum_{t=1}^{m} E(X_t \mid \underline{D}_t), \tag{27}$$

and

$$V_m' = \frac{1}{m}\sum_{t=1}^{m} N_t - \frac{1}{m}\sum_{t=1}^{m} E(N_t \mid \underline{D}_t), \tag{28}$$

then by Theorem 2, $\lim_{m\to\infty} V_m = \lim_{m\to\infty} V_m' = 0$ with probability 1. But from (26) we have

$$\frac{\sum_{t=1}^{m} X_t}{k\sum_{t=1}^{m} N_t + m(i - \alpha_j)} \leqq L(c_j) + \frac{V_m - kL(c_j)V'_m}{\frac{k}{m}\sum_{t=1}^{m} N_t + (i - \alpha_j)}, \tag{29}$$

and (25) follows upon taking the lim $\sup_{m\to\infty}$ of both sides of (29).

If we now let

$$d_j^* = \text{integer nearest to} \begin{cases} \dfrac{k^2\sqrt{(i-\alpha_j)/k+1} - k^2}{i - \alpha_j}, & i \neq \alpha_j \\ k/2, & i = \alpha_j \end{cases} \quad (j = 4, 5); \tag{30}$$

then we have

THEOREM 4: *If the production process alternates between producing* d_4^* [d_5^*] *defective items in blocks of size k during partial inspection and all non-defective items during* 100 *per cent inspection, then for* CSP-4 [CSP-5]

$$\limsup_{m\to\infty} \frac{\sum_{t=1}^{m} X_t}{k\sum_{t=1}^{m} N_t + m(i - \alpha_j)}$$

equals $L_4(c)$ [$L_5(c)$] *(approximately, due to the discreteness of* d_4^* *and* d_5^**) and hence the* AOQL *is given by* $L_4(c)$ [$L_5(c)$].

PROOF: This result follows immediately from (16) and Theorems 2 and 3.

We remark that it is easily verified by differentiation that $L(c_5) \leqq \lim_{c\to 0} L(c_5) = 1/4$, so that the AOQL $\leqq \frac{1}{4}$ for CSP-5 for any choice of i and k. We further remark that if defective items found when on 100 per cent inspection are not replaced by good items but are discarded, the previously derived results are still applicable, i.e., the AOQL is still given approximately by $L(c_j)$. If, under the CSP-4 procedure, a unit found defective while on sampling is also discarded together with the remaining $(k - 1)$ items and not replaced, the previously derived results are also applicable provided that α_j is set equal to k.

4. CSP-4 and CSP-5 under control. This section will be devoted to determining the Average Outgoing Quality (AOQ) function and the AOQL for the CSP-4 and CSP-5 procedures under the assumption of the existence of a state of statistical control.

The AOQ function is defined as

$$\begin{aligned} \text{AOQ}_j &= \limsup_{m\to\infty} \frac{\sum_{t=1}^{m} X_t}{k\sum_{t=1}^{m} N_t - m\alpha_j + \sum_{t=1}^{m} M_t} \\ &= \limsup_{m\to\infty} \frac{\sum_{t=1}^{m} X_t/m}{k\sum_{t=1}^{m} N_t/m - \alpha_j + \sum_{t=1}^{m} M_t/m} \qquad (j = 4, 5) \end{aligned} \tag{31}$$

where

$$\alpha_j = \begin{cases} k - 1, & j = 4 \\ 0, & j = 5 \end{cases}$$

Under the assumption of the existence of a state of statistical control at level p, the law of large numbers becomes applicable so that the AOQ function can be expressed as

$$\text{AOQ}_j = \frac{E(X_t)}{kE(N_t) - \alpha_j + E(M_t)} \qquad (j = 4, 5). \tag{32}$$

It is easily verified that

$$E(M_t) = \frac{1 - q^i}{pq^i} \tag{33}$$

$$E(N) = \frac{1}{p} \tag{34}$$

and

$$E(X_t) = (k - 1)q \tag{35}$$

where $q = 1 - p$. Hence,

$$\text{AOQ}_4 = \frac{(k - 1)(q^{i+1} - q^{i+2})}{1 + q^{i+1}(k - 1)} = \frac{(k - 1)pq^{i+1}}{1 + (k - 1)q^{i+1}}. \tag{36}$$

The maximizing value of q for a fixed i and k is given by solving for q the expression

$$(k - 1)\, q^{i+2} + (i + 2)\, q = (i + 1). \tag{37}$$

Denote this value by $q_{\text{max-4}}$. The AOQL can then be written as

$$\text{AOQL}_4 = 1 - q_{\text{max-4}} \frac{(i + 2)}{(i + 1)} \tag{38}$$

or, solving for $q_{\text{max-4}}$, the expression

$$q_{\text{max-4}} = (1 - \text{AOQL}_4) \frac{(i + 1)}{(i + 2)} \tag{39}$$

is obtained. Substituting this expression for $q_{\text{max-4}}$ into (37) and solving for k, the relationship between k and i for a fixed AOQL is obtained, i.e.,

$$k = 1 + \left(\frac{i + 2}{i + 1}\right)^{i+2} \frac{(i + 1)\ \text{AOQL}_4}{(1 - \text{AOQL}_4)^{i+2}}. \tag{40}$$

For fixed k and i, the expression for the AOQL for the CSP-4 procedure assuming control never exceeds the AOQL which is obtained without making any assumptions about the behavior of the process. However, the differences are much smaller for this procedure than for the CSP-1 procedure.

Similarly, for CSP-5, the AOQ function can be written as

$$\text{AOQ}_5 = \frac{[q^{i+1} - q^{i+2}](k-1)}{1 + q^i(k-1)} = \frac{(k-1)pq^{i+1}}{1+(k-1)q^i}. \tag{41}$$

The maximizing value of q for a fixed i and k is given by solving for q the expression

$$2(k-1)q^{i+1} - (k-1)q^i + (i+2)q = i+1. \tag{42}$$

Denote this value by $q_{\text{max-5}}$. The AOQL can then be written as

$$\text{AOQL}_5 = \frac{(i+1)q_{\text{max-5}} - (i+2)q^2_{\text{max-5}}}{i} \tag{43}$$

or, solving for $q_{\text{max-5}}$, the expression

$$q_{\text{max-5}} = \frac{(i+1) + \sqrt{(i+1)^2 - 4i(i+2)\,\text{AOQL}_5}}{2(i+2)} \tag{44}$$

is obtained.

Substituting this expression for $q_{\text{max-5}}$ into (42) and solving for k, the relationship between k and i for a fixed AOQL is obtained, i.e.,

$$k = 1 + \frac{(i+1) - (i+2)q_{\text{max-5}}}{2q^{i+1}_{\text{max-5}} - q^i_{\text{max-5}}}. \tag{45}$$

Curves of constant AOQL derived from expressions (40), (44), and (45) are given in Figure 1.

5. CSP-1 without assuming control and using probability sampling. In this section, CSP-1 will be studied without assuming control but using a sampling procedure such that while on partial inspection, every item will be inspected with probability $1/k$, or passed without inspection with probability $(1-1/k)$. The notation of Sections 2 and 3 will be used, but for this problem k need not be an integer but may be any number >1.

If we let N_t^* denote the number of items contributed to the production stream during the tth partial inspection cycle, then the AOQL is defined, as before, as the smallest number L with the property that for every strategy of nature the probability is zero that

$$\limsup_{m\to\infty} \frac{\sum_{t=1}^m X_t}{\sum_{t=1}^m N_t^* + \sum_{t=1}^m M_t} > L. \tag{46}$$

To obtain the AOQL it is again sufficient to consider the special class of strategies of nature such that $M_t = i$ for all t, and we must investigate the quantity

$$\limsup_{m\to\infty} \frac{\sum_{t=1}^m X_t}{\sum_{t=1}^m N_t^* + mi}, \tag{47}$$

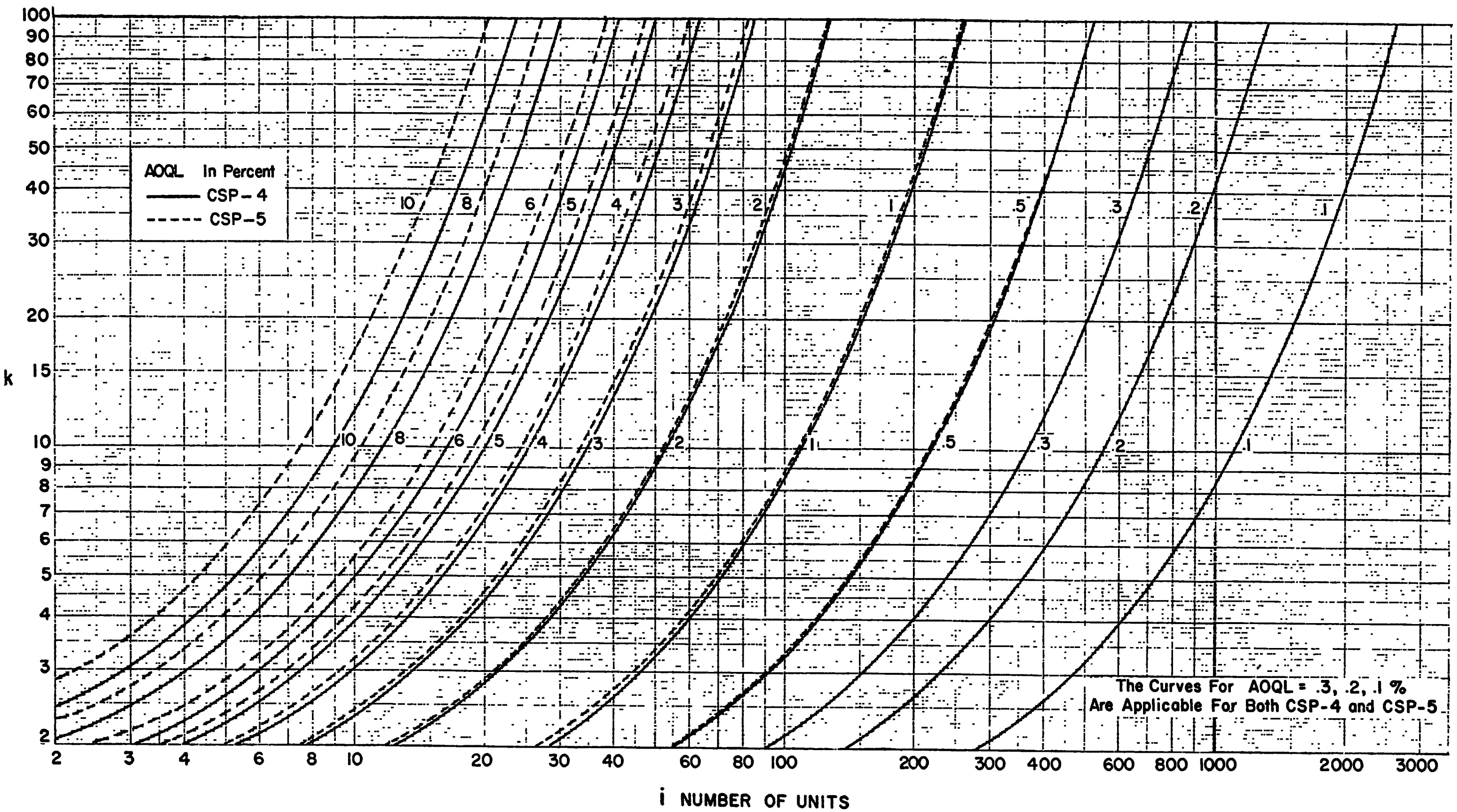

FIG. 1. Curves for Determining Values of k and i for A Given Value of AOQL for CSP-4 and CSP-5 under Control.

for such strategies. For this problem a (randomized) strategy of nature may be characterized by a double sequence of possibly dependent random variables $\{P_{st}\}$ where $0 \leqq P_{st} \leqq 1$ with probability 1 for all s, t and where P_{st} is interpreted as the probability that the sth item in the tth partial inspection cycle is defective. As before we restrict our attention to strategies for which an infinite number of partial inspection cycles will occur with probability 1.

Let R_t be the number of items passed until (and including) the first item inspected during the tth cycle of partial inspection. Then the R_t's are independently and identically distributed random variables with $E(R_t) = k$ and, furthermore, $N_t^* \geqq R_t$ for each t. Hence, by the Strong Law of Large Numbers

$$(48) \qquad \liminf_{m\to\infty} \frac{1}{m}\sum_{t=1}^{m} N_t^* \geqq \lim_{m\to\infty} \frac{1}{m}\sum_{t=1}^{m} R_t = k, \qquad \text{with probability 1,}$$

for any strategy $\{P_{st}\}$ of nature.

We now prove two theorems which enable us to characterize the behavior of the numerator of (47).

THEOREM 5: *For any strategy of nature* $\{P_{st}\}$

$$(49) \qquad E(X_t \mid \underline{P}_t) = k - 1,$$

with probability 1 *for all* t, *where* $\underline{P}_t = \{P_{1t}, P_{2t}, \cdots\}$.

PROOF: If all s, t we define

$$(50) \qquad Z_{st} = \begin{cases} 1, & \text{if the } s\text{th item in the } t\text{th cycle contributes a defective to the output,} \\ 0, & \text{otherwise,} \end{cases}$$

then for all t we may represent X_t by

$$(51) \qquad X_t = \sum_{s=1}^{\infty} Z_{st}.$$

Furthermore, since the probability that the sth item reached during the tth partial inspection cycle is either not inspected or inspected and found non-defective is given by $(1 - P_{st}/k)$, we have for all s, t

$$(52) \qquad E(Z_{st} \mid \underline{P}_t) = \left(1 - \frac{1}{k}\right) P_{st} \prod_{j=1}^{s-1} \left(1 - \frac{P_{jt}}{k}\right),$$

where the empty product is interpreted as 1. Hence,

$$(53) \qquad E(X_t \mid \underline{P}_t) = \left(1 - \frac{1}{k}\right) \sum_{s=1}^{\infty} P_{st} \prod_{j=1}^{s-1} \left(1 - \frac{P_{jt}}{k}\right).$$

We now establish the following equation for all $r \geqq 1$ by induction:

$$(54) \qquad \sum_{s=1}^{r} P_{st} \prod_{j=1}^{s-1} \left(1 - \frac{P_{jt}}{k}\right) = k\left[1 - \prod_{j=1}^{r} \left(1 - \frac{P_{jt}}{k}\right)\right].$$

The equation clearly holds for $r = 1$, and if it is assumed true for $r = n$ then for $r = n + 1$ the left hand side becomes

$$k\left[1-\prod_{j=1}^{n}\left(1-\frac{P_{jt}}{k}\right)\right]+P_{n+1,t}\prod_{j=1}^{n}\left(1-\frac{P_{jt}}{k}\right) = k\left[1-\prod_{j=1}^{n+1}\left(1-\frac{P_{jt}}{k}\right)\right], \tag{55}$$

and the proof by induction is complete.

We now remark that if the number of partial inspection cycles occurring is to be infinite with probability 1, then we must have $\lim_{r\to\infty} P\{N_t^* > r\} = 0$ for each t, which implies that

$$\lim_{r\to\infty} P\{N_t^* > r \mid \underline{P}_t\} = \lim_{r\to\infty}\prod_{j=1}^{r}\left(1-\frac{P_{jt}}{k}\right) = 0, \tag{56}$$

with probability 1 for all strategies $\{P_{st}\}$ under consideration. The desired result (49) now follows from (53), (54) and (56).

THEOREM 6. *For any strategy of nature* $\{P_{st}\}$

$$E(X_t^2) \leqq 2(k-1)^2 + (k-1) \tag{57}$$

for all t.

PROOF: As in Theorem 5 we have

$$E(X_t^2 \mid \underline{P}_t) = 2\sum_{v=2}^{\infty}\sum_{w=1}^{v-1} E(Z_{vt}Z_{wt} \mid \underline{P}_t) + \sum_{s=1}^{\infty} E(Z_{st} \mid \underline{P}_t), \tag{58}$$

and for $v > w$

$$\begin{aligned} E(Z_{vt}Z_{wt} \mid \underline{P}_t) &= \left[\prod_{s=1}^{w-1}\left(1-\frac{P_{st}}{k}\right)\right]\left[P_{wt}\left(1-\frac{1}{k}\right)\right] \\ &\quad \cdot\left[\prod_{s=w+1}^{v-1}\left(1-\frac{P_{st}}{k}\right)\right]\left[P_{vt}\left(1-\frac{1}{k}\right)\right] \\ &= \left(1-\frac{1}{k}\right)^2 P_{vt}P_{wt}\prod_{s\neq w}^{v-1}\left(1-\frac{P_{st}}{k}\right) \\ &= k\left(1-\frac{1}{k}\right)^2\frac{P_{vt}P_{wt}}{k-P_{wt}}\prod_{s=1}^{v-1}\left(1-\frac{P_{st}}{k}\right). \end{aligned} \tag{59}$$

Hence noting (49) of Theorem 5 we may write (58) as

$$E(X_t^2 \mid \underline{P}_t) = 2k\left(1-\frac{1}{k}\right)^2\sum_{v=2}^{\infty}P_{vt}\prod_{s=1}^{v-1}\left(1-\frac{P_{st}}{k}\right)\sum_{w=1}^{v-1}\frac{P_{wt}}{k-P_{wt}} + (k-1). \tag{60}$$

We now establish the following equation for all $r \geqq 2$ by induction:

$$\sum_{v=2}^{r}P_{vt}\prod_{s=1}^{v-1}\left(1-\frac{P_{st}}{k}\right)\sum_{w=1}^{v-1}\frac{P_{wt}}{k-P_{wt}} = k\left[1-\left(1+\sum_{w=1}^{r}\frac{P_{wt}}{k-P_{wt}}\right)\prod_{s=1}^{r}\left(1-\frac{P_{st}}{k}\right)\right]. \tag{61}$$

It is easily verified that for $r = 2$ both sides of (61) are equal to $P_{1t} P_{2t}/k$. If (61) is assumed to be true for $r = n$ then for $r = n + 1$ the left hand side may written as

$$k\left[1 - \left(1 + \sum_{w=1}^{n} \frac{P_{wt}}{k - P_{wt}}\right) \prod_{s=1}^{n} \left(1 - \frac{P_{st}}{k}\right) + \frac{P_{n+1,t}}{k} \prod_{s=1}^{n} \left(1 - \frac{P_{st}}{k}\right) \sum_{w=1}^{n} \frac{P_{wt}}{k - P_{wt}}\right]$$

$$(62) \quad = k\left[1 - \left(1 - \frac{P_{n+1,t}}{k}\right) \sum_{w=1}^{n} \frac{P_{wt}}{k + P_{wt}}\right) \prod_{s=1}^{n} \left(1 - \frac{P_{st}}{k}\right)\right]$$

$$= k\left[1 - \left(\frac{k}{k - P_{n+1,t}} + \sum_{w=1}^{n} \frac{P_{wt}}{k + P_{wt}}\right) \prod_{s=1}^{n+1} \left(1 - \frac{P_{st}}{k}\right)\right]$$

$$= k\left[1 - \left(1 + \sum_{w=1}^{n+1} \frac{P_{wt}}{k + P_{wt}}\right) \prod_{s=1}^{n+1} \left(1 - \frac{P_{st}}{k}\right)\right],$$

which is the right hand side of (61) with $r = n + 1$, so that the proof by induction is complete.

Now (60) and (61) imply that

$$(63) \quad E(X_t^2 \mid \underline{P}_t) \leqq 2k^2 \left(1 - \frac{1}{k}\right)^2 + (k - 1) = 2(k - 1)^2 + (k - 1),$$

and the desired result (57) follows. An examination of (61) shows that if the P_{st}'s are (for example) bounded away from zero then equality holds in (57).

We now prove the main result of this section.

Theorem 7. For CSP-1 with probability sampling the AOQL is given by

$$(64) \quad L = \frac{k - 1}{k + i},$$

and this value of L is achieved by (47) when nature's strategy is to produce all defective items during partial sampling and all non-defective items during 100% sampling.

Proof: The results of Theorems 5 and 6 together with the argument used in Theorem 2 imply that

$$(65) \quad \lim_{m \to \infty} \frac{1}{m} \sum_{t=1}^{m} X_t = k - 1, \qquad \text{with probability } 1,$$

for any strategy $\{P_{st}\}$ of nature. This result together with (48) implies that

$$(66) \quad L \leqq \frac{k - 1}{k + i}.$$

The fact that equality holds in (66) follows by applying the Strong Law of Large Numbers to the quantities

$$\frac{1}{m}\sum_{t=1}^{m} N_t^* \quad \text{and} \quad \frac{1}{m}\sum_{t=1}^{m} X_t \tag{67}$$

for the case where nature uses the strategy described in the Theorem above.

REFERENCES

[1] H. F. DODGE, "A sampling inspection plan for continuous production," *Ann. Math. Stat.*, Vol. 14 (1943), pp. 264–279.

[2] GEORGE J. RESNIKOFF, "Some modifications of the Lieberman-Solomon multi-level continuous sampling plan, MLP" Technical Report No. 26, Applied Math. and Stat. Laboratory, Stanford University, Stanford California, 1956.

[3] GERALD J. LIEBERMAN, "A note on Dodge's continuous inspection plan" *Ann. Math. Stat.*, Vol. 24 (1953), pp. 480–484.

[4] J. L. DOOB, *Stochastic Processes*, John Wiley and Sons, Inc., New York, 1953.