function δ_o, and that

$$(15) \qquad \liminf_{j \to \infty} r^{(k)}(\theta, \delta_{i_j}) \geqq r^{(k)}(\theta, \delta_o)$$

for each $k(k = 1, 2, \cdots)$ and all $\theta \, \varepsilon \, \Omega$. Furthermore, the decision function δ_o can be taken to be in the class A. It will be shown that

$$(16) \qquad r(\theta, \delta_o) \leqq r(\theta, \bar{\delta}) \quad \text{for all } \theta \, \varepsilon \, \Omega.$$

Choose and fix $\theta \, \varepsilon \, \Omega$. It follows from (13) that (16) will be proven if it can be shown that

$$(17) \qquad r^{(k)}(\theta, \delta_o) \leqq r(\theta, \bar{\delta}) \quad \text{for} \quad k = 1, 2, \cdots.$$

Accordingly, let k be any positive integer and let $\epsilon > 0$ be an arbitrary positive number. By (15), an integer i can be chosen large enough so that $i \geqq k$ and

$$(18) \qquad r^{(k)}(\theta, \delta_i) > r^{(k)}(\theta, \delta_o) - \epsilon.$$

Hence, from (18), (11), (14), and (12),

$$r^{(k)}(\theta, \delta_o) - \epsilon < r^{(k)}(\theta, \delta_i) \leqq r^{(i)}(\theta, \delta_i) \leqq r^{(i)}(\theta, \bar{\delta}) \leqq r(\theta, \bar{\delta}).$$

Since ϵ was arbitrary, $r^{(k)}(\theta, \delta_o) \leqq r(\theta, \bar{\delta})$. This completes the proof.

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A PROBLEM IN SURVIVAL[1]

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1. Introduction. Suppose that at a given time an individual has certain resources. These are used up at a specified rate, but from time to time "opportunities" arrive; at an opportunity a decision is made and the resources are changed—increased or decreased—in a random manner depending on the decision. If the resources ever fall to zero, the individual "perishes." The problem is to make the decision at each opportunity which will minimize the probability of ultimately perishing.

A model for approximating certain situations of this type is described below and the optimal policy is established. The problem is well suited for treatment by means of the "principle of optimality" [1], and an elegant theorem due to Blackwell [2].

2. The Model. Let $x(t)$ be the resources of the individual at time t. Should $x(t) = 0$ for the first time at t^*, let $x(t) = 0$ for $t \geq t^*$. Let the rate at which resources are used be a constant which without loss of generality may be taken to be unity. Thus in time t the resources will be reduced by an amount t. Opportunities are distributed in time in accordance with a Poisson process with constant intensity which may be taken to be unity, again without loss of generality, so that on the average one opportunity arrives per unit time. Each such opportunity is characterized by a family of distributions which depends on the resources available when the opportunity arrives. A decision consists of selecting a distribution from this family. Then a number w is randomly chosen according to this distribution and the resources are changed instantaneously by that amount so that if an opportunity arrives at t and w is chosen, $x(t+) = x(t) + w$. The family of distributions available if an opportunity arrives when the resources are x, $\mathfrak{F}_x$, consists of all distributions $\{F(w; x)\}$ on the interval $[-x, \infty)$ with fixed positive expectation μ. Thus the individual cannot lose more than all his resources at the time the opportunity arrives. Since the individual perishes with probability 1 if $\mu \leq 1$, it will be assumed that $\mu > 1$.

A policy is a rule R specifying for every x the choice of a single distribution $F(w; x, R) \; \varepsilon \; \mathfrak{F}_x$ to be used should an opportunity arrive when the amount of resources is x. (Clearly the policy does not depend on time.)

It will be shown that the policy which is optimal in the sense that it minimizes the probability of perishing consists of always choosing the distribution with zero variance; i.e., the individual always prefers a sure thing among the class of risks with equal expectations.

3. The Optimal Policy. Let t_0 be the time at which the problem starts and let t_i, $i = 1, 2, \cdots$, be the time at which the ith opportunity arrives. Let x_0 be the capital at t_0 and let x_i be the resources at t_i *after* they are changed by the outcome of the venture at that time. In case $x(t) = 0$ at some time $t^* \leq t_i$, then $x_i = 0$, $x_{i+1} = 0$, $\cdots$. For convenience define $F_0(y; x, R) = F(y - x; x, R)$ and let $F_0(y; 0, R) = 1$ for $y \geq 0$. Then, for a fixed policy R, the sequence $x_0, x_1, \cdots$ forms a Markov process on the state space $[0, \infty)$ with constant transition distribution,

$$(1) \quad G_{R,x}(y) = \Pr\{x_{i+1} \leq y \mid x_i = x, R\} = \begin{cases} 1 & \text{for } x = y = 0, \\ e^{-x} + \int_0^x F_0(y; x - t, R)e^{-t}\, dt \\ & \text{for } x > 0, y \geq 0. \end{cases}$$

Let S^n be the event that $x_i > 0$ for $i = 0, 1, \cdots, n$. Let F^n be the event that $x_i = 0$ for at least one $i \leq n$. Let $P(F^n \mid x, R) = 1 - P(S^n \mid x, R)$ be the prob-

ability of F^n when $x_0 = x$ and policy R is used. Let

$$(2) \qquad p(x, R) = \lim_{n \to \infty} P(F^n \mid x, R).$$

The principle of optimality provides a necessary condition for a policy R^* to minimize $p(x, R)$:

$$
\begin{aligned}
(3) \quad p(x, R^*) &= \min_R \left\{ \int_0^\infty p(y, R^*) \, dG_{R,x}(y) \right\} \\
&= \min_R \left\{ e^{-x} p(0, R^*) + \int_0^x \int_0^\infty p(y, R^*) \, dF_0(y; x - t, R) e^{-t} \, dt \right\}.
\end{aligned}
$$

This relation is satisfied by

$$(4) \qquad p(x, R^*) = e^{-\alpha x},$$

where α is the positive solution to the equation $1 - \alpha = e^{-\alpha \mu}$. A policy R^* corresponding to (4) is given by

$$(5) \qquad F(w; x, R^*) = F_0(w + x; x, R) = \begin{cases} 1 \text{ for } w \geqq \mu, \\ 0 \text{ for } w < \mu. \end{cases}$$

A proof that policy (5) yields (4) can be obtained from Kendall [3].

For equation (4) to be a solution to (3) requires

$$(6) \qquad p(x, R^*) = \int_0^\infty p(y, R^* \, dG_{R^*,x}(y)$$

and

$$(7) \qquad p(x, R^*) \leqq \int_0^\infty p(y, R^*) \, dG_{R,x}(y).$$

Equation (6) is the less informative condition on $p(x, R^*)$ since it must hold for any probability that depends only on the state of the process. However, if (6) fails, certainly $p(x, R^*)$ is *not* the desired function. From (4) and (5) the right side of (6) is

$$e^{-x} + \frac{1 - e^{-(1-\alpha)x}}{1 - \alpha} e^{-\alpha(x+\mu)},$$

and using the condition $1 - \alpha = e^{-\alpha \mu}$, this reduces to $e^{-\alpha x}$, showing (6) is satisfied.

Equation (4) will satisfy (7) if, for any distribution F_0 with expectation $m = x - t + \mu$.

$$(8) \qquad \varphi(\alpha) = \int_0^\infty e^{-\alpha(y-m)} \, dF_0(y) \geqq 1.$$

Unless F_0 is degenerate at m, $\varphi(\alpha)$ is convex and has a unique minimum. Differentiation is permissible and shows that the minimum is achieved at $\alpha = 0$. But $\varphi(0) = 1$. In case F_0 is degenerate at m, $\varphi(\alpha) = 1$.

On the other hand, for any distribution F_0 minimizing (8), $\varphi(\alpha) = 1$ and $\int_0^\infty e^{-\alpha y} \, dF_0(y) = e^{-\alpha m}$, hence the uniqueness theorem for Laplace transforms implies F_0 is degenerate at m. Consequently, among those policies R for which $p(x, R)$ satisfies (3), $p(x, R) = e^{-\alpha x}$ only for $R = R^*$ given by (5). This means that if R^* is optimal, it is also unique.

As it happens, any constant satisfies (3), as well as equation (4), so that it is not yet clear that (4) provides an optimal policy. However, by means of the previously mentioned theorem of Blackwell, optimality can readily be established. A convenient, special version of this theorem is given below as Theorem 1. Theorem 2 provides a condition under which the hypotheses of Theorem 1 are satisfied.

Let $x_0, x_1, \cdots$ be a Markov process with arbitrary state space Ω and transition probabilities $P_{x,R}(A) = \Pr \{x_{n+1} \subseteq A \mid x_n = x \text{ and policy } R \text{ is used at } x_n\}$ defined for every x, every set $A \subseteq \Omega$, and every policy R. Suppose that for every R there is a certain class of states T for which $P_{x,R}(T) = 1$ for $x \subseteq T$. Let F^n by the event that $x_i \subseteq T$ for some $i \leq n$. Let S^n be the event that $x_0, x_1, \cdots, x_n \subset \Omega - T$. Let $p(x, R) = \lim_{n \to \infty} P(F^n \mid x, R) = \lim_{n \to \infty} \Pr \{F^n \mid x_0 = x \text{ and } R \text{ is used at } x_0, x_1, \cdots\}$. Let $p_1(x, R^N) = \lim_{n \to \infty} P(F^n \mid x, R^N) = \lim_{n \to \infty} \Pr \{F^n \mid x_0 = x \text{ and } R \text{ is used } N \text{ times, at } x_0, x_1, \cdots, x_{N-1}, \text{ and } R^* \text{ is used thereafter}\}$.

THEOREM 1. *If* (i) $p(x, R^*)$ *satisfies the equation*

$$p(x, R^*) = \min_R \left\{ \int_\Omega p(y, R^*) \, dP_{R,x}(y) \right\}$$

and (ii) *for an arbitrary policy R,*

$$\lim_{N \to \infty} p_1(x, R^N) = p(x, R),$$

then $p(x, R^*) \leq p(x, R)$; i.e., R^* is optimal.

PROOF. Let $P_{R,x}^i(A) = \Pr \{x_i \subseteq A \mid x_0 = x \text{ and } R \text{ is used at } x_0, x_1, \cdots, x_{i-1}\}$; that is, $P_{R,x}^i(A) = \int_\Omega \cdots \int_\Omega \int_A dP_{R,x_{i-1}}(x_i) \, dP_{R,x_{i-2}}(x_{i-1}) \cdots dP_{R,x}(x_1)$. Then

$$(9) \qquad p_1(x, R^N) = \int_\Omega p(y, R^*) \, dP_{R,x}^N(y).$$

But $p(y, R^*) \leq \int_\Omega p(z, R^*) \, dP_{R,y}(z)$ by (i). Using this inequality in (9),

$$(10) \qquad p_1(x, R^N) \leq \int_\Omega \int_\Omega p(z, R^*) \, dP_{R,y}(z) \, dP_{R,x}^N(y) = p_1(x, R^{N+1}).$$

Thus

$$p(x, R^*) = p(x, R^0) \leq p_1(x, R^1) \leq p_1(x, R^2) \leq \cdots p_1(x, R^N) \cdots.$$

This sequence is non-decreasing and has a limit. By (ii),

$$p(x, R^*) \leq \lim_{N \to \infty} p_1(x, R^N) = p(x, R).$$

THEOREM 2. *If there is a monotone sequence of sets in* Ω, $\Omega_1 \supseteq \Omega_2 \supseteq \cdots$, *such that* (i) $\lim_{k \to \infty} \sup_{x \subset \Omega_k} p(x, R^*) = 0$ *and* (ii) $P_{R,x}(T) \geqq \gamma_k > 0$ *for* $x \subseteq \Omega - \Omega_k$ *and for every* R, *then* $\lim_{N \to \infty} p_1(x, R^N) = p(x, R)$.

PROOF. Since $S^{n+N} \subseteq S^N$,

$$p(x, R) - p_1(x, R^N) = \lim_{n \to \infty} \{P(S^{n+N} \mid x, R^N) - P(S^{n+N} \mid x, R)\}$$

$$(11) \quad = \lim_{n \to \infty} \{P(S^{n+N} \mid S^{N-1}, x, R^N) - P(S^{n+N} \mid S^{N-1}, x, R)\} P(S^{N-1} \mid x, R)$$

$$= \left\{ \int_\Omega p(y, R) \, dP^N(y) - \int_\Omega p(y, R^*) \, dP^N(y) \right\} P(S^{N-1} \mid x, R),$$

where $P^N(A) = \Pr\{x_N \subseteq A \mid S^{N-1}, x_0 = x, \text{ and } R \text{ is used at } x_0, x_1, \cdots, x_{N-1}\}$. Suppose $p(x, R) = 1$. Then $\lim_{N \to \infty} P(S^{N-1} \mid x, R) = 0$ and $\lim_{N \to \infty} p_1(x, R^N) = 1$. Suppose $p(x, R) = a < 1$. Since

$$\left\{ 1 - \int_\Omega p(y, R) \, dP^N(y) \right\} P(S^{N-1} \mid x, R) = 1 - a.$$

and $P(S^{N-1} \mid x, R)$ is non-increasing and tends to $1 - a$,

$$(12) \quad \lim_{N \to \infty} \int_\Omega p(y, R) \, dP^N(y) = 0.$$

It is also true that

$$\lim_{N \to \infty} \int_\Omega p(y, R^*) \, dP^N(y) = 0.$$

Let α and ϵ be arbitrary positive numbers. By (i) there exists a k_α such that for $k \geqq k_\alpha$, $p(x, R^*) \leqq \alpha/2$ for $x \subset \Omega_k$. By (ii), $p(x, R) \geqq \gamma_k > 0$ for $x \subseteq \Omega - \Omega_k$. From (12) there exists an N_ϵ such that for $N \geqq N_\epsilon$, $\int_\Omega p(y, R) \, dP^N(y) \leqq \epsilon$. Take $\epsilon = \gamma_{k_\alpha} \alpha/2$. Then

$$(13) \quad \gamma_{k_\alpha} \frac{\alpha}{2} \geqq \int_{\Omega - \Omega_{k_\alpha}} p(y, R) \, dP^{N_\epsilon}(y) \geqq \gamma_{k_\alpha} \int_{\Omega - \Omega_{k_\alpha}} dP^{N_\epsilon}(y)$$

and

$$(14) \quad \int_\Omega p(y, R^*) \, dP^{N_\epsilon}(y) \leqq \int_{\Omega - \Omega_{k_\alpha}} dP^{N_\epsilon}(y) + \frac{\alpha}{2} \int_{\Omega_{k_\alpha}} dP^{N_\epsilon}(y).$$

Canceling γ_{k_α} in (13) and using (14), $\int_\Omega p(y, R^*) \, dP^{N_\epsilon}(y) \leqq \alpha$. This completes the proof of Theorem 2.

To apply Theorems 1 and 2 to the model under consideration, $\Omega \equiv [0, \infty)$, T consists of the single point $x = 0$, and $P_{R,x}$ is taken to be $G_{R,x}$ defined by (1). The sequence of sets $\Omega_1, \Omega_2, \cdots$, can be a sequence of intervals $\{[x_i, \infty)\}$, $x_{i+1} - x_i \geqq \delta > 0$, $i = 1, 2, \cdots$. Since $p(x, R^*) = e^{-\alpha x}$, condition (i) of Theorem 2 is satisfied, as is condition (ii), since $P_{R,x}(T)$ is then at least e^{-x_k} for $x \subseteq \Omega - \Omega_k$. Theorem 1 may then be applied and R^* defined by (5) is optimal.

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FIRST PASSAGE TIME FOR A PARTICULAR GAUSSIAN PROCESS

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1. Introduction. Let $x(t)$ be a stationary Gaussian process with $Ex(t) = 0$ and $E[x(t)x(t')] = \rho(t - t')$. Denote by $Q_a(T \mid x_0) \, dT$ the conditional probability that for $t > 0$, $x(t)$ first assumes the value a in the interval $T \leq t \leq T + dT$ given that $x(0) = x_0$. It is well known that the determination of the first passage time probability $Q_a(T \mid x_0) \, dT$ is not an easy matter in general. To the author's knowledge, $Q_a(T \mid x_0)$ is known explicitly for stationary Gaussian processes with continuous spectral densities only in the Markovian case $\rho(\tau) = e^{-|\tau|}$. See [1], [2], [3] and [4]. This note points out that an elementary solution exists for the process with covariance

$$(1) \qquad \rho(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leq 1 \\ 0, & |\tau| \geq 1 \end{cases}$$

for $0 \leq T \leq 1$.

2. Markoff-Like Property. The determination of the first passage time probability density $Q_a(T \mid x_0)$ for the process with covariance (1) follows from a peculiar Markoff-like property it possesses which may be described roughly as follows. Let $0 < t_1 < t_2 < 1$ be two instants in the unit interval. Denote the open interval (t_1, t_2) by A and the set $(0, t_1) \cup (t_2, 1)$ by B. Then for the process at hand, given the values of $x(t_1)$ and $x(t_2)$, events defined on A are statistically independent of events defined on B.

More precisely, we show the following. Let

$$0 < t_1 < t_2 < \cdots < t_k < \cdots < t_l < \cdots < t_n < 1.$$

Then

$$(2) \qquad \begin{aligned} &p(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{l-1}, x_{l+1}, \cdots, x_n \mid x_k, x_l) \\ &= p(x_1, \cdots, x_{k-1}, x_{l+1}, \cdots, x_n \mid x_k, x_l) p(x_{k+1}, \cdots, x_{l-1} \mid x_k, x_l). \end{aligned}$$