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ON A PROPERTY OF A TEST FOR THE EQUALITY OF TWO NORMAL DISPERSION MATRICES AGAINST ONE-SIDED ALTERNATIVES[1]

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1. Introduction and Summary. The purpose of this paper is to establish the monotonicity property of some tests suggested by Roy and Gnanadesikan [2] for the problem of testing the null hypothesis of equality of two dispersion matrices against some specific alternatives. If Σ_1 and Σ_2 denote the dispersion matrices of two non-singular p-variate normals and $\gamma_1, \gamma_2, \cdots, \gamma_p$ denote the characteristic roots (all positive) of $\Sigma_1\Sigma_2^{-1}$, then the null hypothesis is H_0 : all γ_i's are equal to unity. The alternative hypotheses to be considered are: (i) $H_1 : \gamma_m > 1$; (ii) $H_2 : \gamma_M < 1$; (iii) $H_3 : \gamma_M > 1$; (iv) $H_4 : \gamma_m < 1$, where γ_m and γ_M denote, respectively, the smallest and the largest of the γ_i.

Let us denote the largest and smallest characteristic roots of any square matrix A by $\text{ch}_{\max}(A)$ and $\text{ch}_{\min}(A)$, respectively.

2. Case I. $H_1 : \gamma_m > 1$. The three-decision procedure suggested in [2] for this case can be expressed, by reducing the problem to the canonical form (cf. [1], pp. 188), in the following way:

(i) Accept H_0 against H_1 if

$$\mathfrak{D}: \text{ch}_{\max}(XX')(YY')^{-1} < \lambda. \tag{2.1}$$

(ii) Accept H_1 against H_0 if

$$W: \text{ch}_{\min}(XX')(YY')^{-1} > \lambda. \tag{2.2}$$

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[1] Part of this work was done while the author was a graduate student in the Department of Statistics, University of North Carolina.

(iii) Make no decision otherwise, where λ is determined such that, for preassigned $\alpha(0 < \alpha < 1)$, $\Pr\{\mathfrak{D} \mid H_0\} = \alpha$; and

$$\underset{p\times n_1}{X} = \begin{bmatrix} x_{11} & \cdots & x_{1n_1} \\ - & \cdots & - \\ x_{p1} & \cdots & x_{pn_1} \end{bmatrix}; \qquad \underset{p\times n_2}{Y} = \begin{bmatrix} y_{11} & \cdots & y_{1n_2} \\ - & \cdots & - \\ y_{p1} & \cdots & y_{pn_2} \end{bmatrix}$$

have, under H_1, the probability distribution (cf. [1] pp. 189).

$$(2.3) \quad (1/[2\pi])^{\frac{1}{2}[p(n_1+n_2)]} \prod_{i=1}^{p} \gamma_i^{-\frac{1}{2}n} \exp\left[-\tfrac{1}{2} \operatorname{tr}\{D_{1/\gamma} XX' + YY'\}\right] dXdY,$$

where $D_{1/\gamma} = \operatorname{diag}(1/\gamma_1, 1/\gamma_2, \cdots, 1/\gamma_p)$. In what follows, we are going to show that the probability of accepting H_0, when H_1 is true, decreases monotonically as each noncentrality parameter, γ_i, separately, increases.

This probability is obtained by integrating (2.3) over the domain $\mathfrak{D}$ of (2.1) and may be written in the form

$$(2.4) \qquad \text{Const.} \int_{\mathfrak{D}^*} \exp[-\tfrac{1}{2} \operatorname{tr}\{[XX' + YY'\}]\, dXdY.$$

Where the integrand is free from the γ's and the domain $\mathfrak{D}^*$ is merely the domain $\mathfrak{D}$ of (2.1) scaled by $(1/\gamma_i)$ in the directions of $x_{i1}, \cdots, x_{in_1}$; $(i = 1, 2, \cdots, p)$ to allow for the change in the intergrand and the implicit equation for the boundary of $\mathfrak{D}$.

But it has been shown in [3] that, for given values of λ, Y and the elements of the matrix X other than those in the ith row, $\mathfrak{D}$ of (2.1) represents a domain in $(x_{i1}, x_{i2}, \cdots, x_{in_1})$ which is an n_1-dimensional ellipsoid with center at the origin. Thus scaling by $1/\gamma_i$ in the directions of x_{ij}'s $(j = 1, \cdots, n_1)$ will produce an ellipsoid completely imbedded in the original one when, as the case is, $\gamma_i > 1$. This imbedded property of the integration domains, since it holds for any i, establishes the required result.

3. Case II. $H_2 : \gamma_m < 1$. By interchanging Σ_1 with Σ_2 and X with Y, this case can be thrown back to Case I and it follows that the probability of accepting H_0 when H_2 is true decreases monotonically as each γ_i decreases.

4. Case III. $H_3 : \gamma_M > 1$. For this case, the following two-decision procedure is given in [2]: Accept H_0 against H_3 if

$$(4.1) \qquad \mathfrak{D}_1 : \operatorname{ch}_{\max} (XX')(YY')^{-1} \leqq \mu',$$

and reject otherwise.

It is the purpose of this section to show that, for given values of all the γ's, other than γ_M, the probability of Type II error of the proposed test decreases monotonically as γ_M increases.

The probability of Type II error is obtained by integrating (2.3) over $\mathfrak{D}_1$. Without loss of generality, let $\gamma_M = \gamma_p$. For given values $(\gamma_1^*, \gamma_2^*, \cdots, \gamma_{p-1}^*)$

of $(\gamma_1, \gamma_2, \cdots, \gamma_{p-1})$ this probability can be expressed, aside from a constant, as

$$\int_{\mathfrak{D}_1^*} \exp\left[(-\tfrac{1}{2})\left\{\sum_{j=1}^{n_1} x_{pj}^2 + \sum_{i=1}^{p-1} 1/\gamma_i^* \sum_{j=1}^{n_1} x_{ij}^2 + \sum_{i=1}^{p}\sum_{j=1}^{n_2} y_{ij}^2\right\}\right] dXdY,$$

where $\mathfrak{D}_1^*$ is merely $\mathfrak{D}_1$ scaled by $1/\gamma_p$. As indicated in Section (2), and since $\gamma_p > 1$, this scaling will produce an ellipsoid completely imbedded in the original one. Hence the probability of Type II error decreases monotonically as $\gamma_p(= \gamma_M)$ increases (conditionally on the other γ_i's).

The result for $H_4 : \gamma_m < 1$ follows immediately from Case III as Case II follows from Case I.

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