and

$$\sup_{F \varepsilon \mathfrak{F}^{(0,1/3,0)}_{[-1,1]}} E\{e^{tX}\} = \tfrac{2}{3} + (\cosh t)/3 \quad \text{and} \quad \inf_{F \varepsilon \mathfrak{F}^{(0,1/3,0)}_{[-1,1]}} E\{e^{tX}\} = (\cosh \sqrt{3}t)/3.$$

It is readily verified that $(\cosh \sqrt{3}\, t)/3 \leqq (\sinh t)/t \leqq \frac{2}{3} + (\cosh t)/3$, where $(\sinh t)/t$ is the moment generating function of the rectangular distribution on $[-1, 1]$.

3. Let $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 6$, i.e., the first three moments of the exponential distribution with mean unity. Then

$$\inf_{F \varepsilon \mathfrak{F}^{(1,2,6)}_{[0,\infty]}} E\{e^{tX}\} = (3 + 2\sqrt{2})(4 + 2\sqrt{2})^{-1} \exp\{(\sqrt{2}t)(1 + \sqrt{2})^{-1}\} + (4 + 2\sqrt{2})^{-1} \exp\{2 + \sqrt{2}t\}.$$

$$F_1^*(x) = \begin{cases} 0, & x < \sqrt{2}(1 + \sqrt{2})^{-1}, \\ (3 + 2\sqrt{2})(4 + 2\sqrt{2})^{-1}, & \sqrt{2}(1 + \sqrt{2})^{-1} \leqq x < 2 + \sqrt{2}, \\ 1, & 2 + \sqrt{2} \leqq x. \end{cases}$$

The supremum does not exist.

## REFERENCES

[1] Harris, Bernard (1959). Determining bounds on integrals with applications to cataloging problems. *Ann. Math. Statist.* **30** 521–548.

[2] Wald, Abraham (1939). Limits of a distribution function determined by absolute moments and inequalities satisfied by absolute moments. *Trans. Amer. Math. Soc.* **46** 280–306.

# ON BOUNDS OF SERIAL CORRELATIONS

By K. C. Chanda

*Washington State University*

**1. Introduction and summary.** The role of serial correlations in time series analysis is well known. Considerable attention has been given to the derivation of their sampling properties when the sample size is both small and large. In all these discussions it has been tacitly assumed that these correlations are bounded between $-1$ and 1. At least, no literature exists which considers it otherwise. Whereas it is true that the serial correlations are all bounded it is not true that the bounds are $-1$ and 1. In fact, in small samples these bounds may very well be lower than $-1$ and higher than 1. To the best of the author's knowledge, this fact has not been mentioned anywhere. The purpose of this note is to discuss this particular aspect.

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**2. Bounds of serial correlations.** Let us for the sake of simplicity define the $s$th order serial correlation $r_s$ by

$$(2.1)\qquad r_s = C_s/C_0 \qquad 0 \leqq s < n$$
$$C_s = \sum_{t=1}^{n-s} x_t x_{t+s}/(n-s)$$

where we assume that the $x_t$'s can take all possible real values between $-\infty$ and $\infty$. It is easy to show that $r_s$ is bounded. In fact

$$(2.2)\qquad |r_s| \leqq \left\{\left(\sum_{t=1}^{n-s} x_t^2 \sum_{t=1}^{n-s} x_{t+s}^2\right)^{\frac{1}{2}} \Big/ \sum_{t=1}^{n} x_t^2\right\} n/(n-s) < n/(n-s)$$

$|r_s|$ can never attain the value $n/(n-s)$ because this implies and is implied by the conditions $x_t = \alpha x_{t+s}$ for all $t(1 \leqq t \leqq n-s)$ where $\alpha$ is an arbitrary real constant and $x_t = 0$ for $t \leqq s$ and $\geqq n - s + 1$, which, however, means that $x_t = 0$ for all $t(1 \leqq t \leqq n)$. On the other hand, it is possible to prove that $\max |r_s| > 1$. We shall consider first the particular case $s = 1$.

With this end in view we now pose the problem of maximizing $\sum_{t=1}^{n-1} x_t x_{t+1}$ subject to $\sum_{t=1}^{n} x_t^2 =$ constant. The $x_t$'s will then satisfy the following normal equations

$$(2.3)\qquad \begin{aligned} x_2 &= \lambda x_1 \\ x_t - \lambda x_{t-1} + x_{t-2} &= 0 \qquad 3 \leqq t \leqq n \\ x_{n-1} &= \lambda x_n \\ \sum_{t=1}^{n} x_t^2 &= \text{constant} \end{aligned}$$

where $\lambda$ is the Lagrangian undetermined multiplier. The solutions have been derived by Grenander and Szegö (1958) and are given by

$$(2.4)\qquad x_t = x_{n+1-t} = \sin t\theta/\sin\theta \qquad 1 \leqq t \leqq n$$

and $\lambda$ satisfies the equation $(4 - \lambda^2)^{\frac{1}{2}}/\lambda = \tan[\pi/(n+1)]$ i.e., $\lambda = 2\cos[\pi/(n+1)]$. Consequently,

$$r_1 = n\{(n - 1 - C_{n-1})\cos\theta + S_{n-1}\sin\theta\}/\{(n - C_n)(n-1)\}$$

where

$$C_n = \sum_{t=1}^{n} \cos 2t\theta,\ S_n = \sum_{t=1}^{n} \sin 2t\theta,\ \theta = \pi/(n-1).$$

We note that $\cos 2(n+1)\theta = 1$, $\cos 2n\theta = \cos 2\theta$, $\sin 2(n+1)\theta = 0$, and $\sin 2n\theta = -\sin 2\theta$. After some simplification, using standard formulae for $C_n$ and $S_n$ and the above relations we can show that

$$(2.5)\qquad r_1 = \{n/(n-1)\}\cos\{\pi/(n+1)\}.$$

TABLE I

*Extreme values of $r_1$ for selected values of $n$*

| $n$ | $r_1$ |
|---|---|
| 3 | $\pm 1.061$ |
| 4 | $\pm 1.079$ |
| 5 | $\pm 1.083$ |
| 6 | $\pm 1.081$ |
| 7 | $\pm 1.078$ |
| 8 | $\pm 1.074$ |

The right hand side expression of (2.5) is always $> 1$ for $n \geqq 3$ because $\cos\{\pi/(n+1)\} \geqq 1 - \frac{1}{2}\{\pi/(n+1)\}^2 > 1 - 1/n$ for $n \geqq 3$. If we take $x_t = (-1)^t \sin t\theta/\sin\theta$, then $r_1 = -\{n/(n-1)\}\cos\{\pi/(n+1)\}$. Table I gives these values of $r_1$ for various values of $n$. Note that

$$d\,|r_1|/dn = \cos[\pi/(n+1)]/(n-1)^2\{\tan[\pi/(n+1)]n(n-1)\pi/(n+1)^2 - 1\}$$

which is negative if $\tan[\pi/(n+1)] < (n+1)^2/[n\pi(n-1)]$. It is easy to show that the inequality holds if $n \geqq 10$ so that from $n = 10$ onward $|r_1|$ steadily decreases and its asymptotic value is $n/(n-1)$.

Consider, now, the general case $s > 1$. The normal equations in this case are

$$\begin{aligned} x_{t+s} &= \lambda x_t & 1 \leqq t \leqq s \\ x_t - \lambda x_{t-s} + x_{t-2s} &= 0 & 2s+1 \leqq t \leqq n \\ x_t &= \lambda x_{t+s} & n-2s+1 \leqq t \leqq n-s. \end{aligned} \tag{2.6}$$

Let $n = ms + u\,(0 \leqq u < s; m \geqq 1)$. If $m = 1$, $n < 2s$ so that the summation $\sum_{t=1}^{n-s} x_t x_{t+s}$ does not involve the quantities $x_{n-s+1}, \cdots, x_s$. Since we are maximizing the sum keeping $\sum_{t=1}^{n} x_t^2 =$ constant the first step of the procedure would then be to put $x_{n-s+1} = \cdots = x_s = 0$. Write $y_j = x_j$, $j \leqq n-s$, $y_{n-s+j} = x_{s+j}$, $1 \leqq j \leqq n-s$, $N = 2n - 2s$, $s' = n - s$. Then

$$r_s = \sum_{t=1}^{N-s'} y_t y_{t+s'} \Big/ \sum_{t=1}^{N} y_t^2 = r_{s'}\,.$$

Note that the reduced value of $n$ viz., $N = 2s'$. Hence we can assume without any loss of generality that in the representation of $n$ above $m \geqq 2$.

Define $\xi_{t,k} = x_{s(t-1)+k}\,(1 \leqq t \leqq m, 1 \leqq k \leqq s)$. The equations (2.6) will then reduce to

$$\begin{aligned} \xi_{2,k} &= \lambda\xi_{1,k} & \\ \xi_{t,k} - \lambda\xi_{t-1,k} + \xi_{t-2,k} &= 0 & 3 \leqq t \leqq m_k + 1 \\ \xi_{m_k,k} &= \lambda\xi_{m_k+1,k} & 1 \leqq k \leqq s \end{aligned} \tag{2.7}$$

TABLE II

*Extreme values of $r_s$ for selected values of $s$ and $n$*

| $n$ | $r_2$ | $n$ | $r_4$ | $n$ | $r_6$ | $n$ | $r_8$ |
|---|---|---|---|---|---|---|---|
| 4 | ±1.000 | 8 | ±1.000 | 12 | ±1.000 | 16 | ±1.000 |
| 5 | ±1.179 | 9 | ±1.273 | 13 | ±1.313 | 17 | ±1.336 |
| 6 | ±1.061 | 10 | ±1.179 | 14 | ±1.237 | 18 | ±1.273 |
| 7 | ±1.133 | 11 | ±1.111 | 15 | ±1.179 | 19 | ±1.221 |
| | | 12 | ±1.061 | 16 | ±1.131 | 20 | ±1.179 |
| | | | | 17 | ±1.093 | 21 | ±1.142 |
| | | | | 18 | ±1.061 | 22 | ±1.111 |
| | | | | | | 23 | ±1.084 |
| | | | | | | 24 | ±1.061 |

where $m_k = m$ or $m - 1$ according as $1 \leqq k \leqq u$ or $u + 1 \leqq k \leqq s$. We have then

$$r_s = n/(n - s) \left\{ \sum_{k=1}^{s} \sum_{t=1}^{m_k} \xi_{t,k}\, \xi_{t+1,k} \Big/ \sum_{k=1}^{s} \sum_{t=1}^{m_k+1} \xi_{t,k}^2 \right\}. \tag{2.8}$$

Note that the set of equations (2.8) is the same as (2.3) except that we maximize $|r_s|$ in two steps viz., (i) maximizing $\sum_{t=1}^{m_k} \xi_{t,k}\xi_{t+1,k}$ subject to $\sum_{t=1}^{m_k+1} \xi_{t,k}^2 =$ constant and (ii) maximizing (i). The first step leads to

$$r_s = n/(n - s)\{\sum_{k=1}^{s} z_k^2 \cos [\pi/(m_k + 2)]/\sum_{k=1}^{s} z_k^2\}$$

where $z_k^2 = \sum_{t=1}^{m_k+1} \xi_{t,k}^2$. This value of $|r_s|$ is now maximized by putting $z_k^2 = 0$ for $u + 1 \leqq k \leqq s$. This, evidently, implies that $\xi_{t,k} = 0$ for $1 \leqq t \leqq m_k + 1$, $u + 1 \leqq k \leqq s$. It is easy to see, then, that

$$\max |r_s| = \begin{cases} n/(n - s) \cos [\pi/(m + 2)] & \text{if } u > 0, \\ m/(m - 1) \cos [\pi/(m + 1)] & \text{if } u = 0, \end{cases} \tag{2.9}$$

where we note that $m = [n/s]$ and $u = n - ms$. Since $n/(n - s) \geqq (m + 1)/m$ for $u > 0$, it follows that max $|r_s|$ has the same value as max $|r_1|$ based on $m$ observations.

**3. Illustrations.** Table II gives the values of a few extreme values of the serial correlations $r_s$ for various values of $s$.

## REFERENCE

GRENANDER, U. and SZEGO, G. (1958). *Toeplitz Forms and their Applications*. Univ. of California Press.