LIMITING DISTRIBUTIONS ASSOCIATED WITH CERTAIN STOCHASTIC LEARNING MODELS

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1. Introduction and summary. The work of Bush and Mosteller [1] on stochastic learning models leads to certain limiting distributions of response probabilities which are of a somewhat different character from those commonly encountered in mathematical statistics. These authors have derived many useful relations between the moments of the distributions but, in deducing the general nature of the distributions themselves, they have relied chiefly on Monte Carlo methods. In the present paper, we consider the two experimenter-controlled events model and, in the "equal alpha" case, derive limiting distributions for certain special cases by inverting the solution of a moment generating functional equation.

2. The moment generating functional equation for the limiting distribution of response probabilities. Consider a Markov process $x_0, x_1, \cdots$ satisfying the following conditions:[1]

(a) x_0 has an arbitrary distribution on $(0, 1)$,

(b) If x_n is given, then $x_{n+1} = a_0 + \alpha_0 x_n$ with probability π_0 and $x_{n+1} = a_1 + \alpha_1 x_n$ with probability π_1,

(c) $\pi_0 + \pi_1 = 1$, $0 \leqq a_j \leqq 1$ and $0 \leqq \alpha_j \leqq 1 - a_j$ $(j = 0, 1)$.

The random variable x_n is called the "response probability on trial n." Thus, for given x_0, x_n has 2^n possible realizations, say, $x_{\nu n}$ $(\nu = 1, 2, \cdots, 2^n)$. Let $P_{\nu n} = \Pr(x_n = x_{\nu n})$ and let $m_{n+1}(\theta) = E\{\exp(x_{\nu,n+1}\theta)\}$ be the moment generating function of the distribution of x_{n+1}. Then

$$\begin{aligned} m_{n+1}(\theta) &= \sum_{\nu=1}^{2^{n+1}} e^{x_{\nu,n+1}\theta} P_{\nu,n+1} \\ &= \sum_{\nu=1}^{2^n} [e^{(a_0+\alpha_0 x_{\nu n})\theta}\pi_0 P_{\nu n} + e^{(a_1+\alpha_1 x_{\nu n})\theta}\pi_1 P_{\nu n}] \\ &= \pi_0 e^{a_0\theta} \sum_{\nu=1}^{2^n} e^{(\alpha_0\theta)x_{\nu n}} P_{\nu n} + \pi_1 e^{a_1\theta} \sum_{\nu=1}^{2^n} e^{(\alpha_1\theta)x_{\nu n}} P_{\nu n} \\ &= \pi_0 e^{a_0\theta} m_n(\alpha_0\theta) + \pi_1 e^{a_1\theta} m_n(\alpha_1\theta). \end{aligned}$$

Karlin [2] has shown that a limiting distribution exists, so letting $n \to \infty$ we obtain the following functional equation for the moment generating function of the limiting distribution of response probabilities:

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[1] We are indebted to a referee for this concise formulation of the process.

$$m(\theta) = \pi_0 e^{a_0\theta} m(\alpha_0\theta) + \pi_1 e^{a_1\theta} m(\alpha_1\theta), \tag{2.1}$$

where $m(\theta) = \lim_{n\to\infty} m_n(\theta)$.

3. Solution of the m.g. functional equation for the equal alpha case. If $\alpha_0 = \alpha_1 = \alpha$, (2.1) becomes

$$m(\theta) = m(\alpha\theta)[\pi_0 e^{a_0\theta} + \pi_1 e^{a_1\theta}],$$

so that

$$m(\alpha\theta) = m(\alpha^2\theta)[\pi_0 e^{a_0\alpha\theta} + \pi_1 e^{a_2\alpha\theta}]$$

giving

$$m(\theta) = [\pi_0 e^{a_0\theta} + \pi_1 e^{a_1\theta}][\pi_0 e^{a_0\alpha\theta} + \pi_1 e^{a_1\alpha\theta}] m(\alpha^2\theta).$$

Continuing this procedure iteratively and assuming $\alpha < 1$, we obtain

$$m(\theta) = \prod_{n=0}^{\infty} [\pi_0 e^{a_0\alpha^n\theta} + \pi_1 e^{a_1\alpha^n\theta}]. \tag{3.1}$$

Let x be the random variable of the limiting distribution of response probabilities. It can be shown ([1], p. 98) that this limiting distribution is such that if

$$\lambda_0 = a_0/(1-\alpha), \qquad \lambda_1 = a_1/(1-\alpha),$$

and if I is any interval disjoint from the closed interval joining λ_0 and λ_1, then Pr $(x \subset I) = 0$. In other words, the limiting response probabilities are trapped between the *limit points* λ_0 and λ_1.

It will be convenient to transform from the limiting random variable x to

$$y = [(\lambda_1 + \lambda_0)/(\lambda_1 - \lambda_0)] - [2x/(\lambda_1 - \lambda_0)], \tag{3.2}$$

so that y is distributed on $(-1, 1)$. From (3.1), the characteristic function of the distribution of y is

$$\begin{aligned}\varphi_y(\theta) &= e^{i\theta(\lambda_1+\lambda_0)/(\lambda_1-\lambda_0)} \\ &\quad \cdot \prod_{n=0}^{\infty} [\pi_0 e^{-2i\theta(1-\alpha)\alpha^n\lambda_0/(\lambda_1-\lambda_0)} + \pi_1 e^{-2i\theta(1-\alpha)\alpha^n\lambda_1/(\lambda_1-\lambda_0)}] \\ &= e^{i\theta(\lambda_1+\lambda_0)/(\lambda_1-\lambda_0)} \\ &\quad \cdot \prod_{n=0}^{\infty} \{e^{-i\theta(1-\alpha)\alpha^n(\lambda_1+\lambda_0)/(\lambda_1-\lambda_0)}[\pi_0 e^{i\theta(1-\alpha)\alpha^n} + \pi_1 e^{-i\theta(1-\alpha)\alpha^n}]\}\end{aligned}$$

which readily reduces to

$$\varphi_y(\theta) = \prod_{n=0}^{\infty} [\pi_0 e^{i\theta(1-\alpha)\alpha^n} + \pi_1 e^{-i\theta(1-\alpha)\alpha^n}]. \tag{3.3}$$

4. Special cases. In this section we illustrate by some simple examples the types of limiting distributions which can arise in the equal alpha case as α ranges from 0 to 1. In all cases $\pi_0 = \pi_1 = 1/2$.

(i) $\alpha = 0$. In this case $\varphi_y(\theta) = \cos\theta$ which is the characteristic function of a binomial distribution where the random variable y takes values $\pm$ 1 with equal probability of 1/2.

(ii) $\alpha = 1/4$. Here the nature of limiting distribution becomes clear from examining the characteristic function

$$\begin{aligned} \varphi_y(\theta) &= \prod_{n=0}^{\infty} \cos\ (3\theta/2^{2n+2}) \\ &= (\sin 3\theta)/3\theta \prod_{n=0}^{\infty} \sec\ (3\theta/2^{2n+1}) \qquad (4.1) \\ &= \lim_{k\to\infty} \psi_k(\theta), \end{aligned}$$

where we have used the identity $\sin t = t\prod_{m=1}^{\infty} \cos\ (t/2^m)$, and where

$$\begin{aligned} \psi_0(\theta) &= (\sin 3\theta)/3\theta \\ \psi_k(\theta) &= (\sin 3\theta)/3\theta \prod_{n=0}^{k-1} \sec\ (3\theta/2^{2n+1}), \qquad k \geqq 1. \qquad (4.2) \end{aligned}$$

From this it is easily shown that

$$\psi_k(\theta) = \psi_{k-1}(\theta/4) \cos\ (3\theta/4), \qquad k = 1, 2, \cdots$$

Let

$$g_k(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_k(\theta) e^{-i\theta y}\, d\theta,$$

be the density function corresponding to the characteristic function $\psi_k(\theta)$. Then

$$g_k(y) = 2[g_{k-1}(4y - 3) + g_{k-1}(4y + 3)], \qquad k = 1, 2, \cdots. \qquad (4.3)$$

From this formula, we can build up a sequence of probability distributions which tend in the limit to the desired distribution. Thus

$$\begin{aligned} g_0(y) &= 1/6 && (-3 < y < 3) \\ &= 0 && (y < -3, y > 3), \\ g_1(y) &= 1/3 && (-3/2 < y < 3/2) \\ &= 0 && (y < -3/2, y > 3/2), \\ g_2(y) &= 2/3 && (-9/8 < y < -3/8, 3/8 < y < 9/8) \\ &= 0 && (y < -9/8, -3/8 < y < 3/8, y > 9/8), \\ g_3(y) &= 4/3 && (-33/32 < y < -27/32, -21/32 < y < -15/32, \\ &&& \quad 15/32 < y < 21/32, 27/32 < y < 33/32) \\ &= 0 && (y < -33/32, -27/32 < y < -21/32, -15/32 < y < 15/32, \\ &&& \quad 21/32 < y < 27/32, y > 33/32), \end{aligned}$$

and so forth.

At the nth stage there are 2^{n-1} intervals of nonzero probability density. Let them be $I_{n,m}$ ($m = 1, 2, \cdots, 2^{n-1}$; $n \geqq 1$). For example, for $n = 3$ there are four such intervals,

$$I_{3,1} = (-33/32, -27/32), \qquad I_{3,2} = (-21/32, -15/32),$$
$$I_{3,3} = (15/32, 21/32), \qquad I_{3,4} = (27/32, 33/32).$$

Each interval $I_{n,m}$ at stage n generates two new intervals at stage $(n + 1)$ in the following manner: divide $I_{n,m}$ into eight equal subintervals $i_{n,m,1}$, $i_{n,m,2}, \cdots, i_{n,m,8}$ from left to right and discard $i_{n,m,1}$, $i_{n,m,4}$, $i_{n,m,5}$ and $i_{n,m,8}$. Then the two new intervals consist of the union of $i_{n,m,2}$ and $i_{n,m,3}$ and the union of $i_{n,m,6}$ and $i_{n,m,7}$. The probability distribution at stage $(n + 1)$ is then obtained by constructing, over the new intervals, identical rectangles of height double that of the preceding stage. From this construction, it follows that if w_n and h_n are respectively the width and height of a rectangle at the nth stage, then $w_n = 3/2^{2(n-1)}$ and $h_n = 2^{n-1}/3$. The total width of the intervals at stage n is $W_n = 2^{n-1} \cdot 3/2^{2(n-1)} = 3/2^{n-1}$. As $n \to \infty$, $W_n \to 0$ and hence the probability is ultimately concentrated on a set of points of Lebesgue measure 0. The close analogy between the construction described above and that of the Cantor Ternary Set suggests that the limiting distribution is concentrated on a nonenumerable Cantor-like set of points of Lebesgue measure zero and that the cumulative distribution function is a continuous increasing function which is not an integral (cf., Titchmarsh [3], p. 329 and p. 366).

From equation (4.3), it is clear that the development is equivalent to starting with a random variable y_0, say, distributed uniformly on $(-3, 3)$ and then proceeding according to the transitions of the original Markov process.[2]

(iii) $\alpha = 1/2$. In this case,

$$\varphi_y(\theta) = \prod_{n=0}^{\infty} 1/2\,(e^{i\theta/2^{n+1}} + e^{-i\theta/2^{n+1}})$$
$$= \prod_{k=1}^{\infty} \cos\,(\theta/2^k) = \sin\theta/\theta,$$

which is the characteristic function of a random variable which is uniform over $(-1, 1)$. This example was noted by Karlin ([2], p. 755).

(iv) $\alpha = 1/2^{\frac{1}{2}}$. Here,

$$\varphi_y(\theta) = \prod_{n=0}^{\infty} \cos\,[(2^{\frac{1}{2}} - 1)\theta/2^{(n+1)/2}]$$
$$= \prod_{n=1}^{\infty} \cos\,[(2^{\frac{1}{2}} - 1)\theta/2^n] \prod_{n=1}^{\infty} \cos\,[(2 - 2^{\frac{1}{2}})\theta/2^n]$$
$$= \{\sin\,[(2^{\frac{1}{2}} - 1)\theta]/(2^{\frac{1}{2}} - 1)\theta\} \cdot \{\sin\,[(2 - 2^{\frac{1}{2}})\theta]/(2 - 2^{\frac{1}{2}})\theta\}.$$

It follows from the last equation that the distribution of y is the convolution of a uniform distribution over $\{-(2^{\frac{1}{2}} - 1), (2^{\frac{1}{2}} - 1)\}$ with a uniform distribu-

[2] We are indebted to a referee for drawing this fact to our attention.

tion over $\{-(2-2^{\frac{1}{2}}),\ (2-2^{\frac{1}{2}})\}$. Thus the density function $f(y)$ of the limiting distribution is given by,

$$
\begin{aligned}
f(y) &= (1+y)/[4(2^{\frac{1}{2}})(2^{\frac{1}{2}}-1)^2] && (-1 < y < -[3-2(2^{\frac{1}{2}})]),\\
&= 1/[2(2^{\frac{1}{2}})(2^{\frac{1}{2}}-1)] && (-[3-2(2^{\frac{1}{2}})] < y < [3-2(2^{\frac{1}{2}})]),\\
&= (1-y)/[4(2^{\frac{1}{2}})(2^{\frac{1}{2}}-1)^2] && ([3-2(2^{\frac{1}{2}})] < y < 1),\\
&= 0 && (y < -1,\ y > 1).
\end{aligned}
$$

(v) $\alpha = 1$. In this case, $\varphi_y(\theta) = 1$, and so,

$$
\begin{aligned}
\Pr(y = 0) &= 1,\\
\Pr(y = w) &= 0 \qquad (w \neq 0).
\end{aligned}
$$

In other words, this is a deterministic case in which all of the probability is ultimately concentrated at the origin.

In summary, as α ranges from 0 to 1, the distribution of y evolves from one which is as dispersed as possible (i.e., binomial with the probability equally concentrated at the extremes of the range of y) to a deterministic distribution with all of the probability concentrated at the midrange of y.

REFERENCES

[1] Bush, Robert R. and Mosteller, Frederick (1955). *Stochastic Models for Learning.* Wiley, New York.

[2] Karlin, Samuel (1953). Some random walks arising in learning models I. *Pacific J. of Math.* **3** 725–756.

[3] Titchmarsh, E. C. (1939). *The Theory of Functions* (2nd ed.). Oxford University Press, London.