A CHARACTERIZATION OF THE CAUCHY DISTRIBUTION

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1. Introduction and preliminaries. Let a random variable (r.v.) ξ be called stable if its distribution function belongs to the class of stable laws. We shall set two r.v.'s equal to each other if they have the same distribution. All r.v.'s appearing in this paper are assumed to be proper, i.e., there is no point x_0 such that $F(x_0 +) - F(x_0) = 1$, where $F(x)$ is the distribution function of a r.v. Let $g(x)$ be a real-valued function of the real variable x, such that $g'(x) > 0$ for all x, $g(x) = Ax + O(1)$, and $g'(x) = A + O(1/x^{\epsilon})$, as $|x| \to \infty$. Here A is any positive (non-zero) constant and $\epsilon > 0$. We will say that the r.v. ξ satisfies *Condition A* if $1/\xi = g(\xi)$.

We prove (Theorem 2.1) that if a r.v. is proper, stable and satisfies Condition A, then it must have the Cauchy distribution, and in this case $g(x)$ is of the form ax for an appropriate constant a. Due to a referee is the theorem (Theorem 2.2) that if a r.v. and its reciprocal are both stable, then it has a Cauchy distribution.

Let ξ be a stable r.v. Then its characteristic function $\phi(t)$ is of the form [1, p. 164]

$$\phi(t) = \exp\{i\gamma t - c|t|^{\alpha}(1 - i\beta \operatorname{sign} t \tan(\pi/2)\alpha)\}, \qquad \alpha \neq 1,$$

$$\phi(t) = \exp\{i\gamma t - c|t|(1 + i(2/\pi)\beta \operatorname{sign} t \ln |t|)\}, \qquad \alpha = 1,$$

where $0 < \alpha \leqq 2$, $-1 \leqq \beta \leqq 1$, $c > 0$ and γ is any real number.

The p.d.f. $f(x) = f(x, \alpha, \beta)$ of ξ exists, since all proper stable laws are continuous and have derivatives of all orders at every point [1, p. 183]. In particular $f(0)$ is finite.

Now, let

$$\eta = (\xi - \gamma)/c^{1/\alpha}, \qquad \alpha \neq 1,$$

$$\eta = (\xi - \gamma)/c - (2/\pi)\beta \ln c, \qquad \alpha = 1.$$

The characteristic function of η is given by

$$\exp\{-|t|^{\alpha}(1 - i\beta \operatorname{sign} t \tan(\pi/2)\alpha)\}, \qquad \alpha \neq 1,$$

$$\exp\{-|t|(1 + i(2/\pi)\beta \operatorname{sign} t \ln |t|)\}, \qquad \alpha = 1.$$

Denote the p.d.f. of η by $p(x, \alpha, \beta) = p(x)$. Finally, the symbol $O(x)$ will always be understood to mean $O(x)$ as $|x| \to \infty$.

2. Theorem.

Theorem 2.1. *The only proper random variables ξ which are both stable and satisfy Condition* A *are the Cauchy variables $C(c, \gamma)$ with* p.d.f.

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$$(1/\pi)[c/((x-\gamma)^2+c^2)].$$

When $\xi = C(c, \gamma)$, *the function* $g(x)$ *occurring in Condition* A *equals* $(c^2+\gamma^2)^{-1}x$, *or equivalently,* $1/\xi = (c^2+\gamma^2)^{-1}\xi$.

It is easy to see that the following lemma holds.

LEMMA 2.1. $h(x) = g^{-1}(x)$ *exists and has a derivative* $h'(x) > 0$. *Further* $h(x) = (1/A)x + O(1)$ *and* $h'(x) = (1/A) + O(1/x^{\epsilon})$.

In order to avoid repetition we assume that in the lemmas which follow, ξ shall be taken to be a stable r.v. which satisfies Condition A.

We observe that $\xi = a\eta + b$ for some $a \neq 0$ and b. Thus

$$f(x) = (1/|a|)p[(x-b)/a].$$

Further the p.d.f.'s of $1/\xi$ and $g(\xi)$ are $(1/x^2)f(x)$ and $h'(x)f(h(x))$. Hence since Condition A holds, we have

$$f(1/x) = x^2h'(x)(1/|a|)p[(h(x)-b)/a]. \tag{2.1}$$

LEMMA 2.2. $\alpha \nless 1$.

Suppose now that $\alpha < 1$. Then [4],

$$p(x) = 1/(\pi|x|)\sum_{n=1}^{\infty} a_n|x|^{-\alpha n}, \qquad \text{as } x \to \pm\infty,$$

where the a_n are constants and $a_1 > 0$.

Hence, from (2.1), using Lemma 2.1 we obtain $f(1/x) = O(x^{1-\alpha})$. This implies that $f(0)$ is infinite which contradicts the fact, pointed out in the preceding section, that $f(0)$ is finite. Hence α cannot be less than one and the lemma is proved.

LEMMA 2.3. $\alpha \ngtr 1$.

Assume that $\alpha > 1$. We have from [4],

$$p(x) = 1/(\pi|x|)\sum_{n=1}^{N} a_n|x|^{-\alpha n} + O(|x|^{-(N+1)\alpha-1}), \tag{2.2}$$

where the a_n are constants and $a_1 \neq 0$.

Hence, since (2.1) holds, we have $f(1/x) = O(x^{1-\alpha})$. This implies that $f(0) = 0$. But the fact that all stable distribution functions are unimodal [2] together with the asymptotic expansion (2.2) ensure that $f(x)$ cannot be zero, for any x. The contradiction thus arrived at shows that our assumption that $\alpha > 1$ is untenable, and the lemma is established.

LEMMA 2.4. *If* $\alpha = 1$, *then* $\beta \neq \pm 1$.

PROOF. Consider the following expressions [4]:

$$p(x + (2/\pi)\beta \ln|x|, 1, \beta) = 1/(\pi|x|)\sum_{k=1}^{N} b_k|x|^{-k} + O(x^{-N-2}), \tag{2.3}$$

where $b_k = \operatorname{Im}\int_0^\infty (e^{-v}v^k/k!)\,(i + i\beta - (2/\pi)\beta\ln v)^k dv$, if x is positive, and $b_k = \operatorname{Im}\int_0^\infty (e^{-v}v^k/k!)\,(i - i\beta + (2/\pi)\beta\ln v)^k dv$, if x is negative.

$$p(x, 1, \beta) = k\exp\{\tfrac{1}{4}\pi|x| - (2/\pi e)e^{\frac{1}{2}\pi|x|}\}\{1 + O(e^{-d|x|})\} \tag{2.4}$$

where k and d are certain positive constants. By [4], the asymptotic expressions for $p(x, 1, \beta)$ are given by (2.3) for $-1 < \beta \leqq 1$ as $x \to \infty$, or for $-1 \leqq \beta < 1$ as $x \to -\infty$; and by (2.4) for $\beta = 1$ as $x \to -\infty$, or for $\beta = -1$ as $x \to \infty$.

We note that $b_1 > 0$. Now, denoting $g(a(x + (2/\pi)\beta \ln |x|) + b)$ by y, we obtain from (2.1),

$$f(1/y) = y^2 h'(y)(1/|a|)\, p(x + (2\beta/\pi) \ln |x|). \tag{2.5}$$

Now suppose that $\beta = 1$. Then, making use of (2.4), Lemma 2.1 and the fact that $g(x) = O(x)$, we derive from (2.5) the fact that $f(0) = 0$. But by the unimodality of stable laws [2], and the asymptotic expansions (2.3) and (2.4) we must have $f(x) > 0$ for all x. Thus β cannot equal 1 and a similar argument shows that it cannot equal -1.

LEMMA 2.5. *If* $\alpha = 1$ *and* $|\beta| < 1$, *then* $\beta = 0$.

PROOF. Suppose that $\beta \neq 0$. Now, let us call β the second parameter of ξ. Then clearly, $-\beta$ is the second parameter of $-\xi$. Further the function $k(x) = -g(-x)$ is such that $k'(x) > 0$, $k(x) = Ax + O(1)$ and $k'(x) = A + O(1/x^{\epsilon})$. Lastly, $1/(-\xi) = -g(\xi) = k(-\xi)$, i.e., $-\xi$ satisfies Condition A. These observations enable us to assume without loss of generality that

$$a\beta > 0. \tag{2.6}$$

We derive next, an asymptotic expansion for $f(1/y)$, where as in the proof of the preceding lemma $y = g(a(x + (2/\pi)\beta \ln |x|) + b)$. The method used for obtaining this expansion is the same as that used in [4] to derive the expansion (2.3).

We have

$$f(1/y) = 1/\pi \operatorname{Re} \int_0^\infty \exp \{ibt - |at|(1 + i(2/\pi)\beta \operatorname{sign} at \ln |at|) - (it/y)\}\, dt.$$

Setting $v = it$, the right hand side becomes

$$-(1/\pi) \operatorname{Im} \int_0^\infty \exp \{i|a|v(1 \pm \beta) - (2/\pi)\beta\, av \ln (|a|v) + bv - (v/y)\}\, dv$$

$$= (1/\pi) \sum_{k=0}^{N} [(-1)^k/k!] b'_{k+1} y^{-k} + O(y^{-N-1}),$$

where $b'_k = \operatorname{Im} \int_0^\infty v^k \exp \{i|a|v(1 \pm \beta) - (2/\pi)\beta\, av \ln (|a|v) + bv\} dv$. The integral on the right exists by virtue of (2.6). Taking $N = 1$, we thus obtain

$$f(1/y) = 1/\pi(b'_1 - b'_2/y) + O(y^{-2}). \tag{2.7}$$

Hence $f(0) = b'_1/\pi$ and

$$x\{f(1/y) - f(0)\} = O(1). \tag{2.8}$$

On the other hand (2.5) and (2.3) hold (see the part of the proof of the preceding lemma up to equation (2.5)). Hence taking into account also Lemma 2.1

and the fact that $g(x) = Ax + O(1)$, we obtain the relation $f(1/y) \sim B + C(\ln|x|/x)$, where B and C are non-zero constants. Therefore $f(0) = B$. But then $x\{f(1/y) - f(0)\} \sim D(\ln|x|)$, where D is a finite and non-zero or an infinite number, which contradicts (2.8). Thus our original assumption that $\beta \neq 0$ is false. Hence $\beta = 0$ and the lemma is established.

We now proceed to prove Theorem 2.1. As a consequence of the preceding lemmas, in order that the r.v. ξ be stable and also satisfy Condition A, it is necessary that $\alpha = 1$ and $\beta = 0$. Hence ξ must have the characteristic function $\exp(i\gamma t - c|t|)$. But this is the characteristic function of $C(c, \gamma)$. Hence $\xi = C(c, \gamma)$. Therefore $1/\xi$ has p.d.f.

$$(1/x^2)f(1/x) = (1/\pi)\{c/[(1 - \gamma x)^2 + c^2x^2]\} = (1/\pi)\{c'/[(x - \gamma')^2 + c'^2]\},$$

where $c' = c/(c^2 + \gamma^2)$ and $\gamma' = \gamma/(c^2 + \gamma^2)$. Thus

$$1/\xi = C(c', \gamma') = [1/(c^2 + \gamma^2)]\, C(c, \gamma) = [1/(c^2 + \gamma^2)]\xi \tag{2.9}$$

and the proof is complete.

The following corollary is an immediate consequence of (2.9):

Corollary. *If a* r.v. *is stable, and its reciprocal has the same distribution function then it must be a Cauchy variable* $C(c, \gamma)$ *with* $c^2 + \gamma^2 = 1$.

The statement and proof of the next theorem is due to a referee.

Theorem 2.2. *If X and its reciprocal are both stable then X is a Cauchy* r.v.

Proof. If X is a r.v. whose density with respect to Lebesgue measure is continuous and positive at the origin, then absolute moments of order $\alpha > 0$ of $1/X$ exist, if and only if $\alpha < 1$. If the density is zero at the origin, and if all the derivatives of the density exist there, then all positive moments of $1/X$ exist. These statements imply that if X is stable and if its reciprocal is stable, then the characteristic exponent α of X must be one. Furthermore, for a stable r.v. with $\alpha = 1$ and $\beta \neq 0$, the density of the reciprocal is not continuous at the origin.

This proves the theorem which is much more general than the corollary to Theorem 2.1 that is given above.

The referee also points out the theorem is false if the word stable occurring in its statement is replaced by "infinitely divisible", since the reciprocal of the stable r.v. with $\alpha = \frac{1}{2}$, $\beta = 1$ is a r.v. having the gamma distribution.

3. Remarks. The author was led to the investigations given above by the following considerations [3]. Let $X_1, \cdots, X_n$ be independent r.v.'s. Let $X = 1/[X_1^{-1} + X_2^{-1} + \cdots + X_n^{-1}]$. It is easy to see that if each $X_i = C(\lambda_i, \mu_i)$ in the notation of Theorem 2.1, then $X = C[\lambda/(\lambda^2 + \mu^2), \mu/(\lambda^2 + \mu^2)]$ with $\lambda = \sum [\lambda_i/(\lambda_i^2 + \mu_i^2)]$ and $\mu = \sum [\mu_i/(\lambda_i^2 + \mu_i^2)]$. The form of the distribution of X and the ease of its determination is due to the stability of the X_i and also the fact that $1/X_i = mX_i$ where m is a suitable constant. It is then natural to inquire whether there exists a random variable Y other than a Cauchy variable such that Y is stable and such that $1/Y = mY + n$ for some constants m and n.

If there were such a one, then it would be easy to find the distribution function of X (if one assumed that all the $X_i = Y$). Indeed $1/X = aY + b$ for suitable a and b. Either Theorem 2.1 or Theorem 2.2 shows however, that no such r.v. Y exists. One wonders whether a r.v. Y which is such that $X = aY + b$ whenever each $X_i = a_iY + b_i$ (the X_i's are assumed independent) necessarily has to be a Cauchy variable.

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