ASYMPTOTIC PROPERTIES OF AN AGE DEPENDENT BRANCHING PROCESS[1]

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0. Introduction and summary. Let $Z(t)$ denote the number of cells at time t which are progeny of a single cell born at $t = 0$, $G(t)$ with $G(0) = 0$ be the lifetime distribution function of each cell, and $h(s) = \sum_{r=0}^{\infty} p_r s^r$, where p_r are constants, $p_r \geqq 0$, $\sum_{r=0}^{\infty} p_r = 1$ be the generating function of the number of cell progeny which replace each cell on completion of its life. Cells develop and proliferate independently of each other. For general $G(t)$ this process is called an age dependent branching process and for $G(t)$ an exponential distribution, a Markov branching process [3].

When the mean number of progeny per cell, $h^{(1)}(1) = 1$, and $h^{(2)}(1) > 0$, $h^{(3)}(1) < \infty$, and $G(t)$ is an exponential distribution with parameter λ, Sevast'yanov [5] showed by study of a differential equation satisfied by $F(s, t) = \sum_{j=0}^{\infty} P[Z(t) = j]s^j$, that $\lim_{t\to\infty} tP[Z(t) > 0] = 2[\lambda h^{(2)}(1)]^{-1}$ and that for $u \geqq 0$,

$$(1) \qquad \lim_{t\to\infty} P[2(\lambda h^{(2)}(1)t)^{-1}Z(t) > u \mid Z(t) > 0] = \exp(-u).$$

Analogous limit theorems for the discrete time case were obtained by Kolmogorov and by Yaglom. See [3], pp. 21–22, 108–109.

It is the purpose of this paper to extend the results of Sevast'yanov to the case of general $G(t)$. In Section 1, Theorem 1 gives the form of the asymptotic moments of such an age dependent branching process by study of an integral equation satisfied by $D(s, t) = 1 - E[\exp(-sZ(t))]$. Chover and Ney [1] have shown that for mild conditions on $G(t)$ and $h(s)$, that $\lim_{t\to\infty} tP[Z(t) > 0] = b$, where b is a strictly positive constant to be defined. In Section 2, this result, together with Theorem 1 yields a conditional limit theorem which generalizes (1). Section 3 contains remarks on an analogous general discrete time result of Mullikin [4].

1. Asymptotic moments. Define $m(t) = E[Z(t)]$ and $M_n(t) = E[Z^n(t)]$, $n = 1, 2, 3, \cdots$. We will need the following lemma.

Lemma. *Let $h^{(1)}(1) = 1$. Then $M_n(t)$ is increasing.*

Proof. For $0 \leqq u \leqq t$, $E[Z(t) \mid Z(u)] = Z(u)$. It is then known that $M_n(t)$ is non-decreasing by Jensen's inequality (see [2], p. 313), and $G(0) = 0$ insures that $Z(t) < \infty$ a.e. ([3], pp. 138–139).

Theorem 1. *Let $h^{(1)}(1) = 1$, $h^{(2)}(1) > 0$, $h^{(n)}(1) < \infty$, $n = 2, 3, 4, \cdots$, and $\int_0^\infty u\, dG(u) \equiv m_G$, where $0 < m_G < \infty$. Then $\lim_{t\to\infty} t^{-(n-1)}M_n(t) = n!b^{-(n-1)}$ for $n = 1, 2, 3, \cdots$, and $b = 2m_G/h^{(2)}(1)$.*

Proof. $G(0) = 0$ insures that $m(t) < \infty$ and $Z(t) < \infty$ a.e. ([3], pp. 138–139).

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Let $R(s, t) = E[\exp(-sZ(t))]$. Using [3], p. 130, we obtain

$$R(s, t) = \exp(-s)[1 - G(t)] + \int_0^t h(R(s, t-u))\, dG(u).$$

Define $D(s, t) = 1 - R(s, t)$. Then

$$D(s, t) = [1 - \exp(-s)][1 - G(t)] + \int_0^t D(s, t-u)\, dG(u) - \sum_{j=2}^{\infty} (-1)^j h^{(j)}(1)(j!)^{-1} \int_0^t [D(s, t-u)]^j\, dG(u).$$

By taking Laplace transforms with respect to t in the above equation, solving for the Laplace transform of D and reinverting, it is found that

$$D(s, t) = [1 - \exp(-s)] - \sum_{j=2}^{\infty} (-1)^j h^{(j)}(1)(j!)^{-1} \int_0^t [D(s, t-u)]^j\, dK(u), \tag{2}$$

where $K(u) = \sum_{n=1}^{\infty} G^{(n)}(u)$, and $G^{(k)}$ denotes the kth convolution of G.

Then, where derivatives are taken with respect to s, $m(t) = D^{(1)}(0, t) = 1$, and in general, $M_n(t) = (-1)^{n+1} D^{(n)}(0, t)$, $n = 2, 3, \cdots$. We obtain that $M_2(t) = 1 + h^{(2)}(1)K(t)$, and $\lim_{t\to\infty} t^{-1} M_2(t) = h^{(2)}(1)/m_G$ ([8], p. 246). Again from (2),

$$M_3(t) = 1 + h^{(3)}(1)K(t) + 3h^{(2)}(1) \int_0^t M_2(t-u)\, dK(u). \tag{3}$$

Let $H(s)$ denote the Laplace transform of a function $H(t)$. Then, by a standard Abelian theorem [10], $\lim_{s\downarrow 0} s^2 M_2(s)K(s) = h^{(2)}(1)/m_G^2$, and by a Tauberian theorem [10] applied to (3) by virtue of the lemma, $\lim_{t\to\infty} t^{-2} M_3(t) = 3(h^{(2)}(1))^2/2m_G^2 = 6b^{-2}$.

The result of the theorem holds for $m(t)$, $M_2(t)$, and $M_3(t)$, and the terms contributing to the asymptotic formulas of these moments are obtained solely from derivatives of the D^2 term in the integrand on the right hand side of (2). Assume by induction that the result holds for $M_n(t)$. Then by the induction hypothesis and standard Abelian and Tauberian theorems along with the lemma applied to the limiting behavior, for $s \downarrow 0$, of the Laplace transforms of the convolutions of $M_k(t)$, $M_j(t)$, and $K(t)$ for $j, k \leqq n$, the asymptotic formula for $M_{n+1}(t)$ is obtained solely from the derivatives of the D^2 term.

Hence, taking derivatives with respect to s, we obtain from Leibnitz's rule for successive differentiation, that

$$(D^2(0, t))^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} D^{(k)}(0, t) D^{(n+i-k)}(0, t) = \sum_{k=1}^{n} \binom{n+1}{k} D^{(k)}(0, t) D^{(n+1-k)}(0, t), \tag{4}$$

since $D(0, t) = 0$. Using the induction hypothesis, by the Abelian theorem applied to the Laplace transform of (4), the right hand side becomes, in absolute value,

$$\sim s^{-(n-1)} \Gamma(n) \sum_{k=1}^{n} \binom{n+1}{k} b^{-(k-1)} k!\, b^{-(n-k)} (n-k+1)!, \qquad \text{for } s \downarrow 0.$$

Then, from (2),

$$(5)\quad \lim_{s \downarrow 0} s^n M_{n+1}(s) = h^{(2)}(1)(2m_G)^{-1}\Gamma(n) \textstyle\sum_{k=1}^{n} \binom{n+1}{k} k!(n-k+1)!b^{-(n-1)} = (n+1)!b^{-n} n\Gamma(n),$$

so that applying the Tauberian theorem to (5) by virtue of the lemma yields

$$\lim_{t\to\infty} t^{-n} M_{n+1}(t) = (n+1)!b^{-n} n\Gamma(n)/\Gamma(n+1) = (n+1)!b^{-n}$$

to complete the proof.

By a somewhat different method, similar results on the asymptotic moments of $N(t)$, the total number of cell births by time t, have been obtained [9].

2. Conditional limit distribution.

THEOREM 2. *Let* $h^{(1)}(1) = 1$, $h^{(2)}(1) > 0$, *and* $h^{(n)}(1) < \infty$, $n = 2, 3, 4, \cdots$. *If* $1 - G(t) = O(t^{-3})$ *for* $t \to \infty$, *then*

$$\lim_{t\to\infty} P[bt^{-1}Z(t) > u \mid Z(t) > 0] = \exp(-u).$$

PROOF. By Theorem 1, $\lim_{t\to\infty} b^{-1}tE[bt^{-1}Z(t)]^n = n!$. By Carleman's theorem on moment sequences [7], $n!$ are the moments of a unique distribution, clearly the exponential distribution with parameter 1. Chover and Ney [1] have shown that for $1 - G(t) = O(t^{-3})$, $t \to \infty$, and $h^{(3)}(1) < \infty$, that $\lim_{t\to\infty} tP[Z(t) > 0] = b$. Since $E[(bt^{-1}Z(t))^n \mid Z(t) > 0] = E[(bt^{-1}Z(t))^n]/P[Z(t) > 0]$, it follows that

$$(6)\quad \lim_{t\to\infty} E[(bt^{-1}Z(t))^n \mid Z(t) > 0] = n! \qquad \text{for} \quad n = 1, 2, \cdots,$$

so that convergence in distribution holds. Hence, from (6), for $u \geqq 0$, $\lim_{t\to\infty} P[bt^{-1}Z(t) > u \mid Z(t) > 0] = \exp(-u)$.

3. Remarks. In [4], it is shown that for branching process in a general state space and discrete time, with a condition corresponding to $h^{(1)}(1) = 1$ and certain compactness and positivity conditions, the limiting conditional distribution is exponential. It is not clear that the results in [4] imply those in this paper, or, if they do, it is not clear that the latter are easily derived from [4].

Sevast'yanov has also shown [6] that (a) $\lim_{t\to\infty} tP[Z(t) > 0] = b$ under the conditions that $h^{(3)}(1) < \infty$ and $\int_0^\infty u^3\, dG(u) < \infty$, which are somewhat stronger than those of Chover and Ney [1]. Sevast'yanov also claims to show [6] that the result of Theorem 2 holds under the conditions that $h^{(3)}(1) < \infty$ and $\int_0^\infty u^3\, dG(u) < \infty$, by study of the integral equation for the probability generating function $F(s, t) = \sum_{j=0}^{\infty} P[Z(t) = j]s^j$ and using (a). However, the proof appears to have a gap. Specifically, in his notation, one obtains from p. 592 of [6] that

$$[Q(t)]^{-1}R(t, \exp(-sQ(t)))$$
$$= \{1 - \exp(-sQ(t))/Q(t)[\gamma t[1 - \exp(-sQ(t))] + 1]\}[1 + \alpha(t, \exp(-sQ(t))]$$

where for each x in the interval $0 \leqq x < 1$, $0 \leqq \alpha(t, x) \leqq K \log(\gamma(1-x)t + 1)\cdot[\gamma(1-x)t + 1]^{-1}$ ([6], p. 590). Hence, as $t \to \infty$, one obtains that $0 \leqq \lim_{t\to\infty} \alpha(t, \exp(-sQ(t))) \leqq K \log(s+1)[s+1]^{-1}$, which does not yield that $\alpha(t, \exp(-sQ(t))) \to 0$ as $t \to \infty$, but which is required for the proof on p. 592 of [6] to be complete.

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