CONSENSUS OF SUBJECTIVE PROBABILITIES: A CONVERGENCE THEOREM[1]

By Torsten Norvig

Wellesley College

We investigate here an 'economic' model with m individuals and n objects. We assume that each individual has a certain endowment and places a certain value on each object, and ask whether one can make dynamic assumptions about the behavior of the individuals which will insure that in the course of time 'social' values will be attached to the objects—values which in some sense represent a consensus of the values given them by individuals.

We consider a simple dynamic mechanism and show that in the course of time it leads to unique 'social' values for each object. The dynamic mechanism, although extremely simple, acts as an efficient 'feed-back' mechanism, adjusting the values towards the 'social' values.

The model can be interpreted [3] as an economic exchange model in which the consumers' preferences are given by linear utility functions. Here, however, we interpret it in terms of the type of consensus represented by the pari-mutuel method of betting on horse races. In this system the final 'track's odds' on a given horse are proportional to the amount bet on that horse.

In formulating the pari-mutuel model we assume that the m individuals involved are bettors, labeled $B_1, \cdots, B_m$, concerned with one particular race involving n horses, labeled $H_1, \cdots, H_n$. We assume further that each B_i has arrived at an estimate of the relative merits of each of the H_j's which he expresses in quantitative terms. Specifically, we are given an $m \times n$ *subjective probability matrix* $P = (p_{ij})$ where p_{ij} is the probability, in the opinion of B_i, that H_j will win the race. We may as well assume that each column of the matrix P contains at least one positive entry. If this were not so then, say $p_{ij} = 0$ for all i, and we could then eliminate H_j from consideration entirely.

Having determined his subjective probability distribution, B_i will now bet the amount b_i, a fixed positive number called B_i's *budget*, in a way which maximizes his *subjective expectation*. This means, of course, that B_i will not necessarily bet the whole amount b_i on that H_j for which p_{ij} is largest. In general, B_i will 'bet the odds', that is he will consider the current track odds, or, more conveniently, the current track probabilities. If these are $\pi_1, \cdots, \pi_n$, he will examine the ratios p_{ij}/π_j and in some way distribute b_i among those H_j for which this ratio is a maximum. We shall refer to this course of action as *B_i's strategy*. It will be convenient to choose the unit of money so that $\sum_{i=1}^{m} b_i = 1$.

Received 18 January 1966; revised 15 June 1966.

[1] This paper is based on the author's thesis done under David Gale at Brown University. Part of the research was supported by a N.S.F. Fellowship. I wish to thank C. J. Everett for pointing out that my proof of a special case can be extended to a proof for the general case.

We shall now arithmetize the conditions which must be satisfied under the pari-mutuel system. (We are using Greek letters to represent unknowns, Latin letters for the given constants of the problem). Let β_{ij} be the amount which B_i bets on H_j. These numbers must satisfy the *budget relation*

$$\sum_{j=1}^{n} \beta_{ij} = b_i, \qquad i = 1, \cdots, m. \tag{1}$$

Next, the *pari-mutuel* condition requires that

$$\sum_{i=1}^{m} \beta_{ij} = \pi_j, \qquad j = 1, \cdots, n, \tag{2}$$

which is simply the statement that the final track probability π_j is proportional to the total amount bet on H_j. Equality holds here because of the normalization of the monetary unit.

Finally, we must express the fact that each B_i is maximizing his expectation. The reader will easily verify that the condition is the following:

$$\text{if} \quad \beta_{il} > 0 \quad \text{then} \quad p_{il}/\pi_l = \max_j (p_{ij}/\pi_j) \tag{3}$$

which states that B_i bets only on those H_j's for which his expectation is a maximum.

Nonnegative numbers π_j and β_{ij} which satisfy (1), (2) and (3) are called *equilibrium probabilities* and *bets*. They are said to give *equilibrium*.

A static pari-mutuel model has been investigated in [1]. In the static model each bettor makes one bet using the strategy described above. One difficulty is immediately apparent in this model. The final track probability π_j is proportional to the amount bet on H_j and thus not known until each B_i has made his bet. On the other hand, each B_i must know $\pi_1, \cdots, \pi_n$ before he can determine his bet. Despite this Eisenberg and Gale proved the following:

THEOREM. *Equilibrium probabilities and bets exist. The equilibrium probabilities are unique.*

In the dynamic model we are considering each bettor makes an infinite sequence of bets using on each iteration the strategy described above. Formally: the sequence $p^k = (\pi_1^k, \cdots, \pi_n^k)$, $k = 1, 2, \cdots$, is generated by the following rules.

(i) $p^1 = (\pi_1^1, \cdots, \pi_n^1)$ is arbitrary except that $\pi_j^1 \geqq 0$, $j = 1, \cdots, n$, and $\sum_{j=1}^{n} \pi_j^1 = 1$.

(ii) $\pi_j^{k+1} = \pi_j^k + \gamma_j^k$ where $\gamma_t^k = \sum_{\mathcal{B}_t} b_i$ with $\mathcal{B}_t = \{i \mid p_{it}/\pi_t^k \geqq p_{ij}/\pi_j^k, j = 1, \cdots, n$, and there does not exist $j < t$ with $p_{it}/\pi_t^k = p_{ij}/\pi_j^k\}$.

This sequence, of course, has the property $\sum_{j=1}^{n} \pi_j^k = k$, for all k. It is clear, however, from the definition of strategy that B_i will place his bet on the same H_j independent of whether he considers the track probabilities or numbers proportional to these.

Note that (ii) is simply the convention that if for a fixed i $\max_j p_{ij}/\pi_j^k$ is attained for more than one subscript, then B_i bets his entire budget b_i on the H_j with the lowest subscript for which this maximum is attained. We prove the following:

THEOREM. $\lim_{k\to\infty} \pi_j^k/k = \pi_j$, $j = 1, \cdots, n$, *where* $\pi_1, \cdots, \pi_n$ *are the unique equilibrium probabilities of the given model.*

It may be noted in passing that this theorem proves a conjecture made by David Gale in [2]. The Gale conjecture is the analogue for linear economic models of the Brown-Robinson 'method of fictitious play' for matrix games.

We first prove the following 'betting lemma.'

LEMMA 1. *If* $\pi_1^k/\pi_1 \leqq \cdots \leqq \pi_n^k/\pi_n$ *then* $\sum_{j=1}^t \gamma_j^k \geqq \sum_{j=1}^t \pi_j$ *for* $k = 1, 2, \cdots$, *and* $t = 1, \cdots, n$.

PROOF. We first note that we can multiply each row of the subjective probability matrix by a positive scalar without changing the betting strategy of any of the bettors. For $i = 1, \cdots, m$ we multiply row i by $b_i/\sum_{j=1}^n p_{ij}\bar{\beta}_{ij}\pi_j^{-1}$ where $\bar{\beta}_{ij}$ is any set of equilibrium bets. It has been shown in [1] that for this new equivalent matrix with entries $P_{ij} = p_{ij}b_i/\sum_{j=1}^n p_{ij}\bar{\beta}_{ij}\pi_j^{-1}$ we have $\pi_j = \max_i P_{ij}$, and that for the new matrix with entries P_{ij} condition (3) takes the simple form

$$\text{if} \quad \beta_{il} > 0 \quad \text{then} \quad P_{il} = \pi_l. \tag{3*}$$

We next note that if $P_{ir} = \pi_r$, as it must for some $r \in \{1, \cdots, n\}$, then B_i bets on some H_j with $j \leqq r$ in view of the ordering assumption $\pi_1^k/\pi_1 \leqq \cdots \leqq \pi_n^k/\pi_n$. For B_i will bet on iteration k on that H_j with the lowest subscript for which $\max_j P_{ij}/\pi_j^k$ is attained and for $j = r + 1, \cdots, n$ we have

$$P_{ir}/\pi_r^k = \pi_r/\pi_r^k \geqq \pi_j/\pi_j^k \geqq P_{ij}/\pi_j^k.$$

Now let $T = \{1, \cdots, t\}$ and consider those B_i who bet on some H_j, $j \in T$, at equilibrium. In view of (3*) we must have $P_{ij} = \pi_j$, some $j \in T$, for these B_i. The sum of their budgets must be greater than or equal to $\sum_{j=1}^t \pi_j$. But by the above remark they each bet on some H_j, $j \in T$, on iteration k. The conclusion of the lemma follows.

We next prove the key lemma.

LEMMA 2. *Let* $\pi = \min_j \pi_j$. *For* $k = 1, 2, \cdots$ *and any set* T *of* t *indices from* $N = \{1, \cdots, n\}$

$$\sum \pi_j^k/\sum \pi_j \geqq k - (1 - \sum \pi_j)/\pi \tag{4}$$

where the sums in (4) *are over the set* T.

PROOF. The proof is by induction on k.

$k = 1$. For $t < n$ (4) is vacuously satisfied since the right hand side is nonpositive. For $t = n$ (4) becomes an equality.

Inductive step. We assume (4). By reindexing the π_j if necessary we may assume $\pi_1^k/\pi_1 \leqq \cdots \leqq \pi_n^k/\pi_n$. We want

$$\sum \pi_j^{k+1}/\sum \pi_j \geqq k + 1 - (1 - \sum \pi_j)/\pi. \tag{5}$$

where the sums in (5) are over any set T of t indices from $N = \{1, \cdots, n\}$.

If $T = \{1, \cdots, t\}$ then

$$\textstyle\sum_T \pi_j^{k+1} / \sum_T \pi_j = \sum_{j=1}^t (\pi_j^k + \gamma_j^k) / \sum_{j=1}^t \pi_j \geqq k + 1 - (1 - \sum_T \pi_j)/\pi$$

by induction hypothesis and Lemma 1. This proves (5) if $T = \{1, \cdots, t\}$.

Otherwise, let l be the smallest subscript such that $l \not\in T$. Let $L = \{j \mid j \in T, j < l\}$ and let $C = \{j \mid j \in T, j > l\}$. Now define

$$\Pi_1 = 0, \qquad \Pi_L = \textstyle\sum_L \pi_j, \qquad \Pi_C = \sum_C \pi_j, \qquad \Pi_T^{k+1} = \sum_T \pi_j^{k+1}.$$

Then

$$\begin{aligned} \Pi_T^{k+1} &= \textstyle\sum_L \pi_j^{k+1} + \sum_C \pi_j^{k+1} \\ &\geqq \textstyle\sum_L \pi_j^k + \sum_L \gamma_j^k + \sum_C \pi_j^k \\ &\geqq \textstyle\sum_L \pi_j^k + \Pi_L + \sum_C \pi_j^k, \end{aligned} \tag{6}$$

since by Lemma 1 $\sum_L \gamma_j^k \geqq \sum_L \pi_j = \Pi_L$. Now by assumption $\pi_j^k/\pi_j \geqq \pi_l^k/\pi_l$ for $j \geqq l + 1$, whence $\pi_j^k \geqq \pi_j(\pi_l^k/\pi_l)$ and $\sum_C \pi_j^k \geqq \Pi_C(\pi_l^k/\pi_l)$. By induction hypothesis $\sum_L \pi_j^k \geqq \Pi_L(k - (1 - \Pi_L)/\pi)$. Hence

$$\begin{aligned} \Pi_T^{k+1} &\geqq \Pi_L\{k - (1 - \Pi_L)/\pi\} + \Pi_L + \Pi_C(\pi_l^k/\pi_l) \\ &= \Pi_L\{k + 1 - (1 - \Pi_L)/\pi\} + \Pi_C(\pi_l^k/\pi_l) \\ &\geqq (\Pi_L + \Pi_C)\{k + 1 - [1 - (\Pi_L + \Pi_C)]/\pi\} \end{aligned} \tag{5*}$$

provided $\Pi_C(\pi_l^k/\pi_l) \geqq \Pi_C\{k + 1 - [1 - (\Pi_L + \Pi_C)]/\pi\} + \Pi_L(\Pi_C/\pi)$. Hence (5*), which is identical to (5), is true if $\pi_l^k \geqq \pi_l\{k + 1 - [1 - (\Pi_L + \Pi_C)]/\pi\} + \Pi_L\pi_l/\pi$. In the opposite case $\pi_l^k < \pi_l\{k + 1 - [1 - (\Pi_L + \Pi_C)]/\pi\} + \Pi_L\pi_l/\pi$. Then, from (6) above

$$\begin{aligned} \Pi_T^{k+1} &\geqq \textstyle\sum_L \pi_j^k + \Pi_L + \sum_C \pi_j^k \\ &= -\pi_l^k + \Pi_L + \textstyle\sum_L \pi_j^k + \pi_l^k + \sum_C \pi_j^k \\ &> -\pi_l\{k + 1 - [1 - (\Pi_L + \Pi_C)]/\pi\} - \Pi_L\pi_l/\pi + \Pi_L \\ &\quad + (\Pi_L + \pi_l + \Pi_C)\{k - [1 - (\Pi_L + \pi_l + \Pi_C)]/\pi\}. \end{aligned} \tag{7}$$

The first part of the inequality follows from the assumption on π_l^k. The second follows from the induction hypothesis applied to the indices $\{1, \cdots, l\} \cup C$. From (7) we get

$$\begin{aligned} \Pi_T^{k+1} &> -\pi_l + \Pi_L + \pi_l^2/\pi + \Pi_C\pi_l/\pi \\ &\qquad + (\Pi_L + \Pi_C)\{k - [1 - (\Pi_L + \Pi_C)]/\pi\} \\ &\geqq (\Pi_L + \Pi_C)\{k + 1 - [1 - (\Pi_L + \Pi_C)]/\pi\} \end{aligned} \tag{5**}$$

provided $-\pi_l + \Pi_L + \pi_l^2/\pi + \Pi_C\pi_l/\pi \geqq \Pi_L + \Pi_C$, or $\Pi_C\pi_l/\pi + \pi_l^2/\pi \geqq \Pi_C + \pi_l$, or $\pi_l(\Pi_C + \pi_l) \geqq \pi(\Pi_C + \pi_l)$, and this is true since $\pi_l \geqq \pi = \min_j \pi_j$. This proves (5**) and thus (5).

THEOREM. $\lim_{k\to\infty} \pi_j^k/k = \pi_j$, $j = 1, \cdots, n$.

PROOF. Reindex on iteration k so that $\pi_1^k/\pi_1 \leqq \cdots \leqq \pi_n^k/\pi_n$. Let $R = \{1, \cdots, r\}$ and recall the definitions:

$$\Pi_R^k = \sum_{j=1}^r \pi_j^k, \qquad \Pi_R = \sum_{j=1}^r \pi_j .$$

We note that if $a, b, c, d > 0$ and $a/b \leqq c/d$ then $a/b \leqq (a + c)/(b + d) \leqq c/d$ whence

$$(7) \quad \pi_1^k/k\pi_1 = \Pi_1^k/k\Pi_1 \leqq \cdots \leqq \Pi_N^k/k\Pi_N = 1 \quad \text{and} \quad \Pi_R^k/k\Pi_R \leqq \pi_{r+1}^k/k\pi_{r+1} .$$

From Lemma 2 with $T = \{1\}$ we have

$$(8) \quad \pi_1^k/k\pi_1 \geqq 1 - (1 - \pi_1)/k\pi \geqq 1 - (1 - \pi)/k\pi = 1 - \epsilon_k ,$$

where $0 < \epsilon_k \to_k 0$. Now, using (7), we have

$$(9) \quad \begin{aligned} \Pi_R^k/k\Pi_R - \Pi_{R-1}^k/k\Pi_{R-1} &= k^{-1}((\Pi_{R-1}^k + \pi_r^k)/(\Pi_{R-1} + \pi_r) \\ &\qquad - \Pi_{R-1}^k/\Pi_{R-1}) \\ &= \pi_r(\pi_r^k/k\pi_r - \Pi_{R-1}^k/k\Pi_{R-1})/ \\ &\qquad (\Pi_{R-1} + \pi_r) \geqq 0. \end{aligned}$$

By (7) and (8) $1 - \epsilon_k \leqq \Pi_{R-1}^k/k\Pi_{R-1} \leqq \Pi_R^k/k\Pi_R \leqq 1$ whence, by (9), $\pi_r^k/k\pi_r - \Pi_{R-1}^k/k\Pi_{R-1} \leqq [(\Pi_{R-1} + \pi_r)/\pi_r]\cdot \epsilon_k \leqq \epsilon_k/\pi$. Hence, for $r = 2, \cdots, n$,

$$|\pi_r^k/k\pi_r - 1| \leqq |\pi_r^k/k\pi_r - \Pi_{R-1}^k/k\Pi_{R-1}| + |1 - \Pi_{R-1}^k/k\Pi_{R-1}| \leqq \epsilon_k/\pi + \epsilon_k .$$

For $r = 1$ we have $|\pi_1^k/k\pi_1 - 1| = 1 - \pi_1^k/k\pi_1 \leqq \epsilon_k$ and the conclusion of the Theorem follows.

The following observations are offered without comment. The author has run the algorithm on computers and found the 'convergence' to be considerably faster than the bound suggested in the proof of the above Theorem, i.e. $|\pi_r^k/k\pi_r - 1| \leqq k^{-1}(\pi^{-2} - 1)$. On the other hand, it is easy to construct examples where the algorithm actually attains the bounds given in Lemma 2.

The author would like to thank the referee for pointing out errors of omission and commission and for his many suggestions valuable in the preparation of this paper.

REFERENCES

[1] EISENBERG, E. and GALE, D. (1959). Consensus of subjective probabilities. The pari-mutuel method, *Ann. Math. Statist.* **30**.
[2] GALE, D. (1957). General equilibrium for linear models. RAND Research Report P-1156,
[3] GALE, D. (1960). *The Theory of Linear Economic Models*. McGraw-Hill. New York.