AN EXTENSION OF THE ROBBINS-MONRO PROCEDURE[1]

By J. H. Venter

Potchefstroom University, South Africa

1. Summary. A refinement of the Robbins-Monro procedure for estimating the root of a regression equation is given. The essential feature of the procedure is that it estimates the slope of the regression function at the root and employs this information to improve on the rate of convergence and the asymptotic variance of the Robbins-Monro procedure.

2. Introduction. Denote by R the real line and suppose that for each $x \in R$ $Y(x)$ is an observable random variable with expectation $M(x)$. Suppose the equation

$$M(x) = 0 \tag{1}$$

has a single root θ which is unknown and is to be estimated by choosing a number of x values and observing the corresponding $Y(x)$'s.

The Robbins-Monro (RM) procedure [7] for solving this problem is: Choose X_1 arbitrarily (it may be random) and generate $\{X_n\}$ by

$$X_{n+1} = X_n - d_n Y_n \tag{2}$$

where $\{d_n\}$ is a sequence of real numbers and $\{Y_n\}$ random variables, the conditional distribution of Y_n given X_n being the same as that of $Y(X_n)$. After n observations (viz. $Y_1, \cdots, Y_n$), the estimate of θ is X_{n+1}.

It was shown by Blum [1] and Dvoretzky [3] that if $\{d_n\}$ satisfies

$$d_n \geqq 0, \qquad \sum d_n^2 < \infty, \qquad \sum d_n = \infty \tag{3}$$

and if the conditions MI, MII and ZI of Section 3 hold, then $X_n \to \theta$ a.s. while, if in addition also $E|X_1|^2 < \infty$, then also $E|X_n - \theta|^2 \to 0$ as $n \to \infty$. It was shown by Sacks [8] that if in addition also MIV (with $s = 1$) and ZII, ZIII hold and if

$$d_n = n^{-1}a^{-1}(1 + o(1)) \tag{4}$$

where

$$0 < a < 2\alpha, \tag{5}$$

then

$$n^{\frac{1}{2}}(X_{n+1} - \theta) \to_{\mathcal{L}} N(0, \sigma^2/a(2\alpha - a)). \tag{6}$$

Received 1 March 1966.

[1] Parts of this paper are taken from the author's Ph.D. dissertation (1963) at the University of Chicago. This research was supported in part by Research Grant No. NSF-G21058 from the Division of Mathematical, Physical and Engineering Sciences of the National Science Foundation.

The asymptotic variance is minimized by taking $a = \alpha$. If $a \geqq 2\alpha$, it is known that the asymptotic variance of $X_{n+1} - \theta$ is of a larger order of magnitude than n^{-1}, [4]. Hence it is essential to ensure that (5) holds and if possible to take $a = \alpha$. However, in practice α will usually be unknown. This raises the question of estimating α and using this estimate to improve on the RM procedure. Such a procedure is discussed in Section 4. Proof of its convergence consists of a generalization of the existing proofs of convergence of the RM procedure. Derivation of its asymptotic distribution by generalization of Sacks' method seems very difficult but a different approach, keyed to the special features of the procedure, yields the required result reasonably easily. An apparently new method of obtaining information on the rate of a.s. convergence is also given.

3. Conditions and preliminaries. The following conditions on $M(x)$ will be needed as referred to:

MI: For each $\epsilon > 0$,

$$\inf_{\epsilon < x - \theta < \epsilon^{-1}} M(x) > 0 \quad \text{and} \quad \sup_{\epsilon < \theta - x < \epsilon^{-1}} M(x) < 0.$$

MII: For some constants K_1 and K_2,

$$|M(x)| \leqq K_1 + K_2 |x - \theta| \quad \text{for all} \quad x \varepsilon R.$$

MIII: $\sup_{x \varepsilon R} \sup_{0 < c < c_0} |M(x + c) - M(x - c)| < \infty$.

MIV: For some $\rho > 0$ and for $|x - \theta| < \rho$,

$$M(x) = \alpha(x - \theta) + f(x) + \delta(x)$$

where α is a positive number and

$$f(x) = \sum_{i=2}^{s} \alpha_i (x - \theta)^i$$

and

$$\delta(x) = o(|x - \theta|^s) \quad \text{as} \quad |x - \theta| \to 0,$$

where $s = 1$ or 2 or $\cdots$ or ∞. In the case $s = 1$, $f(x) \equiv 0$ and in the case $s = \infty$, ρ is the radius of convergence of the power series $f(x)$ while $\delta(x) \equiv 0$ for $|x - \theta| < \rho$.

In the following conditions we write

$$(7) \qquad Z(x) = Y(x) - M(x) \quad \text{and} \quad \sigma^2(x) = E\,|Z(x)|^2.$$

ZI: $\sup_{x \varepsilon R} \sigma^2(x) < \infty$.

ZII: $\sigma^2(x) \to \sigma^2(\theta)$ as $x \to \theta$.

ZIII: $\lim_{r \to \infty} \lim_{\epsilon \downarrow 0} \sup_{|x - \theta| \leqq \epsilon} E I_{[|Z(x)| > r]} |Z(x)|^2 = 0$, where I_A is the indicator function of A.

We will also need the following lemmas:

LEMMA 1. *If* $\{\xi_n\}$ *is a real sequence satisfying*

$$\xi_{n+1} = (1 - a_n)\xi_n + b_n$$

where $a_n \geqq 0$, $a_n \to 0$, $\sum a_n = \infty$ *and* $\sum b_n$ *converges, then* $\xi_n \to 0$ *as* $n \to \infty$.

PROOF. Suppose that $\limsup \xi_n > \liminf \xi_n$. We consider first the case $\limsup \xi_n > 0$. Choose numbers c and d with $\limsup \xi_n > d > c > \max(0, \liminf \xi_n)$. For each integer n we find two integers $m = m(n) \geqq n$ and $p = p(n) \geqq 0$ such that $\xi_m \leqq c$, $\xi_{m+p+1} \geqq d$ and $c < \xi_{m+j} < d$ for $j = 1, \cdots, p$. Let ϵ be such that $0 < \epsilon < d - c$. Choose n so large that for all p, $|\sum_m^{m+p} b_j| \leqq \epsilon$, and $a_m \leqq \frac{1}{2}$. Then

$$\begin{aligned}\xi_{m+p+1} - \xi_m &= -(a_m\xi_m + a_{m+1}\xi_{m+1} + \cdots + a_{m+p}\xi_{m+p}) \\ &\qquad\qquad + (b_m + \cdots + b_{m+p}) \\ &\leqq -a_m\xi_m + \epsilon.\end{aligned} \tag{8}$$

Hence $d - \xi_m \leqq -a_m\xi_m + \epsilon$ or $\xi_m \geqq (d - \epsilon)/(1 - a_m) > 0$ and therefore also, from (8), $\xi_{m+p+1} - \xi_m \leqq \epsilon$. But this contradicts $\epsilon < d - c \leqq \xi_{m+p+1} - \xi_m$. Hence, if $\limsup \xi_n > \liminf \xi_n$ we cannot have $\limsup \xi_n > 0$. By a similar argument we rule out the possibility of having $\liminf \xi_n < 0$. We conclude that $\limsup \xi_n = \liminf \xi_n$ so that $\xi_n \to \xi$ where $-\infty \leqq \xi \leqq +\infty$. Suppose that $\xi > 0$. Let $0 < K < \xi$. Then for some integer n_0 and for all $n \geqq n_0$, $\xi_n \geqq K$. Then, by recursion,

$$\begin{aligned}\xi_n &= \xi_{n-1} - a_{n-1}\xi_{n-1} + b_{n-1} \\ &= \xi_{n_0} - (a_{n_0}\xi_{n_0} + \cdots + a_{n-1}\xi_{n-1}) + (b_{n_0} + \cdots + b_{n-1}) \\ &\leqq \xi_{n_0} - K(a_{n_0} + \cdots + a_{n-1}) + (b_{n_0} + \cdots + b_{n-1}).\end{aligned}$$

Then the convergence of $\sum b_n$ and the fact that $\sum a_n = \infty$, imply that for n large enough we must have $\xi_n < K$ which is a contradiction. Similarly we rule out the possibility $\xi < 0$. The lemma follows.

LEMMA 2. *Let $\{V_n\}$ be a sequence of random variables and $\{\mathcal{B}_n\}$ a sequence of σ-fields such that $\{V_1, \cdots, V_{n-1}\}$ is measurable with respect to $\mathcal{B}_n$ for $n > 1$.*

(i) *If $\sum EV_n^2 < \infty$ and $\sum E[V_n \mid \mathcal{B}_n]$ converges* a.s. *then $\sum V_n$ converges* a.s.

(ii) *If $\sum b_n^{-2}EV_n^2 < \infty$ with $b_n \uparrow \infty$, then*

$$b_n^{-1} \sum_{k=1}^n \{V_k - E[V_k \mid \mathcal{B}_k]\} \to 0 \quad \text{a.s. as} \quad n \to \infty.$$

PROOF. Slight extensions of parts of Theorems D and E, p. 387 of [6].

4. The procedure. It is assumed throughout that known constants a and b are such that

$$0 < a < \alpha < b < \infty. \tag{9}$$

Let $\{c_n\}$ and $\{d_n\}$ be positive sequences, X_1 arbitrary and $\{X_n\}$ such that

$$X_{n+1} = X_n - d_nA_n^{-1}\tfrac{1}{2}(Y_n' + Y_n'') \tag{10}$$

where $\{Y_k', Y_k''; k = 1, 2, \cdots\}$ are random variables with the conditional distribution of Y_n', Y_n'' given $\{Y_k', Y_k''; k = 1, \cdots, n - 1\}$ independent and identical to that of $Y(X_n + c_n)$ and $Y(X_n - c_n)$ respectively; further, A_n is an estimate of α defined as follows: let

$$B_n = n^{-1} \sum_{j=1}^{n} (Y_j' - Y_j'')/2c_j \tag{11}$$

and

$$\begin{aligned} A_n &= a && \text{if } B_n < a \\ &= B_n && \text{otherwise} \\ &= b && \text{if } B_n > b. \end{aligned} \tag{12}$$

The properties of this procedure will be established for the cases

$$d_n = n^{-1}(1 + O(n^{-\frac{1}{2}})) \tag{13}$$

$$c_n = cn^{-\gamma}(1 + o(1)), \qquad c > 0, \qquad 0 < \gamma < \tfrac{1}{2}, \tag{14}$$

only, which seem to be of most interest. We will denote by $K_3, K_4, \cdots$ constants chosen to suit the context in which they appear.

THEOREM 1a. *If* MI, MII, MIII *and* ZI *hold, then* $X_n \to \theta$ a.s. *If also* $E|X_1|^2 < \infty$, *then* $E|X_n - \theta|^2 \to 0$ *as* $n \to \infty$.

PROOF. Let

$$Z_n' = Y_n' - M(X_n + c_n), \qquad Z_n'' = Y_n'' - M(X_n - c_n). \tag{15}$$

Substituting into (10),

$$X_{n+1} = X_n - S_n(X_n) + U_n \tag{16}$$

where

$$S_n(x) = d_n A_n^{-1}\tfrac{1}{2}\{M(x + c_n) + M(x - c_n)\} \tag{17}$$

and

$$U_n = -d_n A_n^{-1}\tfrac{1}{2}(Z_n' + Z_n''). \tag{18}$$

From (15), ZI and (12), we have

$$E|U_n|^2 \leqq K_3 d_n^2 \tag{19}$$

so that (13) implies $\sum E|U_n|^2 < \infty$.

Let $\mathcal{B}_n$ be the σ-field in the underlying probability space induced by $\{X_1, Y_k', Y_k''; k = 1, \cdots, n-1\}$. Then it is easy to see that

$$\{X_1, \cdots, X_n; A_1, \cdots, A_{n-1}\}$$

is measurable with respect to $\mathcal{B}_n$. Now we wish to show that

$$\sum |E[U_n \mid \mathcal{B}_n]| < \infty \text{ a.s.} \tag{20}$$

By conditional independence and (15),

$$E[A_{n-1}^{-1}(Z_n' + Z_n'') \mid \mathcal{B}_n] = 0 \text{ a.s.}$$

Hence

$$\begin{aligned}-E[U_n \mid \mathcal{B}_n] &= nd_nE[(n^{-1}A_n^{-1} - (n-1)^{-1}A_{n-1}^{-1})\tfrac{1}{2}(Z_n' + Z_n'') \mid \mathcal{B}_n] \\ &= (n-1)^{-1} d_nE[((n-1)A_{n-1} - nA_n) \\ &\qquad \cdot \tfrac{1}{2}(Z_n' + Z_n'')A_n^{-1}A_{n-1}^{-1} \mid \mathcal{B}_n].\end{aligned}$$

From (12) and (11) it follows that

$$\begin{aligned}|(n-1)A_{n-1} - nA_n| &\leqq |(n-1)B_{n-1} - nB_n| \\ &= \tfrac{1}{2}c_n^{-1}|Y_n' - Y_n''| \\ &\leqq \tfrac{1}{2}c_n^{-1}(K_4 + |Z_n' - Z_n''|)\end{aligned} \tag{21}$$

for all n large enough, where we have used (15) and MIII. Hence, applying ZI, (12) and (14), we get

$$|E[U_n \mid \mathcal{B}_n]| \leqq (n-1)^{-1} d_nc_n^{-1}K_5 \sim K_6n^{-2+\gamma} \tag{22}$$

as $n \to \infty$. (20) therefore follows. From (19), (20) and Lemma 2 it follows that $\sum U_n$ converges a.s. With this fact in hand Blum's argument [1] can be extended to show that $X_n \to \theta$ a.s. Alternatively, it is straight forward but tedious to show that Theorem 3 of [9] contains this one as a special case. We will not give the details here.

Theorem 1b. *If* MI, MII, MIII, MIV (*with* $s \geqq 2$) *and* ZI *hold and if* $s\alpha > \gamma b$ *then* $A_n \to \alpha$ a.s. *The condition* $s\alpha > \gamma b$ *vanishes if* $s = \infty$ *in* MIV.

Proof. For simplicity of writing we suppose that $\theta = 0$. It suffices to show that $B_n \to \alpha$ a.s. Substituting (15) and MIV into (11) we get

$$\begin{aligned}B_n - \alpha &= n^{-1}\textstyle\sum_{j=1}^n [f(X_j + c_j) - f(X_j - c_j)]/2c_j \\ &\quad + n^{-1}\textstyle\sum_{j=1}^n [\delta(X_j + c_j) - \delta(X_j - c_j)]/2c_j \\ &\quad + n^{-1}\textstyle\sum_{j=1}^n (Z_j' - Z_j'')/2c_j\,.\end{aligned} \tag{23}$$

From the definition of $f(x)$ in MIV and the result of Theorem 1a it is readily seen that the first term on the right in (23) tends to zero a.s. as $n \to \infty$. Also from ZI, (14) and Lemma 2 it follows that the last term tends to zero a.s. as $n \to \infty$. In the case $s = \infty$ in MIV, the facts that $X_n \to \theta$ a.s. and $c_n \to 0$ imply that the second term on the right in (23) also tends to zero a.s. It remains to deal with this term for the case $s < \infty$. Now, by MIV,

$$\begin{aligned}\delta(X_n + c_n)/c_n &= [\delta(X_n + c_n)/(X_n + c_n)^s][(X_n + c_n)^s/c_n] \\ &= o(1)[X_n^s/c_n + o(1)] \text{ a.s.}\end{aligned}$$

A similar fact holds for $\delta(X_n - c_n)/c_n$ and it follows that it will be sufficient to show that $X_n^sc_n^{-1} \to 0$ a.s., or, in view of (14), that

$$n^{\gamma/s}X_n \to 0 \quad \text{a.s. as} \quad n \to \infty. \tag{24}$$

Now, from MIV straightforward manipulations show that

$$(25)\quad \tfrac{1}{2}[M(X_n + c_n) + M(X_n - c_n)] = \alpha X_n[1 + \epsilon_{1n}] + \alpha_2 c_n^2[1 + \epsilon_{2n}]$$

where, in view of Theorem 1a and (14),

$$(26)\qquad \epsilon_{1n} \to 0 \quad \text{and} \quad \epsilon_{2n} \to 0 \text{ a.s.}$$

Putting (15) and (25) into (10), multiplying both sides by $(n + 1)^{\gamma/s}$ and rearranging terms, we get

$$(27)\quad (n + 1)^{\gamma/s} X_{n+1} = \{1 - n^{-1}(\alpha A_n^{-1} - \gamma/s + \epsilon_{3n})\} n^{\gamma/s} X_n + V_n$$

where

$$(28)\qquad V_n = -(n + 1)^{\gamma/s} d_n c_n^2 A_n^{-1}(1 + \epsilon_{2n})\alpha_2 + (n + 1)^{\gamma/s} U_n$$

with U_n as in (18) and with $\epsilon_{3n} \to 0$ a.s. From (19), (13) and $\gamma/s < 1$, we get

$$(29)\qquad \sum E|(n + 1)^{\gamma/s} U_n|^2 < \infty,$$

while from (22) and $\gamma + \gamma/s \leqq 2\gamma < 1$, we get

$$(30)\qquad \sum |E[(n + 1)^{\gamma/s} U_n \mid \mathcal{B}_n]| < \infty \text{ a.s.}$$

(29) and (30) together with Lemma 2 implies that $\sum (n + 1)^{\gamma/s} U_n$ converges a.s. and since $\gamma/s < 2\gamma$ it follows from (28) that

$$(31)\qquad \sum V_n \text{ converges a.s.}$$

Also in (27) we have $\alpha A_n^{-1} \geqq \alpha/b > \gamma/s$ by the condition stated in the formulation of this theorem. Hence an a.s. application of Lemma 1 to (27) shows that (24) holds and the theorem follows.

Remarks. The requirement $0 < a < \alpha$ can usually be satisfied in practice by taking a small enough. Theorem 1b and the theorems to follow require that $\alpha < b < s\alpha/\gamma$. If $s = \infty$ in MIV, this places no upper bound on b and the requirement can be satisfied in practice by taking b large enough. The choice for γ would usually be small (e.g. $\gamma = \frac{1}{4}$; see the remarks following Theorem 3) so that even for moderate values of s in MIV the upper bound $s\alpha/\gamma$ on b would be large enough to make it negligible in practice.

Theorem 2. *If* MI, MII, MIII, MIV (*with* $s \geqq 2$) *and* ZI *hold and if* $s\alpha > b\gamma$, *then*

(a) *for any number* λ *such that*

$$(32)\qquad 0 \leqq \lambda < \min(\tfrac{1}{2}, 2\gamma)$$

we have

$$(33)\qquad X_n - \theta = o(n^{-\lambda}) \text{ a.s.};$$

(b) *for any number* μ *such that*

$$(34)\qquad 0 < \mu < \tfrac{1}{2} - \gamma$$

we have

$$A_n - \alpha = o(n^{-\gamma}) + o(n^{-\mu}) \quad \text{a.s. } as \quad n \to \infty. \tag{35}$$

PROOF. Taking $\theta = 0$ for simplicity, the equivalent of (27) is obtained as before, viz.

$$(n + 1)^{\lambda} X_{n+1} = \{1 - n^{-1}(\alpha A_n^{-1} - \lambda + \epsilon_{3n})\} n^{\lambda} X_n + W_n \tag{36}$$

where

$$W_n = -(n + 1)^{\lambda} d_n c_n^2 A_n^{-1}(1 + \epsilon_{3n})\alpha_2 + (n + 1)^{\lambda} U_n. \tag{37}$$

Now (32) ensures that the equivalents of (29) and (30) hold and that

$$\sum W_n \text{ converges a.s.} \tag{38}$$

Also, since according to Theorem 1b, $\alpha A_n^{-1} - \lambda \to 1 - \lambda > 0$, an application of Lemma 1 shows that (33) is true.

Further, it is readily seen that the first two terms on the right of (23) are at most $o(c_n)$ a.s. as $n \to \infty$ while it follows from Lemma 2 that the third term is $o(n^{-\mu})$ a.s. It follows then that (35) is true.

THEOREM 3. *If* MI, MII, MIII, MIV (*with* $s \geqq 2$), ZI, ZII *and* ZIII *hold and if*

$$\tfrac{1}{4} < \gamma < \tfrac{1}{2} \tag{39}$$

and $s\alpha > b\gamma$, *then we have*

$$n^{\frac{1}{2}}(X_n - \theta) \to_{\mathfrak{L}} N(0, \sigma^2/2\alpha^2) \tag{40}$$

and

$$n^{\frac{1}{2}} c_n(A_n - \alpha) \to_{\mathfrak{L}} N(0, \sigma^2/2(1 + 2\gamma)). \tag{41}$$

If instead of (39) *we have*

$$\gamma = \tfrac{1}{4}, \tag{42}$$

then

$$n^{\frac{1}{2}}(X_n - \theta) \to_{\mathfrak{L}} N(-2\alpha_2 c^2/\alpha, \sigma^2/2\alpha^2) \tag{43}$$

and

$$n^{\frac{1}{4}}(A_n - \alpha) \to_{\mathfrak{L}} N(0, \sigma^2/3c^2). \tag{44}$$

PROOF. For simplicity we take $\theta = 0$ and abbreviate

$$\epsilon_n = \tfrac{1}{2}\{M(X_n + c_n) + M(X_n - c_n)\} - \alpha X_n. \tag{45}$$

From MIV,

$$\epsilon_n = \alpha_2(X_n^2 + c_n^2) + o(X_n^2) + o(c_n^2) \quad \text{a.s. as} \quad n \to \infty. \tag{46}$$

Substituting (15) into (10) and using the abbreviations just introduced, we have

(47) $nX_{n+1} = (n-1)X_n + (1 - nd_nA_n^{-1}\alpha)X_n - nd_nA_n^{-1}\epsilon_n + nU_n$

with U_n as defined by (18). Iterating back to $n = 1$ and dividing by $n^{\frac{1}{2}}$,

(48) $$n^{\frac{1}{2}}X_{n+1} = Q_{1n} - Q_{2n} + Q_{3n}$$

where

(49) $$Q_{1n} = n^{-\frac{1}{2}}\textstyle\sum_{k=1}^{n}(1 - k\,d_kA_k^{-1}\alpha)X_k\ ;$$

(50) $$Q_{2n} = n^{-\frac{1}{2}}\textstyle\sum_{k=1}^{n}k\,d_kA_k^{-1}\epsilon_k\ ;$$

(51) $$Q_{3n} = n^{-\frac{1}{2}}\textstyle\sum_{k=1}^{n}kU_k\,.$$

For (40) it will suffice to show that

(52) $$Q_{1n} \to 0, \qquad Q_{2n} \to 0 \quad \text{a.s.}$$

and

(53) $$Q_{3n} \to_{\mathfrak{L}} N(0, \sigma^2/2\alpha^2).$$

Now

$$\begin{aligned} 1 - k\,d_kA_k^{-1}\alpha &= A_k^{-1}\{A_k - \alpha + O(k^{-\frac{1}{2}})\} \\ &= A_k^{-1}\{o(k^{-\gamma}) + o(k^{-\mu})\} \text{ a.s.} \end{aligned}$$

as $k \to \infty$, where we have used (13) and (35). Using (33) and (49) we then have

$$Q_{1n} = o(n^{\frac{1}{2}-\lambda-\gamma}) + o(n^{\frac{1}{2}-\lambda-\mu}) = o(1) \text{ a.s.}$$

since λ can be taken arbitrarily close to $\frac{1}{2}$ according to (32) and (39) while both γ and μ are positive. Further, from (46), (33) and (14)

$$d_kkA_k^{-1}\epsilon_k = o(k^{-2\lambda}) + O(k^{-2\gamma}) \text{ a.s.}$$

and hence from (50)

$$Q_{2n} = o(n^{\frac{1}{2}-2\lambda}) + O(n^{\frac{1}{2}-2\gamma}) = o(1) \text{ a.s.}$$

according to (32) and (39). Hence (52) holds.

Further, let $\mathfrak{B}_n$ be as in the proof of Theorem 1a and put

(54) $$t_n = E[nU_n \mid \mathfrak{B}_n], \qquad T_n = nU_n - t_n\,.$$

Then, from (22), $t_n = O(n^{-1+\gamma})$ a.s. and hence

(55) $$n^{-\frac{1}{2}}\textstyle\sum_{k=1}^{n}t_k = O(n^{\gamma-\frac{1}{2}}) = o(1) \text{ a.s.}$$

as $n \to \infty$. Hence (53) will hold if we show that

(56) $$n^{-\frac{1}{2}}\textstyle\sum_{k=1}^{n}T_k \to_{\mathfrak{L}} N(0, \sigma^2/2\alpha^2).$$

This is done by an application of a slight extension of Lemma 6, p. 377 of [8] or Theorem C, p. 377 of [6]. Since the details are analogous to that given in [8] we will not present it here.

For (43) the only change in the proof is in the evaluation of the limit of Q_{2n}. With the help of (46) it is easily seen that if $\gamma = \frac{1}{4}$, then $Q_{2n} \to 2c^2\alpha_2/\alpha$ a.s. as $n \to \infty$ so that (43) follows.

Finally (41) and (44) are established by applying the same methods to (23), and making use of the fact that A_n and B_n differ only at most for a finite number of indices n a.s.

REMARKS. If γ is chosen $<\frac{1}{4}$, the bias in the estimate X_{n+1} of θ will dominate the error. The choice $\gamma = \frac{1}{4}$ therefore seems the most suitable in practice. It will be noticed that making c small will decrease the bias in (43) but increase the variance in (44) and vice versa. The best choice must achieve a compromise here. In some situations in practice, e.g. the quantal response estimation problem [10], it is usually the case that $\alpha_2 = 0$ and that MIV holds with $s \geqq 3$. In this case γ may be chosen as small as $\frac{1}{6}$ before a bias term in the asymptotic distribution of $n^{\frac{1}{2}}(X_n - \theta)$ appears while $n^{\frac{1}{3}}(A_n - \alpha)$ has an asymptotic distribution. Finally we note that the fact that 2 observations per step are required by the extended RM procedure does not put this procedure at a disadvantage compared to the old RM procedure: after n steps ($2n$ observations) its variance is still achieving the minimum value of the old RM procedure after $2n$ steps ($2n$ observations), as $n \to \infty$ according to e.g. (40).

5. Concluding remarks.

1. In practice one would not use the estimate A_n of α exactly as defined in (11) and (12) but would omit some of the earlier terms in the sum in (11) as n increases in order to get rid of large biases that may be present in these early terms.

2. An estimate of σ^2 would usually be required if the results of Theorem 3 are to be used for constructing approximate confidence intervals for θ. An example of an estimate for σ^2 is

$$\hat{\sigma}_n^2 = (2n)^{-1}\sum_{k=1}^{n}\{[Y_k' - A_n(X_k + c_k - X_{n+1})]^2 + [Y_k'' - A_n(X_k - c_k - X_{n+1})]^2\}.$$

It can be shown that this is weakly consistent under the conditions of Theorem 3 and also strongly consistent under an additional condition such as $\sup_x E|Z(x)|^4 < \infty$. Another possibility is to take two observations at each of $X_n - c_n$ and $X_n + c_n$ and to take their respective averages as the Y_n', Y_n'' in (10) while using their differences for estimation of σ^2.

3. The ideas developed here for the RM procedure can be carried over to the so-called Kiefer-Wolfowitz procedure for estimating the maximum (or minimum) of a regression function [5]. In this case one would take three observations on each step, viz. at $X_n + c_n$, X_n and $X_n - c_n$, and use the appropriate second order differences of the observations to estimate the second order derivative of the regression function in the maximum (or minimum). This estimate would

then be used to determine the next estimate X_{n+1} of the maximum (minimum) in a way analogous to that of the extended RM procedure discussed here.

4. Estimation of the slope of M at θ was also considered by Burkholder [2]. There the effect of feeding the estimate of the slope back into the recursive relation generating the estimate of θ was not investigated.

REFERENCES

[1] Blum, J. R. (1954). Approximation methods which converge with probability one. *Ann. Math. Statist.* **25** 382–386.

[2] Burkholder, D. L. (1956). On a class of stochastic approximation processes. *Ann. Math. Statist.* **27** 1044–1059.

[3] Dvoretzky, A. (1956). On stochastic approximation. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **1** 39–55. Univ. of California Press.

[4] Hodges, J. L., Jr. and Lehmann, E. L. (1956). Two approximations to the Robbins-Monro process. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **1** 95–104. Univ. of California Press.

[5] Kiefer, K. and Wolfowitz, J. (1952). Stochastic estimation of the maximum of a regression function. *Ann. Math. Statist.* **23** 462–466.

[6] Loève, M. (1960). *Probability Theory* (2nd Edition). Van Nostrand, Princeton.

[7] Robbins, H. and Monro, S. (1951). A stochastic approximation method. *Ann. Math. Statist.* **22** 400–407.

[8] Sacks, J. (1958). Asymptotic distributions of stochastic approximation procedures. *Ann. Math. Statist.* **29** 373–405.

[9] Venter, J. H. (1966). On Dvoretzky stochastic approximation theorems. *Ann. Math. Statist.* **37** 1534–1544.

[10] Venter, J. H. (1963). On stochastic approximation methods. Doctoral dissertation. University of Chicago.